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A SECRETARY PROBLEM WITH DOUBLE CHOICES

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Abstract We consider the situation in which the decision-maker is allowed to have two choices and he must choose either the best or the second best from the group of N applicants. The optimal stopping rule and the maximum probability of choosing either of them are derived.

1. Introduction

In this note we consider a version of optimal stopping problem, often referred to as secretary problem, beauty contest problem, dowry problem and Googol, in for example, Lindley [4], Chow et al. [1] and Gilbert and Mosteller [2]. The basic framework of these original problems can be stated as follows. N applicants appear before the decision-maker sequentially in a random order. Therefore all permutations are supposed to be equally likely. Each of these applicants can be ranked according to some quality by the decision-maker(1 being the best and N the worst). As each applicant appears, the decision-maker observes her rank relative to those preceding her and must decide whether to accept(choose) her or to reject her and continue to interview further applicants. The optimal stopping rule is the one which maximizes the probability of choosing the best applicant.

Mucci [5] and [6] treat the problems with generalized payoff function and obtain the asymptotic forms for the optimal payoff and the stopping rule by the analysis of related differential equations. In his papers the decision-maker is, however, allowed only one choice. Gilbert and Mosteller [2] and Sakaguchi [7] consider the multi choice problems in which the decision-maker who is allowed $r(r \geqq 2)$ choices succeeds, if either of his choices is the best. Another example in [8] of multi choice problem is to choose both the best and the second best from a group of applicants who have the right to refuse the offer with a known and fixed probability.

The problem we consider in this paper is that the decision-maker is allowed two choices and succeeds if either of his choices is the best or the second best. If he has not accepted until the last two applicants then he is forced to accept both of them. Our problem is a kind of multi choice problem and a generalization of the Gusein-Zade model. We wish to find the stopping rule which maximizes the probability of success.

We give the name "1-candidate" to any applicant who is best among the applicants that have already appeared and give the name "2-candidate" to any applicant who is second best among the applicants that have already appeared. When we need not distinguish between them, we call them simply "candidate". Then it is obvious that any applicant who is not a candidate can not be chosen. Let N be the number of applicants. For $1 \leq t \leq N$, we denote by $(t,i), i=1,2$, the situation in which the decision-maker is facing the t-th applicant who happens to be a i-candidate and he is still allowed two choices and by $(t,ij), i,j=1,2$, the situation in which the decision-maker, who has already chosen the i-candidate among the first t-1 applicants as the first choice, is facing the t-th applicant who happens to be a j-candidate.

2. Formulae and Their Consequences

Given the t-th applicant is a candidate, the conditional probability that the s-th(s>t) is the earliest candidate and, at the same time, earliest i-candidate(i=1,2) is $t(t-1)/s(s-1)(s-2)$ which we denote by π_{ts}. Hence, the probability that no candidate will appear after the t-th stage is $1-2\sum_{s=t+1}^{N}\pi_{ts} = t(t-1)/N(N-1)$. We denote by $u_t^{(i)}$ and $v_t^{(ij)}$ the probabilities of success under an optimal policy when starting from situations (t,i) and (t,ij) respectively. The probability of success is $t(2N-t-1)/N(N-1)$ when the decision-maker chooses the 1-candidate among the first t applicants and terminates the process and the probability of success is $t(t-1)/N(N-1)$ when he chooses the 2-candidate and terminates the process. So we now obtain the following recurrence relations.

$$(2.1) \qquad v_t^{(21)} = \max\begin{cases} A: & \dfrac{t(2N-t-1)}{N(N-1)} \\ R: & \displaystyle\sum_{s=t+1}^{N} \pi_{ts}\left(v_s^{(21)} + v_s^{(22)}\right) \end{cases} \qquad (3 \leq t \leq N-1;\ v_N^{(21)}=1)\ ,$$

$$(2.2)\qquad v_t^{(22)} = \max\begin{cases} A: & \frac{t(t-1)}{N(N-1)} \\ R: & \sum_{s=t+1}^{N} \pi_{ts}(v_s^{(21)} + v_s^{(22)}) \end{cases} \quad (3 \leq t \leq N-1;\ v_N^{(22)}=1),$$

$$(2.3)\qquad v_t^{(11)} = \max\begin{cases} A: & \frac{t(2N-t-1)}{N(N-1)} \\ R: & \frac{t(t-1)}{N(N-1)} + \sum_{s=t+1}^{N} \pi_{ts}(v_s^{(21)} + v_s^{(22)}) \end{cases} \quad (2 \leq t \leq N-1;\ v_N^{(11)}=1),$$

$$(2.4)\qquad v_t^{(12)} = \frac{t(t-1)}{N(N-1)} + \sum_{s=t+1}^{N} \pi_{ts}(v_s^{(11)} + v_s^{(12)}) \quad (2 \leq t \leq N-1;\ v_N^{(12)}=1),$$

$$(2.5)\qquad u_t^{(1)} = \max\begin{cases} A: & \frac{t(t-1)}{N(N-1)} + \sum_{s=t+1}^{N} \pi_{ts}(v_s^{(11)} + v_s^{(12)}) \\ R: & \sum_{s=t+1}^{N} \pi_{ts}(u_s^{(1)} + u_s^{(2)}) \end{cases} \quad (1 \leq t \leq N-1;\ u_N^{(1)}=1),$$

$$(2.6)\qquad u_t^{(2)} = \max\begin{cases} A: & \frac{t(t-1)}{N(N-1)} + \sum_{s=t+1}^{N} \pi_{ts}(v_s^{(21)} + v_s^{(22)}) \\ R: & \sum_{s=t+1}^{N} \pi_{ts}(u_s^{(1)} + u_s^{(2)}) \end{cases} \quad (2 \leq t \leq N-1;\ u_N^{(2)}=1),$$

where A and R symbolically represent acceptance and rejection respectively. (2.4) comes from the fact that it does no good to stop in situation (t,12). The system of equations (2.1)-(2.6) can be solved recursively to yield the optimal stopping rule and the maximum probability $u_1^{(1)}$.

We put

$$(2.7)\qquad v_t \equiv \sum_{s=t+1}^{N} \pi_{ts}(v_s^{(21)} + v_s^{(22)}),$$

then we have by (2.1) and (2.2)

$$(2.8)\qquad \begin{aligned} v_{t-1} &= \frac{1}{t}(v_t^{(21)} + v_t^{(22)}) + \frac{t-2}{t}v_t \\ &= v_t + \frac{1}{t}[\max(\frac{t(2N-t-1)}{N(N-1)} - v_t, 0) + \max(\frac{t(t-1)}{N(N-1)} - v_t, 0)] \end{aligned}$$

and hence v_t is non-increasing in t. Considering that $t(2N-t-1)/N(N-1) > t(t-1)/N(N-1)$, for $1<t<N$, and that both of these functions are increasing in t, we can summarize the optimal policy in situation (t,2j), j=1,2, as follows.

Theorem 1. There exists a pair of integers s_1^* and s_2^*, $1 \leq s_1^* \leq s_2^* \leq N$, such that the optimal policy in situation $(t,2j)$, $j=1,2$, is to accept the current candidate and stop immediately if and only if $t \geq s_j^*$, where

$$s_1^* = \min[\ t \mid \frac{t(2N-t-1)}{N(N-1)} \geq v_t\ ] \tag{2.9}$$

and

$$s_2^* = \min[\ t \mid \frac{t(t-1)}{N(N-1)} \geq v_t\ ]\ . \tag{2.10}$$

Gilbert and Mosteller, and Gusein-Zade have shown this theorem. The explicit expression of v_t was given in [3] and the values of s_1^* and s_2^* were given in Table 6 of [2].

Lemma 1. $v_t^{(11)}$ is increasing in t and we have, for $1<t\leq N$,

$$v_t^{(11)} = \frac{t(t-1)}{N(N-1)} + v_t \geq \frac{t(2N-t-1)}{N(N-1)}\ . \tag{2.11}$$

Proof: It is easily seen by (2.1),(2.2) and (2.7)

$$\begin{aligned}\frac{t(t-1)}{N(N-1)} + v_t &= \frac{t(t-1)}{N(N-1)} + \sum_{s=t+1}^{N} \pi_{ts}(v_s^{(21)} + v_s^{(22)}) \\ &\geq \frac{t(t-1)}{N(N-1)} + \sum_{s=t+1}^{N} \pi_{ts}[\frac{s(2N-s-1)}{N(N-1)} + \frac{s(s-1)}{N(N-1)}] \\ &= \frac{t(2N-t-1)}{N(N-1)}\ .\end{aligned}$$

Thus we have shown, with (2.3), the latter half of the lemma. Showing that $v_t^{(11)}$ is increasing in t is equivalent to showing that $v_t^{(11)}-v_{t-1}^{(11)}>0$. (2.7) and (2.11) give

$$v_t^{(11)}-v_{t-1}^{(11)} = \frac{1}{t}[(v_t^{(11)}-v_t^{(21)}) + (v_t^{(11)}-v_t^{(22)})]\ . \tag{2.12}$$

Hence, considering (2.9) and (2.10), we can rewrite (2.12) as

$$v_t^{(11)}-v_{t-1}^{(11)} = \begin{cases} \frac{1}{t}\frac{2t(t-1)}{N(N-1)}\ , & 2\leq t<s_1^* \\ \frac{1}{t}[(v_t^{(11)}-\frac{t(2N-t-1)}{N(N-1)}) + \frac{t(t-1)}{N(N-1)}]\ , & s_1^*\leq t<s_2^* \\ \frac{1}{t}[(v_t^{(11)}-\frac{t(2N-t-1)}{N(N-1)}) + v_t]\ , & s_2^*\leq t\leq N\ . \end{cases} \tag{2.13}$$

In either case, (2.13) is positive. Thus the lemma is proved.

Lemma 1 and (2.4) show that, as far as the applicant chosen as the first choice still remains a candidate, the decision-maker should not accept the current candidate as the second choice. We can rewrite (2.4) as

$$v_t^{(12)} = \frac{1}{t+1}(v_{t+1}^{(11)} + tv_{t+1}^{(12)})\ . \tag{2.14}$$

We have the following lemma.

Lemma 2. $v_t^{(12)}$ is increasing in t and we have, for $1<t<N$, $v_t^{(12)}>v_t^{(11)}$.

Proof: We show $v_t^{(12)}>v_t^{(11)}$ by backward induction. It is easily checked $v_{N-1}^{(12)}>v_{N-1}^{(11)}$. Suppose that $v_{t+1}^{(12)}>v_{t+1}^{(11)}$, then, by (2.14) and Lemma 1, we soon have $v_t^{(12)}>v_{t+1}^{(11)}>v_t^{(11)}$. Hence, the induction is completed. Applying this result to (2.14), we also have $v_t^{(12)}<v_{t+1}^{(12)}$. Thus the lemma is proved.

Now put

$$(2.15) \qquad u_t \equiv \sum_{s=t+1}^{N} \pi_{ts}(u_s^{(1)} + u_s^{(2)}) ,$$

then, by (2.5) and (2.6), u_t turns out to be non-increasing in t. Hence, (2.5) and (2.6), combined with Lemma 2, lead us to the following theorem.

Theorem 2. There exists a pair of integers d_1^* and d_2^*, $1\leqq d_1^*\leqq d_2^*\leqq N$, such that the optimal policy in situation (t,i), i=1,2, is to accept the current candidate if and only if $t\geqq d_i^*$, where

$$(2.16) \qquad d_1^* = \min[\ t \mid v_t^{(12)}\geqq u_t]$$

and

$$(2.17) \qquad d_2^* = \min[\ t \mid v_t^{(11)}\geqq u_t] .$$

Ultimately optimal stopping policy of our problem is given by Theorem 1,2, Lemma 1 and (2.4). Table 1 gives the values of s_1^*, s_2^*, d_1^* and d_2^* and the maximum probability $u_1^{(1)}$ for some values of N.

Table 1.

N	s_1^*	s_2^*	d_1^*	d_2^*	$u_1^{(1)}$
4	2	(3)4	(1)2	(2)3	0.9167
5	(2)3	4	2	3	0.9167
6	3	5	2	3	0.9000
7	3	5	2	4	0.8810
8	4	6	2	4	0.8705
9	4	7	3	5	0.8585
10	4	7	3	5	0.8551
20	8	14	5	9	0.8248
50	18	34	11	22	0.8056
100	35	67	22	43	0.7995
200	70	134	44	85	0.7965
500	174	334	108	211	0.7946
1000	348	667	216	421	0.7940
∞	0.3470N	0.6667N	0.2150N	0.4201N	0.7934

where the values of s_1^* and s_2^* are reproduced from Table 6 in Gilbert and Mosteller(1966).

Following the method proposed by [5], [6] and [8], we finally derive the asymptotic values of s_1^*, s_2^*, d_1^*, and d_2^*, when N tends to infinity, by the analysis of the corresponding differential and integral equations. Write $v_t^{(ij)}$, $u_t^{(i)}$, v_t and u_t, as $f^{(ij)}(t/N)$, $g^{(i)}(t/N)$, p(t/N) and q(t/N) respectively, and let N and t go to infinity, setting $t/N=x$. Then, by (2.1), (2.2), (2.11), (2.12) and (2.14), we have for $0 \leq x \leq 1$

$$f^{(21)}(x) = \max(x(2-x),\ p(x)), \tag{2.18}$$

$$f^{(22)}(x) = \max(x^2,\ p(x)), \tag{2.19}$$

$$p(x) = f^{(11)}(x) - x^2, \tag{2.20}$$

$$f^{(11)\prime}(x) = \frac{2}{x} f^{(11)}(x) - \frac{1}{x}(f^{(21)}(x) + f^{(22)}(x)), \tag{2.21}$$

$$f^{(12)\prime}(x) = \frac{1}{x}(f^{(12)}(x) - f^{(11)}(x)), \tag{2.22}$$

where $f^{(21)}(1)=f^{(22)}(1)=f^{(11)}(1)=f^{(12)}(1)=1$ and $p(1)=1$.

These equations (2.18)-(2.22) can be solved and especially $f^{(11)}(x)$ and $f^{(12)}(x)$ become

$$f^{(11)}(x) = \begin{cases} x^2+\alpha_1(2-\alpha_1), & 0 \leq x \leq \alpha_1 \\ 2x(x-\log x+\log \alpha_2), & \alpha_1 \leq x \leq \alpha_2 \\ x(2-x), & \alpha_2 \leq x \leq 1 \end{cases} \tag{2.23}$$

and

$$f^{(12)}(x) = \begin{cases} -x^2+(\alpha_1^2-2\alpha_1+1-2\log \alpha_2)x+\alpha_1(2-\alpha_1), & 0 \leq x \leq \alpha_1 \\ x(2-2x-2\log \alpha_2+(\log x/\alpha_2)^2), & \alpha_1 \leq x \leq \alpha_2 \\ x(x-2\log x), & \alpha_2 \leq x \leq 1 \end{cases} \tag{2.24}$$

where $\alpha_2=2/3$ and $\alpha_1 \simeq 0.3470$ is the unique root in $(0,\alpha_2)$ of the equation

$$x-\log x = 1-\log \alpha_2 . \tag{2.25}$$

This coincides with the results of [2] and [3]. Similarly we have, by (2.5), (2.6) and (2.15), for $0 \leq x \leq 1$

$$g^{(1)}(x) = \max(f^{(12)}(x),\ q(x)), \tag{2.26}$$

$$g^{(2)}(x) = \max(f^{(11)}(x),\ q(x)), \tag{2.27}$$

$$q(x) = \int_x^1 \frac{x^2}{y^3} (g^{(1)}(y) + g^{(2)}(y))dy, \tag{2.28}$$

where $g^{(1)}(1)=g^{(2)}(1)=1$ and $q(1)=0$.

Applying (2.23) and (2.24) to (2.26)-(2.28), we reach,after tedious calculations,

$$\begin{cases} q(\beta_1), & 0 \leq x \leq \beta_1 \\ x^2+[(\alpha_1-1)^2\log \alpha_1-2\log \alpha_2\log \beta_2-(\log \alpha_1-\log \beta_2)\log^2\alpha_2 \\ +(\log^2\alpha_1-\log^2\beta_2)\log \alpha_2-\frac{1}{3}(\log^3\alpha_1-\log^3\beta_2)]x \\ +(1-\alpha_1^2+2\log \alpha_1)x\log x+\alpha_1(2-\alpha_1), & \beta_1 \leq x \leq \alpha_1 \end{cases}$$

$$
(2.29)\quad q(x)=\begin{cases} 2x^2+(\log^2\alpha_2\log\beta_2+2\log\alpha_2-\log\alpha_2\log^2\beta_2 \\ -2\log\alpha_2\log\beta_2+\frac{1}{3}\log^3\beta_2)x-\frac{1}{3}x\log^3 x \\ +(\log\alpha_2)x\log^2 x+(2\log\alpha_2-2-\log^2\alpha_2)x\log x, & \alpha_1\leqq x\leqq\beta_2 \\ -3x^2+(\log^2\alpha_2-2\log\alpha_2+2)x+x\log^2 x-2(\log\alpha_2)x\log x, & \beta_2\leqq x\leqq\alpha_2 \\ -2x\log x, & \alpha_2\leqq x\leqq 1 \end{cases}
$$

where $\beta_2 \underset{=}{\sim} 0.4201$ is the unique root in (0,1) of the equation

$$
(2.30)\qquad (\log x-\log\alpha_2)^2+2-5x-4\log\alpha_2+2\log x=0
$$

and $\beta_1 \underset{=}{\sim} 0.2150$ is also the unique root in $(0,\beta_2)$ of the equation

$$
(2.31)\qquad \begin{aligned} &2x+(1-\alpha_1^2+2\log\alpha_1)\log x+[(1-\alpha_1^2)+(\alpha_1^2-2\alpha_1+3)\log\alpha_1 \\ &-2\log\alpha_2\log\beta_2-(\log\alpha_1-\log\beta_2)\log^2\alpha_2 \\ &+(\log^2\alpha_1-\log^2\beta_2)\log\alpha_2-\frac{1}{3}(\log^3\alpha_1-\log^3\beta_2)]=0 . \end{aligned}
$$

Summarizing the above results we reach the following lemma.

Lemma 3. We have asymptotically $s_1^* \underset{=}{\sim} \alpha_1 N$, $s_2^* \underset{=}{\sim} \alpha_2 N$, $d_1^* \underset{=}{\sim} \beta_1 N$, and $d_2^* \underset{=}{\sim} \beta_2 N$. And then $q(\beta_1)$ in (2.29), maximum probability of success, becomes approximately 0.7934.

3. Remark

A generalization of our problem would be as follows. For any positive integers r and k, the decision-maker is allowed r choices and succeeds if either of his choices is one of k best.

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