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AN ALGORITHM FOR A PARTIALLY CHANCE-CONSTRAINED E-MODEL

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Abstract This paper considers an E-model having a random linear inequality constraint and provides an algorithm for solving it. An original problem P is first transformed into deterministic equivalent problem P'. For solving P', a subsidiary problem P (μ) with a parameter μ is defined. The dual-like relation between P and P (μ) is clarified. Then an algorithm for solving P (μ) is proposed. This algorithm is based on the parametric quadratic programming technique. Next fully utilizing this algorithm for P (μ), main algorithm is constructed. Finally, applicability of the above dual-like relation to other nonlinear problems is suggested.

1. Introduction

Many types of chance-constrained programming problems have been considered [1-6,8,9] since Charnes and Cooper [1] introduced chance constraints into mathematical programming problems. This paper considers an E-model having a random linear inequality constraint and provides an algorithm for solving it. Few problems with these stochastic constraints have been investigated so far and even few for solution algorithms.

In Section 2 the problem P and its deterministic equivalent problem P' are formulated. Section 3 introduces subsidiary problem P(μ) parametrized with μ and derives useful relations between P' and P(μ). Section 4 gives Algorithm 1 for solving P(μ) based on the parametric procedure [5]. Section 4 also proves validity and finiteness of Algorithm 1. Section 5 introduces another type of the subsidiary problem P^R and provides the main Algorithm 2 for solving P' utilizing Algorithm 1 and properties of P^R. The validity and finiteness of Algorithm 2 are proved in the same section 5. After an illustrative example in Section 6, Section 7 concludes this paper and discusses further development.

2. Problem Formulation

This paper considers the following problem P.

$$\text{P:} \quad \text{Maximize} \quad E(c^T x)$$

$$\text{subject to} \quad Prob\{ a^T x \leqq b\} \geqq \alpha \tag{2.1}$$

$$A_1 x \leqq B_1 \;, \qquad x \geqq 0 \;,$$

where T and E mean transpose and expectation respectively; $a=(a_1,\cdots,a_n)^T$ is an n-dimensional random vector and distributed according to multivariate normal distribution with variance-covariance matrix W and mean vector $(E(a_1),\cdots,E(a_n))^T$; b is distributed according to a normal distribution with mean $E(b)$ and variance σ_0^2; a_i, $i=1,2,\ldots,n$, and b are mutually independent; $c=(c_1, c_2,\cdots,c_n)^T$ is an n-dimensional random vector with mean $(E(c_1),\cdots,E(c_n))^T$; A_1 is an m×n matrix; B_1 is an m-dimensional vector; $x=(x_1,\cdots,x_n)^T$ is an n-dimensional decision variable vector; $\alpha(>\frac{1}{2})$ is a probability level at least with which constraint $a^T x \leqq b$ must hold.

We assume that W is a positive definite matrix and $E(a_i)$, $E(c_i)$, $i=1,\ldots,n$, and $E(b)$ are finite. (Hereafter, $(E(a_1),\cdots,E(a_n))^T$ and $(E(c_1),\cdots,E(c_n))^T$ are denoted simply with $E(a)$ and $E(c)$ respectively.)

The problem P is equivalent to the following deterministic problem P'. (See Appendix 1 for the details.)

$$\text{P':} \quad \text{Maximize} \quad E(c)^T x$$

$$\text{subject to} \quad E(a)^T x + K_\alpha(\sigma_0^2 + x^T W x)^{\frac{1}{2}} \leqq E(b)$$

$$A_1 x \leqq B_1 \;, \qquad x \geqq 0 \;,$$

where K_α is a quantile of order α of the standard normal distribution function F, i.e., $K_\alpha = \mathrm{F}^{-1}(\alpha) > 0$.
Moreover we assume that the set

$$S \triangleq \{x \mid E(a)^T x + K_\alpha(\sigma_0^2 + x^T W x)^{\frac{1}{2}} \leqq E(b),\quad A_1 x \leqq B_1,\ x \geqq 0\ \}$$

is not empty and bounded. As is easily shown, S is a convex set and so P' is a convex programming problem.

3. Subsidiary Problem P(μ) and Its Relation to P'

Let x^* and μ^* denote an optimal solution and the optimal value of P' respectively. To solve P', subsidiary problem P(μ) is defined as follows.

$$\text{P}(\mu):\qquad \mathit{Minimize}\quad E(a)^T x + K_\alpha(\sigma_0^2 + x^T W x)^{\frac{1}{2}}$$

$$\mathit{subject\ to}\quad E(c)^T x \geqq \mu\ ,\qquad A_1 x \leqq B_1\ ,\qquad x \geqq 0\ .$$

Denoting the optimal solution[†] and the optimal value of P(μ) with $x(\mu)$ and $z(\mu)$ respectively, then we can derive the following relation between P(μ) and P'.

Theorem 1. If $x(\mu)$ satisfies

$$E(a)^T x(\mu) + K_\alpha(\sigma_0^2 + x(\mu)^T W x(\mu))^{\frac{1}{2}} = E(b) \quad \text{and} \quad E(c)^T x(\mu) = \mu\ ,$$

then $x(\mu)$ is also an optimal solution of P'.

Proof: The Kuhn-Tucker condition (KTP) of P' is as follows [7].

$$\text{KTP}:\qquad v - pE(a) - K_\alpha p \frac{Wx}{(\sigma_0^2 + x^T Wx)^{\frac{1}{2}}} - A_1^T q = -E(c)$$

$$E(a)^T x + K_\alpha(\sigma_0^2 + x^T Wx)^{\frac{1}{2}} + s_0 = E(b)$$

$$A_1 x + s = B_1 \qquad v^T x + s^T q + s_0 p = 0$$

$$v,\ x,\ s,\ q,\ s_0,\ p \geqq 0\ ,$$

where v is an n-dimensional vector; s, q are m-dimensional vectors; s_0, p are scalars. On the other hand, the Kuhn-Tucker condition (KTP(μ)) of P(μ) becomes as follows [7].

[†] As is easily proved, P(μ) is a strictly convex programming problem and so $x(\mu)$ is unique.

$$\text{KTP}(\mu): \quad \bar{v} - (E(a) + \frac{K_\alpha Wx}{(\sigma_0^2 + x^T Wx)^{\frac{1}{2}}}) - A_1^T \bar{q} = -\bar{r}E(c)$$

$$E(c)^T x - \bar{s}_0 = \mu \qquad A_1^T x + \bar{s} = B_1$$

$$\bar{v}^T x + \bar{s}_0 \bar{r} + \bar{q}^T \bar{s} = 0 \qquad \bar{s}_0,\ \bar{v},\ x,\ \bar{s},\ \bar{r},\ \bar{q} \geqq 0 \ ,$$

where $\bar{v}$ is an n-dimensional vector; $\bar{s}$, $\bar{q}$ are m-dimensional vectors; $\bar{s}_0$, $\bar{r}$ are scalars. Let

$$X(\mu) \underline{\triangle} (x(\mu)^T,\ \bar{v}(\mu)^T,\ \bar{q}(\mu)^T,\ \bar{r}(\mu)^T,\ \bar{s}_0(\mu)^T,\ \bar{s}(\mu)^T)^T$$

denote the solution of KTP(μ). Since $E(c)^T x(\mu) = \mu$ means $\bar{s}_0(\mu) = 0$, $\bar{r}(\mu)$ must be positive. By the positivity of $\bar{r}(\mu)$ and the condition

$$E(a)^T x(\mu) + K_\alpha (\sigma_0^2 + x(\mu)^T Wx(\mu))^{\frac{1}{2}} = E(b) \ ,$$

the solution of KTP is constructed from $X(\mu)$ as follows:

$$v = \bar{v}(\mu)/\bar{r}(\mu), \quad p = 1/\bar{r}(\mu), \quad q = \bar{q}(\mu)/\bar{r}(\mu), \quad s_0 = 0, \quad x = x(\mu), \quad s = \bar{s}(\mu) \ .$$

(Indeed this solution satisfies KTP.) Since P' and P(μ) are strictly concave programming problem and strictly convex programming problem respectively, the feasible solution of KTP and KTP(μ) are optimal solutions of P' and P(μ) respectively. Therefore $x(\mu)$ satisfying conditions of this theorem is the optimal solution of P'. □

Moreover the following properties of P(μ) can be derived.

Property 1. $z(\mu)$ is a convex function of μ.

Proof: For $\mu_1 < \mu_2$, $0 < \lambda < 1$ and $\bar{\lambda} \underline{\triangle} 1 - \lambda$,

$$\lambda z(\mu_1) + \bar{\lambda} z(\mu_2) - z(\lambda\mu_1 + \bar{\lambda}\mu_2) = \lambda\{E(a)^T x(\mu_1) + K_\alpha(\sigma_0^2 + x(\mu_1)^T Wx(\mu_1))^{\frac{1}{2}}\}$$

$$+ \bar{\lambda}\{E(a)^T x(\mu_2) + K_\alpha(\sigma_0^2 + x(\mu_2)^T Wx(\mu_2))^{\frac{1}{2}}\}$$

$$-\{E(a)^T x(\lambda\mu_1 + \bar{\lambda}\mu_2) + K_\alpha(\sigma_0^2 + x(\lambda\mu_1 + \bar{\lambda}\mu_2)^T Wx(\lambda\mu_1 + \bar{\lambda}\mu_2))^{\frac{1}{2}}\}$$

$$= \lambda E(a)^T x(\mu_1) + \bar{\lambda} E(a)^T x(\mu_2) - E(a)^T x(\lambda\mu_1 + \bar{\lambda}\mu_2)$$

$$+\lambda K_\alpha\{\sigma_0^2+x(\mu_1)^T Wx(\mu_1)\}^{\frac{1}{2}}+\bar{\lambda}K_\alpha\{\sigma_0^2+x(\mu_2)^T Wx(\mu_2)\}^{\frac{1}{2}}$$

$$-K_\alpha\{\sigma_0^2+x(\lambda\mu_1+\bar{\lambda}\mu_2)^T Wx(\lambda\mu_1+\bar{\lambda}\mu_2)\}^{\frac{1}{2}}$$

$$\overset{\dagger}{\geqq} E(a)^T(\lambda x(\mu_1)+\bar{\lambda}x(\mu_2))-E(a)^T x(\lambda\mu_1+\bar{\lambda}\mu_2)$$

$$+K_\alpha\{\sigma_0^2+(\lambda x(\mu_1)+\bar{\lambda}x(\mu_2))^T Wx(\lambda x(\mu_1)+\bar{\lambda}x(\mu_2))\}^{\frac{1}{2}}$$

$$-K_\alpha\{\sigma_0^2+x(\lambda\mu_1+\bar{\lambda}\mu_2)^T Wx(\lambda\mu_1+\bar{\lambda}\mu_2)\}$$

$\geqq 0$ (by the feasibility of $\lambda x(\mu_1)+\bar{\lambda}x(\mu_2)$ and optimality of $x(\lambda\mu_1+\bar{\lambda}\mu_2)$ for $P(\lambda\mu_1+\bar{\lambda}\mu_2)$). □

Property 2. $z(\mu)$ is a nondecreasing function of μ.

Proof: It is clear from the fact that the feasible region of $P(\mu)$ becomes smaller as μ increases. □

Theorem 2. Without any loss of generality, we can always assume $\bar{s}_0(\mu)=0$.

Proof: If there exists a $\hat{\mu}$ such that $\bar{s}_0(\hat{\mu})>0$, then $z(\mu)=z(\hat{\mu})$ and $x(\mu)=x(\hat{\mu})$ for any $\hat{\mu}+\bar{s}_0(\hat{\mu})\geqq\mu\geqq\hat{\mu}$ since $\bar{s}_0(\hat{\mu})>0$ implies

$$E(c)^T x(\hat{\mu})\geqq\hat{\mu}+\bar{s}_0(\hat{\mu}) \tag{3.1}$$

and (3.1) means that $x(\hat{\mu})$ is optimal for any μ among $\hat{\mu}+\bar{s}_0(\hat{\mu})\geqq\mu\geqq\hat{\mu}$ from Property 2. Convexity of $z(\mu)$ shows that this occurs only the first portion of $z(\mu)$. Since $S\neq\phi$ implies $z(\hat{\mu})\leqq E(b)$, this portion can be excluded from further consideration by Theorem 1. That is, we can assume $\bar{s}_0(\mu)=0$ without any loss of generality. □

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†by

$$K_\alpha\{\sigma_0^2+x(\mu_1)^T Wx(\mu_1)\}^{\frac{1}{2}}+\bar{\lambda}K_\alpha\{\sigma_0^2+x(\mu_2)Wx(\mu_2)\}^{\frac{1}{2}}$$

$$\geqq K_\alpha\{\sigma_0^2+(\lambda x(\mu_1)+\bar{\lambda}x(\mu_2))^T W(\lambda x(\mu_1)+\bar{\lambda}x(\mu_2))\}^{\frac{1}{2}}$$

as is shown in Appendix 2.)

Theorem 2 permits us to assume that $E(c)^T x=\mu$ in Theorem 1 and shows $z(\mu) > z(\mu')$ for $\mu > \mu'$ as byproduct. That is, Property 2 is strengthened as follows.

Property 2' There exists $\check{\mu}$ such that for any $\mu \geqq \check{\mu}$, $z(\mu)$ is monotonically increasing function of μ.

Now we must check whether μ such that $z(\mu) = E(b)$ exists or not. For this purpose, let

$$\tilde{\mu} \triangleq max\{E(c)^T x \ |A_1 x \leqq B_1, \quad x \geqq 0\},$$

Note that $\tilde{\mu}$ may not exist. If $\tilde{\mu}$ exists, then $x(\mu)$ for $\mu > \tilde{\mu}$ does not exist. Moreover if $E(b) > z(\tilde{\mu})$ holds, μ such that $z(\mu) = E(b)$ is not defined. But in this case, $x(\tilde{\mu})$ becomes an optimal solution of P' as is easily shown.

Property 3. μ such that $z(\mu) = E(b)$ (that is, the optimal value of P') is unique if it exists.

Proof: This is clear from $z(\check{\mu}) \leqq E(b)$ and Property 2'. Note that $z(\check{\mu}) \leqq E(b)$ is derived from $S \neq \phi$. □

4. Algorithm 1 for Solving P(μ)

In order to solve P(μ), we introduces in this section an auxiliary parametrized problem $P^R(\mu)$.

$$P^R(\mu): \qquad Minimize \ RE(a)^T x + \frac{1}{2} K_\alpha (\sigma_0^2 + x^T Wx)$$

$$subject\ to \quad E(c)^T x \geqq \mu \ , \quad A_1 x \leqq B_1 \ , \quad x \geqq 0 \ .$$

Note that the feasible region of $P^R(\mu)$ coincides with that of P(μ). Let $x^R(\mu)$ and $z^R(\mu)$ denote the optimal solution[†] and optimal value of $P^R(\mu)$.

Theorem 3. $x^R(\mu)$ is the optimal solution $x(\mu)$ of P(μ) if it satisfies

$$R^2 = (\sigma_0^2 + x^R(\mu)^T Wx^R(\mu)).$$

Proof: Each $P^R(\mu)$ is a convex programming problem and corresponding Kuhn-Tucker condition $KTP^R(\mu)$ becomes as follows.

$$KTP^R(\mu): \qquad \hat{v} - K_\alpha Wx + \hat{r}E(c) - A_1^T \hat{q} = RE(a)$$

[†] $P^R(\mu)$ is a strictly convex function and so $x^R(\mu)$ is unique.

$$A_1 x + \hat{s} = B_1 \qquad E(c)^T x - \hat{s}_0 = \mu$$

$$\hat{v}^T x + \hat{s}^T \hat{q} + \hat{r}\hat{s}_0 = 0 \qquad \hat{v},\ x,\ \hat{s},\ \hat{q},\ \hat{r},\ \hat{s}_0 \geqq 0 \quad .$$

If $x^R(\mu)$ satisfies $R = (\sigma_0^2 + x^R(\mu)^T W x^R(\mu))^{\frac{1}{2}}$, then $X(\mu)$ can be constructed from a solution $X^R(\mu) \triangleq (x^R(\mu),\ \hat{v}^R(\mu),\ \hat{q}^R(\mu),\ \hat{r}^R(\mu),\ \hat{s}_0^R(\mu),\ \hat{s}^R(\mu))$ of KTP$^{\text{R}}$(μ) as follows.

$$X(\mu): \qquad x(\mu) = x^R(\mu), \quad \bar{v}(\mu) = \hat{v}^R(\mu)/R, \quad \bar{q}(\mu) = \hat{q}^R(\mu)/R, \quad \bar{r}(\mu) = \hat{r}^R(\mu)/R,$$

$$\bar{s}_0(\mu) = \hat{s}_0(\mu), \quad \bar{s}(\mu) = \hat{s}^R(\mu) \ .$$

Indeed the solution constructed as above satisfies KTP(μ) as is easily checked. Therefore $x^R(\mu)$ becomes an optimal solution of P(μ). □

Property 4. $z^R(\mu)$ is a monotonically increasing function of μ.

Proof: For $R' < R$, since $x^R(\mu)$ is an optimal solution of P$^{\text{R}}$(μ),

$$RE(a)^T x^R(\mu) + \frac{1}{2}K_\alpha(\sigma_0^2 + x^R(\mu)^T W x^R(\mu)) \leqq RE(a)^T x^{R'}(\mu) + \frac{1}{2}K_\alpha(\sigma_0^2 + x^{R'}(\mu)^T W x^{R'}(\mu))$$

holds. This implies

$$(4.1) \qquad R\{E(a)^T x^R(\mu) - E(a)^T x^{R'}(\mu)\} + \frac{1}{2}K_\alpha\{x^R(\mu)^T W x^R(\mu) - x^{R'}(\mu)^T W x^{R'}(\mu)\} \leqq 0.$$

Similarly, from the optimality of $x^{R'}(\mu)$,

$$R'E(a)^T x^{R'}(\mu) + \frac{1}{2}K_\alpha\{\sigma_0^2 + x^{R'}(\mu)^T W x^{R'}(\mu)\} \leqq R'E(a)^T x^R(\mu) + \frac{1}{2}K_\alpha\{\sigma_0^2 + x^R(\mu)^T W x^R(\mu)\},$$

or

$$(4.2) \qquad R'\{E(a)^T x^R(\mu) - E(a)^T x^{R'}(\mu)\} + \frac{1}{2}K_\alpha\{x^R(\mu)^T W x^R(\mu) - x^{R'}(\mu)^T W x^{R'}(\mu)\} \geqq 0$$

holds. (4.1) and (4.2) together show

$$(R - R')\{E(a)^T x^R(\mu) - E(a)^T x^{R'}(\mu)\} \leqq 0,$$

that is, from $R < R'$

$$E(a)^T x^R(\mu) \leqq E(a)^T x^{R'}(\mu)$$ □

Property 6. $x^R(\mu)^T W x^R(\mu)$ is a nondecreasing function of R.

Proof: For $R < R'$

$$(4.3)\quad R'E(a)^T x^{R'}(\mu)+\frac{1}{2}K_\alpha\{\sigma_0^2+x^{R'}(\mu)^T Wx^{R'}(\mu)\} \leqq R'E(a)^T x^R(\mu)+\frac{1}{2}K_\alpha\{\sigma_0^2+x^R(\mu)^T Wx^R(\mu)\} \;.$$

Since

$$(4.4)\qquad E(a)^T x^{R'}(\mu) \geqq E(a)^T x^R(\mu)$$

from Property 5, (4.3) and (4.4) together imply

$$x^{R'}(\mu)^T Wx^{R'}(\mu) \leqq x^R(\mu)^T Wx^R(\mu) \qquad \square$$

We define $R(\mu) \triangleq \{\sigma_0^2 + x(\mu)^T Wx(\mu)\}^{\frac{1}{2}}$. $R(\mu)$ is a target point of R of $P^R(\mu)$. Next Theorem 4 provides useful informations about $R(\mu)$ even if $R \neq R(\mu)$.

Theorem 4.

(i) $R > R(\mu) \longleftrightarrow R^2 - \{\sigma_0^2 + x^R(\mu)^T Wx^R(\mu)\} > 0$

(ii) $R < R(\mu) \longleftrightarrow R^2 - \{\sigma_0^2 + x^R(\mu)^T Wx^R(\mu)\} < 0$

(iii) $R = R(\mu) \longleftrightarrow R^2 - \{\sigma_0^2 + x^R(\mu)^T Wx^R(\mu)\} = 0$.

Proof: For each $x^R(\mu)$, $x^R(\mu)^T Wx^R(\mu) < \infty$ holds since $P^R(\mu)$ has the same feasible region as $P(\mu)$ and boundedness of S implies $E(a)^T x^R(\mu) > -\infty$. Therefore, from Property 6 there exists a sufficiently large $\bar{R}$ such that for $R > \bar{R}$, $x^R(\mu)Wx^R(\mu) = constant$. The continuity of $x^R(\mu)^T Wx^R(\mu)$ with respect to R can be derived from the continuity of $x^R(\mu)$ with respect to R. Therefore, Mean-Value Theorem, Theorem 3 and uniqueness of $x(\mu)$ together prove Theorem 4. $\square$

Now we are ready to solve $P(\mu)$, fully utilizing $P^R(\mu)$. Generally $X^R(\mu)$ depends upon μ and R, which determine a basic matrix B. Being based on B, there exist constant vectors d_B', e_B', g_B' and a certain interval $L_B(\mu) \leqq R \leqq U_B(\mu)$ determined by the basic matrix B and μ, and so $X^R(\mu)$ can be written as below.

$$X^R(\mu) = Rd_B' + \mu e_B' + g_B' \qquad (L_B(\mu) \leqq R \leqq U_B(\mu)).$$

Moreover, taking x part of $X^R(\mu)$, we can write down as

$$x^R(\mu) = Rd_B + \mu e_B + g_B \quad ,$$

using d_B, e_B and g_B (x part of d'_B, e'_B and g'_B respectively).

By above discussion, the condition

$$R^2 = \sigma_0^2 + x^R(\mu)^T W x^R(\mu)$$

is equivalent to the condition that one of roots of the equation,

$$(d_B^T W d_B - 1)R^2 + 2(\mu e_B + g_B)^T W d_B R + (\mu e_B + g_B)^T W(\mu e_B + g_B) + \sigma_0^2 = 0,$$

exists on the interval $[L_B(\mu), U_B(\mu)]$. Hereafter let us refer this equation to Q-equation. The roots of Q-equation are as follows:

(Case a) $d_B^T W d_B = 1$

$$R = \frac{-\{(\mu e_B + g_B)^T W(\mu e_B + g_B) + \sigma_0^2\}}{2(\mu e_B + g_B)^T W d_B}$$

(Case b) $d_B^T W d_B \neq 1$

$$R = \frac{-(\mu e_B + g_B)^T W d_B \pm \sqrt{D}}{d_B^T W d_B - 1} \quad ,$$

where $D = \{(\mu e_B + g_B)^T W d_B\}^2 - (d_B^T W d_B - 1)\{(\mu e_B + g_B)^T W(\mu e_B + g_B) + \sigma_0^2\}$.

Remark 1. $R \geqq \sigma_0$ only must be checked for $R^2 = \sigma_0^2 + x^R(\mu)^T W x^R(\mu)$ since W is positive definite.

Let $K^\mu(R) \underline{\Delta} \sigma_0^2 + x^R(\mu)^T W x^R(\mu) - R^2$. Then if $K^\mu(L_B(\mu)) \geqq 0$ and $K^\mu(U_B(\mu)) \leqq 0$, one root of Q-equation exists in the interval $[L_B(\mu), U_B(\mu)]$.

[Algorithm 1 for solving P(μ)]

Step 1: Set $R_\ell \leftarrow \sigma_0$, $R_u \leftarrow M$ (M is a sufficiently large positive number) and $R \leftarrow R_0$ (R_0 is an arbitrary number such that $R_0 \geqq \sigma_0$). Solve $P^R(\mu)$ and B, d_B, e_B, g_B, $L_B(\mu)$ and $U_B(\mu)$. Go to Step 2.

Step 2: If $K^\mu(L_B(\mu)) < 0$, set $R_u \leftarrow L_B(\mu)$ and $R \leftarrow (R_u + R_\ell)/2$, and go to Step 4; If $K^\mu(L_B(\mu)) = 0$, set $x(\mu) = L_B(\mu)d_B + \mu e_B + g_B$ and terminate; If $K^\mu(L_B(\mu)) > 0$, go to Step 3.

Step 3: If $K^\mu(U_B(\mu)) < 0$, solve Q-equation, find roots β_1, β_2 and go to Step 5; If $K^\mu(U_B(\mu)) = 0$, set $x(\mu) = U_B(\mu)d_B + \mu e_B + g_B$ and terminate; If $K^\mu(U_B(\mu)) > 0$, set $R_\ell \leftarrow U_B(\mu)$ and $R \leftarrow (R_\ell + R_u)/2$, and go to Step 4.

Step 4: Solve $P^R(\mu)$ and find B, d_B, e_B, g_B, $L_B(\mu)$ and $U_B(\mu)$. Return to Step 2.

Step 5: If $\beta_1(\beta_2)$[†] belongs to $[L_B(\mu), U_B(\mu)]$, set $x(\mu) = \beta_1 d_B + \mu e_B + g_B$ ($x(\mu) = \beta_2 d_B + \mu e_B + g_B$) and terminate.

Remark 2. (i) If $K^\mu(L_B(\mu)) < 0$, $K^\mu(U_B(\mu)) < 0$ necessarily holds by Theorem 4. Thus the test for $K^\mu(U_B(\mu))$ is to be omitted. While, if $K^\mu(U_B(\mu)) > 0$, $K^\mu(L_B(\mu)) > 0$ holds and the test for $K^\mu(L_B(\mu))$ is also omitted. (ii) $[L_B(\mu), U_B(\mu)] \subseteqq [R_\ell, R_u]$ and $U_B(\mu) - L_B(\mu) \leqq \frac{1}{2}(R_u - R_\ell)$ hold except the first $[L_B(\mu), U_B(\mu)]$.

Theorem 5. Algorithm 1 terminates after finite iterations and upon termination, it finds $x(\mu)$.

Proof: (Finiteness) After each calculation of Step 4, five cases (a) ∿ (e) as illustrated in Figure 1a ∿ Figure 1e are possible. In case (d)((e)), it is clear that

$$x(\mu) = L_B(\mu)d_B + \mu e_B + g_B \quad (\ x(\mu) = U_B(\mu)d_B + \mu e_B + g_B\ )$$

holds. In case (c), either β_1 or β_2 (but not both) must belong to the interval $[L_B(\mu), U_B(\mu)]$ according to the continuity and Mean Value Theorem with respect to $K^\mu(R)$. Thus in cases (c) ∿ (e), Algorithm 1 terminates. In cases (a) and (b), neither β_1 nor β_2 belongs to the interval $[L_B(\mu), U_B(\mu)]$ by Theorem 4. First note that

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[†] If $d_B^T W d_B = 1$, then we consider $\beta_1 = \beta_2 = -\dfrac{(\mu e_B + g_B)^T W(\mu e_B + g_B) + \sigma_0^2}{2(\mu e_B + g_B)^T W d_B}$.

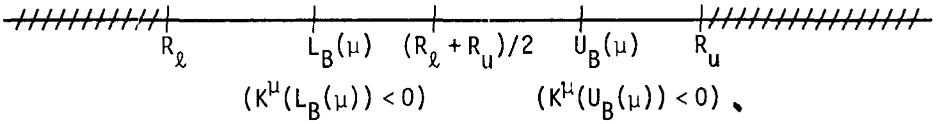

Figure 1a. Case (a)

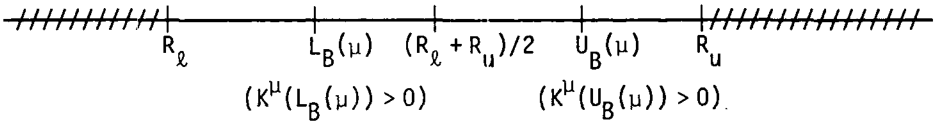

Figure 1b. Case (b)

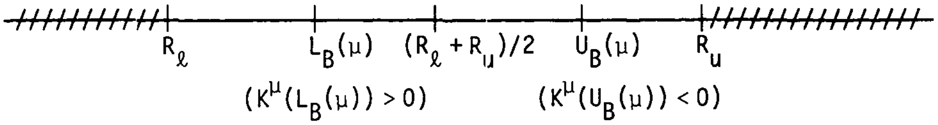

Figure 1c. Case (c)

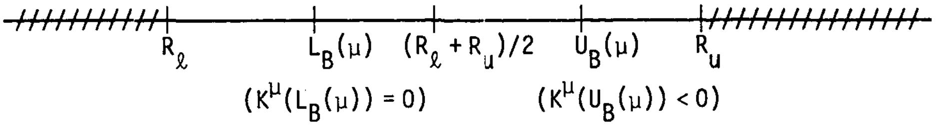

Figure 1d. Case (d)

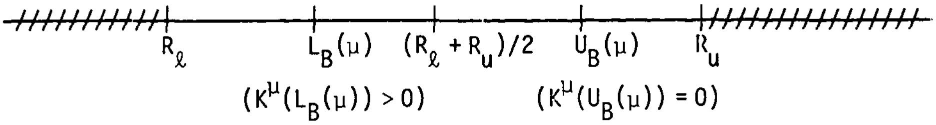

Figure 1e. Case (e)

(4.5) $\quad L_B(\mu) \leqq (R_\ell + R_u)/2 \leqq U_B(\mu)$

holds as is easily known from the updating procedure of R in Step 2 or Step 3.
Case a: R_u is set to $L_B(\mu)$ since $K^\mu(L_B(\mu)) < 0$.
Case b: R_ℓ is set to $U_B(\mu)$ since $K^\mu(U_B(\mu)) > 0$,
In any case, it follows from (4.5) that the difference $R_u - R_\ell$ is at least halved except the first execution of Step 2 and Step 3. Therefore , after finite iterations, case (c), (d) or (e) occurs since $R(\mu)$ belongs to a certain interval $[L_B(\mu),\ U_B(\mu)]$ with $U_B(\mu) - L_B(\mu) > 0$.[†]

(Validity) Termination condition itself assures validity of Algorithm 1. □

5. Main Algorithm for Solving p'

Let B[††] denote the optimal basic matrix of KTP(μ), that is, let $x(\mu) = \beta(\mu)d_B + \mu e_B + g_B$. (Of course, $\beta(\mu) = R(\mu)$, but for convenience, we denote $R(\mu)$ with $\beta(\mu)$.) In turn, we solve the inequality

$$L_B(\mu) \leqq \beta(\mu) \leqq U_B(\mu) \quad \text{and} \quad x(\mu) \geqq 0$$

with respect to μ and denote this region of μ with $I(B)$. $I(B)$ is the set of μ where B becomes the optimal basic matrix of KTP(μ) and for μ on $I(B)$ we can write down $x(\mu) = \beta(\mu)d_B + \mu e_B + g_B$. In other words, shape of $z(\mu)$ with respect to μ on $I(B)$ is determined. If $z(\mu)$ on $I(B)$ crosses $z(\mu) = E(b)$, then the optimal solution will be found. For this purpose, let

$$\bar{\mu}_B \underset{=}{\Delta} sup\{\mu | \mu \in I(B)\} \quad \text{and} \quad \mu'_B \underset{=}{\Delta} sup\{\mu | \mu \in I(B),\ z(\mu) \underset{=}{\Delta} E(b)\}.$$

When $\mu'_B = \mu^*$,

$$x^* = \beta(\mu'_B)d_B + \mu'_B e_B + g_B$$

holds. While in case that $\mu'_B < \mu^*$, we have to continue the search for μ^*. Now define another type subsidiary problem P^R with a parameter $R \geqq \sigma_0$.

[†] Even in the degenerate case, i.e., $L_B(\mu) = U_B(\mu)$, there exists another base $\bar{B}$ such that $L_{\bar{B}}(\mu) = U_B(\mu)$ (or $U_{\bar{B}}(\mu) = L_B(\mu)$) and $U_{\bar{B}}(\mu) - L_{\bar{B}}(\mu) > 0$ according to the theory of the parametric quadratic programming. Therefore without any loss of generality, $U_B(\mu) - L_B(\mu) > 0$ can be assured.

[††] Rigorously speaking, this B must be denoted with $B(\mu)$.

P^R: $\quad$ *Maximize* $E(c)^T x$

$$\text{subject to} \quad E(a)^T x \leqq E(b) - K_\alpha R\ , \quad A_1 x \leqq B_1\ , \quad x \geqq 0.$$

Let x^R and μ^R denote an optimal solution and the optimal value of P^R respectively.

Proposition 1. If an optimal solution x^{σ_0} of P^{σ_0} satisfies

$$E(a)^T x^{\sigma_0} + K_\alpha\{\sigma_0^2 + (x^{\sigma_0})^T W x^{\sigma_0}\}^{\frac{1}{2}} \leqq E(b)\ ,$$

then x^{σ_0} become an optimal solution of P'.

Proof: Since any $x \in S$ satisfies

$$K_\alpha(\sigma_0^2 + x^T W x)^{\frac{1}{2}} \geqq K_\alpha \sigma_0,$$

P^{σ_0} is a relaxation problem of P'. Therefore by the assumption $x^{\sigma_0} \in S$, it is clear that x^{σ_0} is also an optimal solution of P'. □

Proposition 2. If x^R satisfies

$$E(a)^T x^R < E(b) - K_\alpha R,$$

that is, there exists a gap between $E(b) - K_\alpha R$ and $E(a)^T x^R$, then $\mu^R = E(c)^T x^R \geqq \mu^*$ holds.

Proof: Assume $\mu^R < \mu^*$, then

$$E(a)^T x^* > E(b) - K_\alpha R$$

holds, for otherwise x^* is feasible for P^R and $\mu^R \geqq \mu^* = E(c)^T x^*$ holds. Now consider $x^\lambda \triangleq \lambda x^R + \bar{\lambda} x^*$. Then

$$E(b) - E(a)^T x^\lambda - K_\alpha R = E(b) - E(a)^T(\lambda x^R + \bar{\lambda} x^*) - K_\alpha R$$

$$= \lambda(E(b) - E(a)^T x^R - K_\alpha R) + \bar{\lambda}(E(b) - E(a)^T x^* - K_\alpha R)$$

$$= \lambda S^R + \bar{\lambda} S^* = \lambda(S^R - S^*) + S^*$$

holds, where $S^R \triangleq E(b) - E(a)^T x^R - K_\alpha R < 0$ and $S^* \triangleq E(b) - E(a)^T x^* - K_\alpha R > 0$. If λ is taken to be $1 > \lambda > \frac{-S^*}{S_{\mathrm{R}}^{R} - S^*} > 0$, then $E(b) - E(a)^T x^\lambda - K_\alpha R > 0$ and $A_1 x^\lambda \geqq B_1$, $x^\lambda \geqq 0$, x^λ is feasible for P^{R}. Besides,

$$E(c)^T x^\lambda = E(c)^T x^R + \bar{\lambda} E(c)^T x^* = \lambda \mu^R + \bar{\lambda} \mu^* > \mu^R$$

and it contradicts optimality of x^R. Therefore $\mu^R \geqq \mu^*$ results. □

Property 7. μ^R is a nonincreasing function of R.

Proof: As R increases, the feasible region of P^{R} reduces. Therefore μ^R is a nonincreasing function of R. □

Property 8. μ^R is a concave function of R.

Proof: For $R_1 > R_2$ and $1 \geqq \lambda \geqq 0$, let $R_\lambda \triangleq \lambda R_1 + \bar{\lambda} R_2$. Then

$$E(a)^T (\lambda x^{R_1} + \bar{\lambda} x^{R_2}) = \lambda E(a)^T x^{R_1} + \bar{\lambda}\, E(a)^T x^{R_2}$$

$$\leqq \lambda (E(b) - K_\alpha R_1) + \bar{\lambda}(E(b) - K_\alpha R_2) = E(b) - K_\alpha (\lambda R_1 + \bar{\lambda} R_2) = E(b) - K_\alpha R_\lambda ,$$

and

$$A_1 (\lambda x^{R_1} + \bar{\lambda} x^{R_2}) \leqq \lambda B_1 + \bar{\lambda} B_1 = B_1, \quad \lambda x^{R_1} + \bar{\lambda} x^{R_2} \geqq 0$$

hold, i.e., $\lambda x^{R_1} + \bar{\lambda} x^{R_2}$ is feasible for $\mathrm{P}^{\mathrm{R}\lambda}$. Since

$$\lambda \mu^{R_1} + \bar{\lambda} \mu^{R_2} = E(c)^T (\lambda x^{R_1} + \bar{\lambda} x^{R_2}),$$

and

$$\lambda \mu^{R_1} + \bar{\lambda} \mu^{R_2} \leqq \mu^{R\lambda} = E(c)^T x^{R\lambda}$$

hold from optimality of $x^{R\lambda}$ for $\mathrm{P}^{\mathrm{R}\lambda}$. Therefore, μ^R is a concave function of R. □

Now let $R^* \triangleq (x^{*T} W x^* + \sigma_0^2)^{\frac{1}{2}}$, then x^* is feasible for $\mathrm{P}^{\mathrm{R}^*}$ and so $\mu^* \leqq \mu^{R^*}$ follows. By Property 8, Property 7 is strengthened as follows.

Property 7'. Except a first portion, μ^R is a monotonically decreasing function of R.

Figure 2 and Figure 3 show the shapes of $z(\mu)$ and μ^R respectively. Note that the optimal value of $\mathrm{P}^{\mathrm{R}(\mu)}$ is not less than μ since $x(\mu)$ is a feasible solution of $\mathrm{P}^{\mathrm{R}(\mu)}$. Now we are ready to describe our main algorithm for solving P'. In the algorithm, the following notations are used.

μ_c: current μ , $\bar{\mu}$: an upper bound of μ^* , $R(x) \triangleq (\sigma_0^2 + x^T W x)^{\frac{1}{2}}$,

B_c: basic matrix corresponding to the current optimal solution,

β_c: current solution of Q-equation

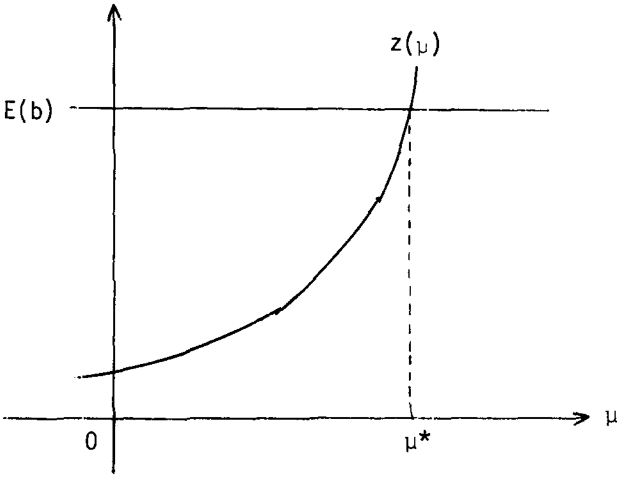

Figure 2. z(μ) v.s. μ

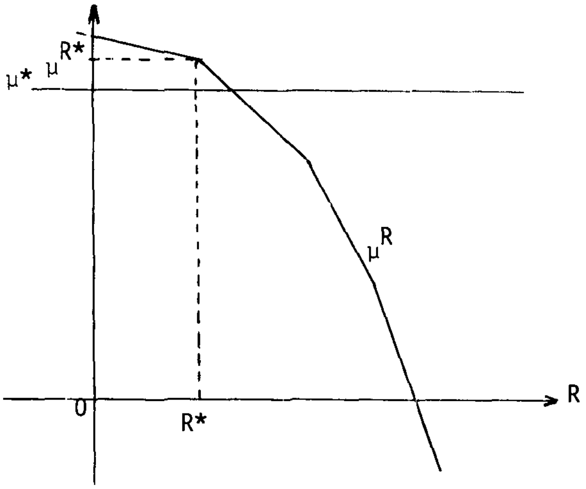

Figure 3. μ^R v.s. R

[Algorithm 2]

Step 0: Calculate $\tilde{\mu}$, solve P($\tilde{\mu}$) and find $x(\tilde{\mu})$ and $z(\tilde{\mu})$ by using Algorithm 1. If $z(\tilde{\mu}) \leqq E(b)$, set $x^* \leftarrow x(\tilde{\mu})$ and terminate. Otherwise set $\mu \leftarrow \tilde{\mu}$, $\underline{R} \leftarrow \sigma_0$ and $\mu_c \leftarrow (-M)$ (M is a sufficiently large number). Go to Step 1.

Step 1: Solve $P(\mu_c)$ and find $x(\mu_c)$, optimal basic matrix B_c and $I(B_c)$. If $\mu^* \in I(B_c)$, set $x^* \leftarrow \beta_c(\mu^*)d_{B_c} + \mu^* e_{B_c} + g_{B_c}$ and terminate; If $\mu^* \notin I(B_c)$ and $\bar{\mu}_{B_c} > \mu^*$, go to Step 2; If $\mu^* \notin I(B_c)$ and $\bar{\mu}_{B_c} < \mu^*$ (in this case $\underline{\mu}_{B_c} = \mu'_{B_c}$), go to Step 3.

Step 2: If $\bar{\mu} > \bar{\mu}_{B_c}$, set $\bar{\mu} \leftarrow \bar{\mu}_{B_c}$ and

$$\mu_c \leftarrow \frac{(\bar{\mu} - \mu'_{B_c})E(b) - \bar{\mu} z(\mu'_{B_c}) + \mu'_{B_c} z(\bar{\mu})}{z(\bar{\mu}) - z(\mu'_{B_c})}$$

and return to Step 1; If $\bar{\mu} \leqq \bar{\mu}_{B_c}$ and $R(x(\mu'_{B_c})) > \underline{R}$, set $\underline{R} \leftarrow R(x(\mu'_{B_c}))$, solve $P^{R(x(\mu'_{B_c}))}$ and^{c}calculate $^{c}\mu^{R(x(\mu'_{B_c}))}$. Go to Step 3; If $\bar{\mu} \leqq \bar{\mu}_{B_c}$ and $\underline{R} \geqq R(x(\mu'_{B_c}))$, then set

$$\mu_c \leftarrow \frac{(\bar{\mu} - \mu'_{B_c})E(b) - \bar{\mu} z(\bar{\mu}) + \mu'_{B_c} z(\bar{\mu})}{z(\bar{\mu}) - z(\mu'_{B_c})}$$

and return to Step 1.

Step 3: Solve $P(\mu^{R(x(\mu'_{B_c}))})$ and calculate $z(\mu^{R(x(\mu'_{B_c}))})$. If $E(b) > z(\mu^{R(x(\mu'_{B_c}))})$ and

$$\frac{(\bar{\mu} - \mu'_{B_c})E(b) - \bar{\mu} z(\bar{\mu}) + \mu'_{B_c} z(\bar{\mu})}{z(\bar{\mu}) - z(\mu'_{B_c})} < \mu^{R(x(\mu'_{B_c}))},$$

then set $\mu_c \leftarrow \mu^{R(x(\mu'_{B_c}))}$ and return to Step 1; If $E(b) = z(\mu^{R(x(\mu'_{B_c}))^{c}})$, set $x^* \leftarrow x(\mu^{R(x(\mu'_{B_c}))})$ and terminate; Otherwise, set

$$\mu_c \leftarrow \frac{(\bar{\mu} - \mu'_{B_c})E(b) - \bar{\mu} z(\mu'_{B_c}) + \mu'_{B_c} z(\bar{\mu})}{z(\bar{\mu}) - z(\mu'_{B_c})}$$

and return to Step 1.

Theorem 6. Algorithm 2 finds x^* at finite iterations.

Proof: (Finiteness) Each $P^R(\mu)$ has a same constraint condition $KTP^R(\mu)$ except parametrized right hand side with respect to R and μ. The number of basic matrices satisfying nonnegativity and complementary condition is finite, and by the theory of parametric quadratic programming, $R(\mu)$ corresponds to an optimal basis $B=B(\mu)$. That is, μ is divided into $I(B)$'s determined by basic matrix B. Algorithm 2 searches for μ^* among those regions $I(B)$ at most once for each $B^{\dagger}$. Therefore finiteness of Algorithm 2 follows from finiteness of the number of $I(B)$.

(Validity) Theorem 2 assures the condition $\bar{s}_0(\bar{\mu})=0$ in Theorem 1. Termination condition that $z(\mu)=E(b)$ assures validity by Theorem 1. □

6. An Example

We consider the following example P.

$$\text{P:} \quad \text{Maximize} \quad E(c_1x_1+c_2x_2)$$

$$\text{subject to} \quad \text{Prob}(\ a_1x_1+a_2x_2 \leqq b\ ) \geqq 0.7,$$

$$3x_1+2x_2 \leqq 18, \quad x_1+2x_2 \leqq 10, \quad x_1,\ x_2 \geqq 0\ ,$$

where $E(c)=(8,\ 6)^T$, $E(b)=32$, $\sigma_0=4$, $E(a)=(5,\ 6)^T$ and $W=\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$. P is transformed into the following deterministic equivalent problem P'.

$$\text{P}': \quad \text{Maximize} \quad 8x_1+6x_2$$

$$\text{subject to} \quad 5x_1+6x_2+0.5(16+x_1^2+x_2^2)^{\frac{1}{2}} \leqq 32,$$

$$3x_1+2x_2 \leqq 18, \quad x_1+2x_2 \leqq 10\ , \quad x_1,\ x_2 \geqq 0\ .$$

Step 0:

$$\tilde{\mu} \triangleq \max\{8x_1+6x_2 \mid 3x_1+2x_2 \leqq 18, \quad x_1+x_2 \leqq 10, \quad x_1,\ x_2 \geqq 0\} = 48.$$

Solve P($\tilde{\mu}$) and find $x(\tilde{\mu})$ and $z(\tilde{\mu})$.

---

† (1) Convexity of $z(\mu)$ implies

$$\mu'_{B_c} < \frac{(\bar{\mu}-\mu'_{B_c})E(b)-\bar{\mu}z(\bar{\mu})+\mu'_{B_c}z(\bar{\mu})}{z(\bar{\mu})-z(\mu'_{B_c})} \leqq \mu^*$$

in Step 2.

(2) For $\mu \in I(B)$, the shape of $z(\mu)$ is known. Therefore, whether $\mu^* \in I(B)$ or $\mu^* \notin I(B)$ is determined by checking the existence of μ such that $z(\mu)=E(b)$ on $I(B)$.

P(48): $$\text{Minimize}\quad 5x_1+6x_2+0.5(16+x_1^2+x_2^2)^{\frac{1}{2}}$$

$$\text{subject to}\quad 8x_1+6x_2 \geqq 48\ (=\tilde{\mu}),\quad 3x_1+2x_2 \leqq 18,\quad x_1+2x_2 \leqq 10,$$

$$x_1,\ x_2 \geqq 0.$$

(Algorithm 1)

Step 1: Set $R_\ell \leftarrow 4$, $R_u \leftarrow M$ and $R \leftarrow 5$.

$\mathrm{P}^R(\tilde{\mu})$: $$\text{Minimize}\ R(5x_1+6x_2)+0.25(16+x_1^2+x_2^2)$$

$$\text{subject to}\ 8x_1+6x_2 \geqq 48,\quad 3x_1+2x_2 \leqq 18,\quad x_1+2x_2 \leqq 10,$$

$$x_1,\ x_2 \geqq 0.$$

$\mathrm{KTP}^R(\tilde{\mu})$: $$\hat{v}_1-0.5x_1+8\hat{r}-3\hat{q}_1-\hat{q}_2=5R,\quad \hat{v}_2-0.5x_2+6\hat{r}-2\hat{q}_1-2\hat{q}_2=6R,$$

$$3x_1+2x_2+\hat{s}_1=18,\quad x_1+2x_2+\hat{s}_2=10,\quad 8x_1+6x_2-\hat{s}_0=\tilde{\mu}(=48),$$

$$x_1\hat{v}_1+x_2\hat{v}_2+\hat{s}_1\hat{q}_1+\hat{s}_2\hat{q}_2+\hat{r}\hat{s}_0=0,$$

$$x_1,\ x_2,\ \hat{v}_1,\ \hat{v}_2,\ \hat{s}_1,\ \hat{s}_2,\ \hat{q}_1,\ \hat{q}_2,\ \hat{r},\ \hat{s}_0 \geqq 0.$$

$X^R(\mu)$ is given as follows:

$$x_1=\frac{\tilde{\mu}}{8},\ x_2=0,\ \hat{v}_1=0,\ \hat{v}_2=\frac{9}{4}R-\frac{3}{64}\tilde{\mu},\ \hat{s}_1=18-\frac{3}{8}\tilde{\mu},\quad \hat{s}_2=10-\frac{\tilde{\mu}}{8},\quad \hat{s}_0=0,$$

$$\hat{q}_1=\hat{q}_2=0,\ \hat{r}=\frac{5}{8}R+\frac{\tilde{\mu}}{128}.$$

$$B=\begin{array}{c}\begin{array}{ccccc} x_1 & \hat{v}_2 & \hat{s}_1 & \hat{s}_2 & \hat{r}\end{array}\\ \begin{bmatrix} -\frac{1}{2} & 0 & 0 & 0 & 8\\ 0 & 1 & 0 & 0 & 6\\ 3 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 1 & 0\\ 8 & 0 & 0 & 0 & 0\end{bmatrix}\end{array},\quad d_B=\begin{pmatrix}0\\0\end{pmatrix},\quad e_B=\begin{pmatrix}\frac{1}{8}\\0\end{pmatrix},\quad g_B=\begin{pmatrix}0\\0\end{pmatrix},$$

$$R \geqq \frac{\tilde{\mu}}{48}\ (=L_B(\tilde{\mu})).$$

Step 2: $K^\mu(L_B(\tilde{\mu}))=16+\frac{\tilde{\mu}^2}{64}-\frac{\tilde{\mu}^2}{(48)^2}>0.$

Step 3: $K^\mu(U_B(\tilde{\mu}))<0.$

Therefore $R(\tilde{\mu})$ exists on $[\frac{\tilde{\mu}}{48},\infty)$ and given as follows.

$$R(\tilde{\mu})=(16+\tilde{\mu}^2/64)^{\frac{1}{2}}$$

Step 5: $$x(\tilde{\mu})=\begin{pmatrix}\frac{1}{8}\tilde{\mu}\\0\end{pmatrix}.$$ Return to Main Algorithm.

Since $z(\tilde{\mu}) = \frac{5}{8}\tilde{\mu} + 0.5\sqrt{16 + \frac{\tilde{\mu}^2}{64}} = 30 + \sqrt{13} > 32 = E(b)$, set $\bar{\mu} \leftarrow 48 (= \tilde{\mu})$ and $\underline{R} \leftarrow 4 (= \sigma_0)$. Go to Step 1.

$$P(\mu_c): \quad \textit{Minimize} \quad 5x_1 + 6x_2 + 0.5\sqrt{16 + x_1^2 + x_2^2}$$

$$\textit{subject to} \ 8x_1 + 6x_2 \geqq \mu_c \ (=-M), \quad 3x_1 + 2x_2 \leqq 18, \quad x_1 + 2x_2 \leqq 10,$$

$$x_1, \ x_2 \geqq 0.$$

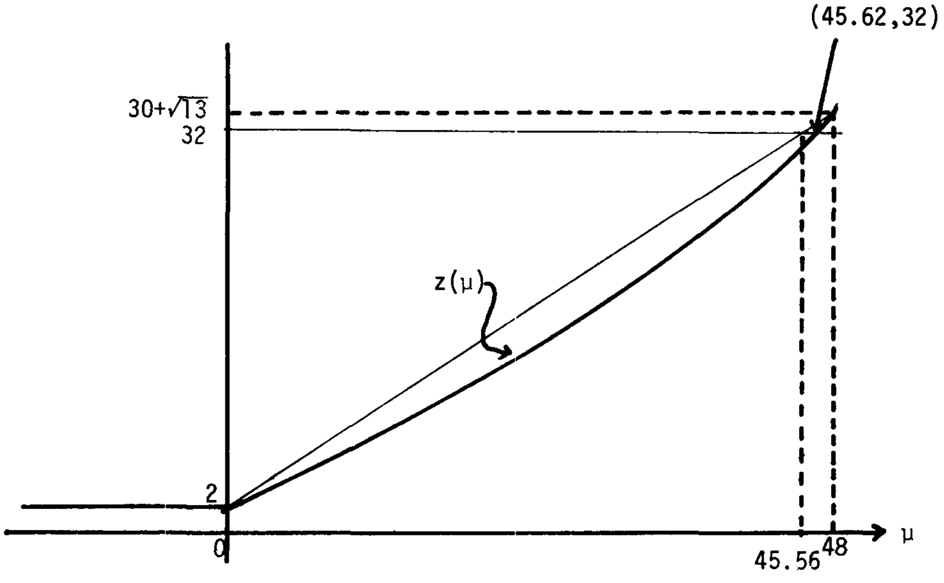

Figure 4. z(μ) v.s. μ

Using Algorithm 1, we obtain $d_B = e_B = g_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $I(B_c) = (-\infty, \ 0)$. Therefore $\bar{\mu}_{B_c} = \mu'_{B_c} = 0$ and $z(\mu) = 2$ on $I(B_c)$. $\bar{\mu}_{B_c} < \mu^*$, i.e., $\mu^* \notin I(B_c)$. Go to Step 3.

Step 3: $x(\mu'_{B_c}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $R(x(\mu'_{B_c})) = \sqrt{16} = 4.0$, $\mu^{R(x(\mu'_{B_c}))} = \mu^4 = \mu^{\sigma_0} = 48$.
$z(48) > E(b) = 32$,

$$\mu_c \leftarrow \frac{(\bar{\mu} - \mu'_{B_c})E(b) - \bar{\mu}z(\mu'_{B_c}) + \mu'_{B_c} z(\bar{\mu})}{z(\bar{\mu}) - z(\mu'_{B_c})} = \left(\frac{(48 - 0) - 48 \times 2 + 0 \times (30 + \sqrt{13})}{30 + \sqrt{13} - 2} \right.$$

$\fallingdotseq 45.5616)$. Return to Step 1.

Step 1: Solving $P(\mu_c)$, we obtain $X(\mu_c)$ given as below.

$$X(\mu_c): \quad x_1 = \frac{\mu_c}{8}, \quad x_2 = 0, \quad \hat{v}_1 = 0, \quad \hat{v}_2 = \frac{9}{4}R - \frac{3}{64}\mu_c, \quad \hat{s}_1 = 18 - \frac{3}{8}\mu_c, \quad \hat{s}_2 = 10 - \frac{\mu_c}{8}, \quad \hat{s}_0 = 0, \quad \hat{q}_1 = \hat{q}_2 = 0, \quad \hat{r} = \frac{5}{8}R + \frac{\mu_c}{128} (R \geqq \frac{1}{48}\mu_c = L_B(\mu_c)).$$

$$R(\mu_c) = \sqrt{16 + \frac{\mu_c^2}{64}}, \quad z(\mu_c) = \frac{5}{8}\mu_c + 0.5\sqrt{16 + \frac{\mu_c^2}{64}}, \quad d_{B_c} = g_{B_c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad e_{B_c} = \begin{pmatrix} \frac{1}{8} \\ 0 \end{pmatrix}$$

$$I(B_c) = \{\mu \mid 0 \leqq \mu \leqq 48\}.$$

Obviously $\mu^* \in I(B_c)$. Thus we solve

$$z(\mu) = \frac{5}{8}\mu + 0.5\sqrt{16 + \frac{\mu_c^2}{64}} = 32.$$

and obtain $\mu^* \fallingdotseq 45.62$ and $x^* = \begin{pmatrix} \frac{1}{8}\mu^* \\ 0 \end{pmatrix} = \begin{pmatrix} 5.70 \\ 0 \end{pmatrix}$. $(E(c)^T x^* = \mu^*.)$

7. Conclusion

This paper discussed a chance-constrained E-Model and provided an algorithm based on the parametric procedure. Problem P was first transformed into the equivalent problem P' and based on Theorem 1, $\mu = \mu^*$ such that $z(\mu) = E(b)$ was searched for systematically by Algorithm 2. Moreover, this paper clarified the dual-like relation between P' and P(μ). This relation seems to be useful to solve other nonlinear programming problems, especially those with a linear objective function but nonlinear constraints.

As is stated in Section 1, algorithms for stochastic programming problems are few. Especially, algorithms for problems with stochastic constraints are even few. P belongs to such a class. Discussion and development of solution algorithms for more general problems are further research problems.

8. Acknowledgement

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References

[1] Charnes, A. and Cooper, W. W.: Chance Constrained Programming. *Management Science*, Vol. 6, (1959), 73-79.

[2] Charnes, A. and Cooper, W. W.: Deterministic Equivalents for Optimizing and Satisficing under Chance Constraints. *Operations Research*, Vol. 11, (1963), 18-39.

[3] Charnes, A. and Kirby, M. L.: Some Special P-Models in Chance Constrained Programming. *Management Science*, Vol. 14, (1967), 183-195.

[4] Eisner, M. J., Kaplan, R. S. and Soden, J. V.: Admissible Decision Rules for the E-Model of Chance-Constrained Programming. *Management Science*, Vol. 17, (1971), 337-353.

[5] Ishii, H., Nishida, T. and Nanbu, Y.: A Generalized Chance Constraint Programming Problem. *Journal of Operations Research Society of Japan*, Vol. 21, (1978), 124-145.

[6] Kataoka, S.: A Stochastic Programming Model. *Econometrica*, Vol. 31, (1963), 181-196.

[7] Mangasarian, O. L.: *Nonlinear Programming*. McGraw-Hill, 1969.

[8] Miller, B. L. and Wagner, H. M.: Chance Constrained Programming with Joint Constraints. *Operations Research*, Vol. 13, (1965), 930-945.

[9] Vajda, S.: *Probabilistic Programming*. Academic Press, 1972.

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Appendix 1. Derivation of P'

The chance constraint in (2.1) can be transformed into the following form by simple subtraction and devision.

$$\text{(A-1)} \qquad Prcb(\ a^T x \leqq b\ ) = Prob\ \{ \frac{a^T x - b - E(a)^T x + E(b)}{\sqrt{\sigma_0^2 + x^T W x}} \leqq \frac{E(b) - E(a)^T x}{\sqrt{\sigma_0^2 + x^T W x}} \} \geqq \alpha$$

Since a is distributed according to $N(E(a), W)$ and b according to $N(E(b), \sigma_0^2)$,

$$\frac{a^T x - b - E(a)^T x + E(b)}{\sqrt{\sigma_0^2 + x^T W x}}$$

can be considered as a normal random variable with zero mean and unit variance (i.e., standard normal distribution). Therefore (A-1) is replaced by

$$\text{(A-2)}\qquad \frac{E(b) - E(a)^T x}{\sqrt{\sigma_0^2 + x^T W x}} \geqq F^{-1}(\alpha),$$

where F is the distribution function of standard normal distribution $N(0, 1)$. (A-2) is further transformed into

$$\text{(A-3)}\qquad E(a)^T x + K_\alpha \sqrt{\sigma_0^2 + x^T W x} \leqq E(b),$$

where $K_\alpha \triangleq F^{-1}(\alpha)$. $E(c^T x)$ is equivalent to $E(c)^T x$ by the linearity of expectation. Above discussion shows P is equivalent to P'.

Appendix 2.

$$\lambda K_\alpha \{\sigma_0^2 + x(\mu_1)^T W x(\mu_1)\}^{\frac{1}{2}} + \bar{\lambda} K_\alpha \{\sigma_0^2 + x(\mu_2)^T W x(\mu_2)\}^{\frac{1}{2}}$$

$$\geqq K_\alpha \{\sigma_0^2 + (\lambda x(\mu_1) + \bar{\lambda} x(\mu_2))^T W(\lambda x(\mu_1) + \bar{\lambda} x(\mu_2))\}^{\frac{1}{2}}$$

Proof: Let D_i, $i=1,2$, and D_λ denote

$\sigma_0^2 + x(\mu_i)^T W x(\mu_i)$, $i=1,2$, and $\sigma_0^2 + (\lambda x(\mu_1) + \bar{\lambda} x(\mu_2))^T W(\lambda x(\mu_1) + \bar{\lambda} x(\mu_2))$

respectively. Then

$$\text{(A-4)}\qquad \lambda K_\alpha D_1^{\frac{1}{2}} + \bar{\lambda} K_\alpha D_2^{\frac{1}{2}} - K_\alpha D_\lambda^{\frac{1}{2}} = \frac{\lambda^2 K_\alpha^2 D_1 + \bar{\lambda}^2 K_\alpha^2 D_2 + 2\lambda\bar{\lambda} K_\alpha^2 D_1^{\frac{1}{2}} D_2^{\frac{1}{2}} - K_\alpha^2 D_\lambda}{\lambda K_\alpha D_1^{\frac{1}{2}} + \bar{\lambda} K_\alpha D_2^{\frac{1}{2}} + K_\alpha D_\lambda^{\frac{1}{2}}}$$

Now nonnegativity of only the numerator in (A-4) must be shown to prove Appendix 2.

$$\text{(A-5)}\qquad \lambda^2 K_\alpha^2 D_1 + \bar{\lambda}^2 K_\alpha^2 D_2 + 2\lambda\bar{\lambda} K_\alpha^2 D_1^{\frac{1}{2}} D_2^{\frac{1}{2}} - K_\alpha^2 D_\lambda = 2\lambda\bar{\lambda} K_\alpha^2 \{-\sigma_0^2 - x(\mu_1)^T W x(\mu_2) + D_1^{\frac{1}{2}} D_2^{\frac{1}{2}}\}$$

(since $D_\lambda = \sigma_0^2 + (\lambda x(\mu_1) + \bar{\lambda} x(\mu_2))^T W(\lambda x(\mu_1) + \bar{\lambda} x(\mu_2)) = \lambda^2 D_1 + \bar{\lambda}^2 D_2 + 2\lambda\bar{\lambda}\sigma_0^2$ $+ 2\lambda\bar{\lambda} x(\mu_1)^T W x(\mu_2)$ by expanding $(\lambda x(\mu_1) + \bar{\lambda} x(\mu_2))^T W(\lambda x(\mu_1) + \bar{\lambda} x(\mu_2))$ and $\sigma_0^2 = (\lambda + \bar{\lambda})^2 \sigma_0^2 = (\lambda^2 + 2\lambda\bar{\lambda} + \bar{\lambda}^2)\sigma_0^2$)

$$(\text{A-6}) \qquad = \frac{2\lambda\bar{\lambda}K_{\alpha}^{2}\{D_1D_2 - (\sigma_0^2 + x(\mu_1)Wx(\mu_2))^2\}}{\sigma_0^2 + x(\mu_1)^T Wx(\mu_2) + D_1^{\frac{1}{2}}D_2^{\frac{1}{2}}}$$

Again only nonnegativity of the numerator in (A-6) must be shown.

$$D_1D_2 - (\sigma_0^2 + x(\mu_1)^T Wx(\mu_2))^2 = \sigma_0^2\{x(\mu_1)^T Wx(\mu_1) + x(\mu_2)^T Wx(\mu_2) - 2x(\mu_1)^T Wx(\mu_2)\}$$

$$+ x(\mu_1)^T Wx(\mu_1) \times x(\mu_2)^T Wx(\mu_2) - (x(\mu_1)^T Wx(\mu_2))^2$$

$$\geqq \sigma_0^2\{x(\mu_1)^T Wx(\mu_1) + x(\mu_2)^T Wx(\mu_2) - 2x(\mu_1)^T Wx(\mu_2)\}.$$

(because $x(\mu_1)^T Wx(\mu_1) \times x(\mu_2)^T Wx(\mu_2) - (x(\mu_1)^T Wx(\mu_2))^2 \geqq 0$ since W is a positive matrix)

$$= \sigma_0^2 (x(\mu_1) - x(\mu_2))^T W (x(\mu_1) - x(\mu_2)) \geqq 0$$

(since W is a positive definite matrix). □