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ON THE EXISTENCE OF THE CORE OF A CHARACTERISTIC FUNCTION GAME WITH ORDINAL PREFERENCES

Shin Ishikawa
and
Kenjiro Nakamura
Tokyo Institute of Technology

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Abstract Nakamura establishes a theorem which gives necessary conditions for a simple game with ordinal preferences to have a nonempty core. The conditions are also sufficient, if the set of alternative outcomes is finite. In the present paper, we will show that Nakamura's method of the proof of this theorem makes it possible to generalize the theorem to an arbitrary characteristic function game with ordinal preferences.

1. Introduction

Nakamura [5] establishes a theorem which gives necessary conditions for a simple game with ordinal preferences to have a nonempty core. The conditions are also sufficient, if the set of alternative outcomes is finite. The theorem has already been successfully applied to social choice theory by Nakamura [6], [7], [8], and Ishikawa and Nakamura [3]. Also, by using the theorem, Peleg [10] has developed a new direction of the study connecting social choice theory to game theory.

In the present paper we generalize the theorem to an arbitrary characteristic function game with ordinal preferences. Rather surprisingly, this generalization will be carried out almost straightforwardly by modifying the proof of the theorem. The result, however, may be expected to have new applications, since the concept of a characteristic function game is clearly much more general than that of a simple game. Moreover it might also be expected to shed new light on general existence problem of a core of a cooperative game without sidepayments. Therefore it will be worthwhile to demonstrate the generalization.

Throughout the paper, the cardinal number of a set A is denoted by $p(A)$ and the symbol of negation by $\sim$. We also explicitly assume the axiom of well-ordering.

Let N be an arbitrary fixed nonempty set of players. A nonempty set of alternative outcomes Ω is given. A binary relation Q on Ω is said to be acyclic, if for any integer $m \geqq 1$, and for any $x_1, x_2, \ldots, x_m \in \Omega$,

$$x_2 \, Q \, x_1, \; x_3 \, Q \, x_2, \; \ldots, \; x_m \, Q \, x_{m-1} \qquad \text{imply} \sim x_1 \, Q \, x_m.$$

When m = 1, we understand that the definition means $\sim x_1 \, Q \, x_1$; i.e., irreflexivity. We denote by D the set of all acyclic strict preference relations on Ω. A function P^N from N to D is called a profile of the players' preference relations. We denote by D^N the set of all profiles.

Following an analog to Aumann and Peleg [1] (See also Peleg [9]) or to Bloomfield [2], we define a characteristic function game with ordinal preferences (or, simply, a characteristic function game).

A characteristic function game (with ordinal preferences) is a triple (N, Ω, v), where v is a mapping from 2^N into 2^Ω such that

(1.1) $\quad v(\emptyset) = \emptyset$,

(1.2) $\quad v(N) = \Omega$.

Let $x \in \Omega$. A coalition S is said to be effective for x, if $x \in v(S)$. We denote by E(x) the set of the effective coalitions for x. By (1.2), N is effective for all $x \in \Omega$, and so $N \in E(x)$ for all $x \in \Omega$. Hence note that E(x) is always nonempty.

Let $G = (N, \Omega, v)$ be a characteristic function game. Let $P^N \in D^N$ and let $x, y \in \Omega$ with $x \neq y$. We say that x dominates y with respect to P^N, denoted by $x \operatorname{dom}(P^N) y$, if there is an $S \subset N$ such that $x \in v(S)$ and $x \, P^i \, y$ for all $i \in S$. Then the core for the game G with respect to P^N is defined by

$$C(G,P^N) = \{x \in \Omega \mid \sim y \operatorname{dom}(P^N) \, x \text{ for all } y \in \Omega\}.$$

(Our definition of a core is the direct extension of that in Nakamura [4].)

2. A Main Theorem

Let $G = (N, \Omega, v)$ be a characteristic function game. For each $x \in \Omega$, we choose $S(x) \in E(x)$. If, for every selection of these sets $\{S(x)\}_{x \in \Omega}$,

$$\bigcap_{x \in \Omega} S(x) \neq \emptyset$$

holds, then we call this property the complete intersection property (CIP).

Now we are able to state the following theorem. The following method of the proof of the theorem is virtually the same as that of the corresponding theorem in Nakamura [5] (Theorem 3.1 in section 3 of the present paper). But for the sake of completeness, we should state the proof, and moreover this proof is somewhat simpler than that in Nakamura [5].

Theorem 2.1. Let $G = (N, \Omega, v)$ be a characteristic function game. If $C(G,P^N) \neq \emptyset$ for all $P^N \in D^N$, then G satisfies CIP. Further if Ω is finite, then this necessary condition is also sufficient.

Proof. Suppose G does not satisfy CIP. Then there is a selection $\{S(x)\}_{x\in\Omega}$ such that $\bigcap_{x\in\Omega} S(x) = \emptyset$. Let $(\Omega, \geqq)$ be a well-ordered set with respect to a linear order $\geqq$. We shall define a preference relation P^i of each player i. Let $i \in N$. Then there is an $\hat{x} \in \Omega$ such that $i \notin S(\hat{x})$. We arbitrarily fix such an $\hat{x}$ and denote it by $\hat{x}^i$. We define $\Omega(\hat{x}^i) = \{x\in\Omega | x<\hat{x}^i\}$. By defining the following lexicographical direct sum:

$$\{\hat{x}^i\} + (\Omega - \Omega(\hat{x}^i) - \{\hat{x}^i\}) + \Omega(\hat{x}^i),$$

we have another well-ordered set $(\Omega, \geqq^i)$. For any $x, y \in \Omega$, we define

$$x \ P^i \ y \quad \text{iff} \quad y <^i x.$$

Here note that $x \ P^i \ \hat{x}^i$ for all $x \in \Omega$ with $x \neq \hat{x}^i$. P^i is clearly a linear order on Ω. Then we can define a profile P^N. Now we shall show that $C(G,P^N) = \emptyset$. Let $x \in \Omega$. We need to consider two cases.

Suppose first that x is not a maximal element of $(\Omega, \geqq)$. Then there is a $y \in \Omega$ which is immediately after x with respect to $\geqq$. For each $i \in S(y)$, $y \neq \hat{x}^i$. Hence it is easy to see that $y \ P^i \ x$. Since $S(y) \in E(y)$, $y \ \mathrm{dom}(P^N) \ x$. Suppose next that x is a maximal element of $(\Omega, \geqq)$. Let x^o be the minimum element of $(\Omega, \geqq)$. For each $i \in S(x^o)$, $x^o \neq \hat{x}^i$. Hence it is easy to see that $x^o \ P^i \ x$. Since $S(x^o) \in E(x^o)$, $x^o \ \mathrm{dom}(P^N) \ x$. Therefore x is always dominated; and hence $C(G,P^N) = \emptyset$.

Finally let Ω be finite and we show the converse. If $C(G,P^N) = \emptyset$ for some $P^N \in D^N$, then we will be able to find some p distinct $x_1,..,x_p$ such that:

$$x_2 \ \mathrm{dom}(P^N) \ x_1, \ x_3 \ \mathrm{dom}(P^N) \ x_2, \ldots, \ x_p \ \mathrm{dom}(P^N) \ x_{p-1}, \ x_1 \ \mathrm{dom}(P^N) \ x_p.$$

Let S_i $(i = 1,..,p)$ be an effective set for each dominance relation. Since each P^i is acyclic, $\bigcap_{i=1}^{p} S_i = \emptyset$. We define $S(x) = S_i$ for $x = x_i$ $(i=1,..,p)$ and $S(x) = N$ for $x \neq x_i$ $(i=1,..,p)$. Clearly $\bigcap_{x\in\Omega} S(x) = \emptyset$, which is a contradiction. Q.E.D.

In the next section we will derive the corresponding theorem in Nakamura [5] from theorem 2.1.

3. Simple Games

Following Shapley [12], we first define a simple game. Let $G = (N, \Omega, v)$ be a characteristic function game, and let ω be a class of coalitions such that:

$$(3.1) \qquad v(S) = \Omega \qquad \text{for all } S \in \omega,$$

$$(3.2) \qquad v(S) = \emptyset \qquad \text{for all } S \notin \omega.$$

The characteristic function game with these requirements is called a simple game and ω is called the class of winning coalitions. Note that $E(x) = \omega$ for all $x \in \Omega$. Since the simple game is specified by N, Ω and ω, we hereafter denote it by $G = (N, \Omega, \omega)$. (Our definition of a simple game is also the same as that in Nakamura [4] and [5].)

Let $G = (N, \Omega, \omega)$ be a simple game. G is weak, if

$$V_G = \cap\{S \mid S \in \omega\} \neq \emptyset.$$

A member of V_G is called a veto player in G.

Let $G = (N, \Omega, \omega)$ be a simple game without veto players. Then we define

$$\Sigma = \{\sigma \subset \omega \mid \cap\{S \mid S \in \sigma\} = \emptyset\}.$$

Σ is nonempty and partitioned into the nonempty classes of numerically equivalent subsets of Σ. The set of equivalence classes is the set of distinct cardinal numbers of σ in Σ, and we denote it by Σ_p. Since Σ_p is a well-ordered set by its natural linear order. Then we define the following cardinal number:

$$\nu(G) = \min \Sigma_p.$$

Theorem 3.1 (Nakamura [5]). Let $G = (N, \Omega, \omega)$ be a simple game. If $C(G,P^N) \neq \emptyset$ for all $P^N \in D^N$, then either G is weak or $\nu(G) > p(\Omega)$. Further if Ω is finite, then the necessary conditions are also sufficient.

In order to prove theorem 3.1, we show that, for a simple game, CIP condition is the same as the following conditions:

$$V_G \neq \emptyset \quad \text{or} \quad \nu(G) > p(\Omega).$$

Proof. Let $G = (N, \Omega, \omega)$ be a simple game. Suppose the above conditions are satisfied. If there is a selection $\{S(x)\}_{x\in\Omega}$, $S(x) \in \omega$, such that $\bigcap_{x\in\Omega} S(x) = \emptyset$, then G is not weak and $\nu(G) \leqq p(\Omega)$, which is impossible. Conversely suppose G satisfies CIP. Suppose also G is not weak. If $\nu(G) \leqq p(\Omega)$, then there exists some $\sigma \subset \omega$ such that $p(\sigma) = \nu(G)$ and $\cap\{S \mid S \in \sigma\} = \emptyset$. Then there is a one to one mapping ψ from σ into Ω. We define $S(x) \in \omega$ for each $x \in \Omega$ as follows:

$$S(x) = \psi^{-1}(x) \quad \text{for } x \in \psi(\sigma),$$

$$S(x) = N \quad \text{for } x \notin \psi(\sigma).$$

By noting $\omega = E(x)$ and $N \in E(x)$ for all $x \in \Omega$, we get

$$\bigcap_{x\in\Omega} S(x) = \emptyset,$$

which contradicts CIP. Q.E.D.

Hence CIP condition is equivalent to the above conditions for a simple game. Hence theorem 3.1 follows from theorem 2.1 immediately.

4. Symmetric Games

In this section we characterize CIP condition for a monotonic symmetric characteristic function game, and we derive a theorem by Polishchuk [11].

Let $G = (N, \Omega, v)$ be a characteristic function game. G is monotonic, if

$$S \subset T \quad \Longrightarrow \quad v(S) \subset v(T).$$

G is symmetric, if

$$p(S) = p(T) \Longrightarrow v(S) = v(T).$$

Lemma 4.1. Let $G = (N, \Omega, v)$ be a monotonic characteristic function game. G is symmetric iff there exists a function h from Ω into the set of cardinal numbers p with $0 < p \leqq p(\Omega)$, such that

$$E(x) = \{S \subset N \mid p(S) \geqq h(x)\} \quad \text{for all } x \in \Omega.$$

Proof. Sufficiency is trivial. Then we show necessity. Let $E(x)_p$ be the set of distinct cardinal numbers of the elements in $E(x)$. Since $E(x)_p$ is a well-ordered set by its natural linear order, there exists the minimum cardinal number with respect to this order. Then we define:

$$h(x) = \min E(x)_p.$$

Let $x \in \Omega$. If $S \in E(x)$, then clearly $p(S) \geqq h(x)$. Conversely suppose $S \subset N$ and $p(S) \geqq h(x)$. There is a $T \in E(x)$ with $h(x) = p(T)$. Since $p(S) \geqq p(T)$, there is a one to one mapping ψ from T into S; and so $p(\psi(T)) = p(T)$ and $\psi(T) \subset S$. Since v is symmetric and monotonic, $v(T) \subset v(S)$. By $x \in v(T)$, $x \in v(S)$; i.e., $S \in E(x)$. Q.E.D.

From theorem 2.1, the following theorem is easily verified by examining CIP condition.

Theorem 4.2. (Polishchuk [11]). Let N and Ω be finite, and let $p(N) = n$.

Let $G = (N, \Omega, v)$ be a monotonic symmetric game and let h be a function determined by lemma 4.1. Then $C(G,P^N) \neq \emptyset$ for all $P^N \in D^N$ iff

$$\sum_{x \in \Omega} [n - h(x)] < n.$$

Remark 4.1. If one develops our discussion with a weak ordering preference relation or a linear ordering preference relation, instead of an acyclic preference relation P^i, then our theorems in this paper are also true.

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Kenjiro NAKAMURA: Department of Humanities and Social Sciences
Tokyo Institute of Technology
2-12-1, Oh-Okayama, Meguro-Ku
Tokyo 152, JAPAN