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A NOTE ON IDLE TIME POLICY WITH REPAIR

Yoshio Tabata
Osaka University

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Abstract The present paper considers a system in which components stochastically deteriorate with age. When the component fails, it is repaired with a specified repair time distribution or left as it is until the next planned replacement opportunity. As a result of this action, an idle time occurs and the cost is incurred during the idle time. The optimal policy which minimizes the total expected cost is derived and the critical point in time to distinguish the above two actions is found by the method of dynamic programming. Some simple examples are discussed to illustrate our model.

1. Introduction

Components of a given system operate for a time and then fail. A preventive maintenance or replacement is worthwhile in reducing the cost of operating a stochastically failing equipment. Since Barlow and Proschan[2] presented a fundamental work on an age replacement policy, much effort has been devoted to this field. The age replacement policy is the one in which a component is replaced at failure or at a specified age, whichever occurs first. Recent advances in optimal control theory led to modelling of this problem as a control problem and application of dynamic programming to obtain optimal maintenance policies. Bellman[4] gave a functional equation for the replacement problem in the discrete case. Descamps[9] and Beckman[3] provided a continuous version to Bellman's problem and successfully solved the functional equation. Sivazlian[10] derived the expression for the long-run total expected discounted cost in replacement problems incorparating the downtime effect associated with the repair state of the failed component. The principal disadvantage of this policy is its complexity since it is necessary to keep detailed records about times of failures or ages of components.

Campbell[5] discussed the comparative of replacing a number of street lamps either all at once or as they failed. This problem is concerned with the so-called block replacement or constant interval policy. Under the constant interval policy the component is replaced at times kT (k = 1, 2, . ..), and at failure. The principal disadvantage of this policy is its wastefulness since we may replace practically new components at the prescribed point in time. Barlow and Hunter[1] introduced the notion of periodic replacement with minimal repair for any intervening failures. Cox[6] considered the model that if the component fails close to the time of the scheduled block replacement, we do not replace until the next planned replacement time. As a result of this policy, an idle time occurs and a cost is incurred due to the failed component remaining idle. A model along these lines for a replacement problem has been described by Crookes[7] and Woodman [11].

In the present paper, we consider a modified constant interval replacement policy with repair and idle time. A critical point in time to repair or to leave the failed component as it is, is derived by dynamic programming. Some simple examples are discussed to illustrate our model and the optimal time below which idle time should be incurred is determined.

2. Model and Formulation

Consider a system in which components stochastically deteriorate with age, that is, continuously components are subject to failure. When the component fails it is repaired with a specified repair time distribution or left as it is until the planned replacement point. As a result the system will be idle for a certain time. The opportunity cost is incurred during the idle time. Our problem is to find the optimal selection of these two actions in order to minimize the total expected cost and to derive the critical point in time to repair or to leave the failed component as it is.

Concentrating on our model, we define the following notations:

$R(t)$ = distribution function of repair time

K = fixed cost of making a repair service

C = cost associated with each unit of idle time

$\lambda(x)\Delta x$ = probability that the component aged x fails between x and $x + \Delta x$

$U(x, y)$ = the expected cost up to the next planned replacement when there is still a time x to go, the component aged y is in the state of failure, and an optimal policy is followed

$V(x, y)$ = the expected cost up to the next planned replacement when there is still a time x to go, the component aged y is in the operable state, and an optimal policy is followed

Consider the situation in which the component aged y is failed when there is still a time x to go. Let us attempt to formulate the principle of optimality by comparing the system at two closely spaced remaining times x and $x-\Delta x$. If the idle time policy is chosen at x, the cost $C\Delta x$ is incurred, the system aged y is still in the state of failure and the remaining time becomes $x-\Delta x$. If the action of repair is chosen at x, either the system turns out to be an operable state or the repair service does not finish until the next planned replacement. Note that the fixed cost K and the cost Ct are incurred during the repair service, where t is a repair time subject to the distribution $R(t)$.

Using the Principle of Optimality, we have the following functional equation for small time interval Δx:

(1) $$U(x, y) = \min \begin{cases} C\Delta x + U(x-\Delta x, y) \\ K + \int_0^x \{ Ct + V(x-t, A) \} dR(t) + Cx\int_x^\infty dR(t) \end{cases}$$

The first term in the bracket represents the cost of idle time and the second one the cost of repair service. When the repair service is over, the age of component is A, a given value which may not exceed the component age prior to failure. It is noted that the case of $A = y$ corresponds to minimal repair and the case of $A = 0$ major repair. For simplicity, we assume that the effect of repair is to set the component back into operation without affecting its age. For this case $A = y$.

If x is small enough, it is clear that the first alternative is preferable. Thus,

$$U(x, y) = C\Delta x + U(x-\Delta x, y).$$

Letting Δx approach zero, the following equation is obtained:

(2) $$\frac{\partial U(x, y)}{\partial x} = C$$

Considering the boundary condition that

$$U(0, y) = 0,$$

the solution for equation (2) is as follows:

(3) $$U(x, y) = C x$$

On the other hand, for $V(x, y)$ we have the following relation:

$$(4)\quad \begin{aligned} &V(x, y) = \lambda(y)\Delta y\ U(x-\Delta y, y) + \{ 1 - \lambda(y)\Delta y \} V(x-\Delta y, y+\Delta y) \\ &V(0, y) = 0 \end{aligned}$$

Using the Taylor expansion and equation (3), we have the quasi-linear partial differential equation.

$$\frac{\partial V(x, y)}{\partial x} - \frac{\partial V(x, y)}{\partial y} = \lambda(y) \{ Cx - V(x, y)\}$$

The solution for this equation is given by

$$(5)\quad \begin{aligned} V(x, y) &= e^{-\int_0^x \lambda(x+y-\xi)d\xi} \left[\int_0^x \lambda(x+y-\zeta)\ C\zeta e^{\int_0^\zeta (x+y-\xi)d\xi}\ d\zeta \right] \\ &= \frac{C}{1 - F(y)} \int_0^x \xi f(x+y-\xi)d\xi \end{aligned}$$

where $F(\cdot)$ and $f(\cdot)$ are the failure distribution and its density, respectively.

Therefore, the functional equation (1) for $U(x, y)$ can be written as

$$(6)\quad U(x, y) = Cx + \min[0; K - CxR(x) + \int_0^x \{ Ct + V(x-t, y)\}\ dR(t)]$$

where $V(x, y)$ is given by equation (5).

It should be noted that equation (6) is valid for the preferential region of idle time. Thus, the critical value of x, for which a repair action should be made, is obtained by setting the second term in the bracket equal to zero and finding its minimum positive root. The existence of such a root is shown as follows: Let

$$\begin{aligned} G_y(x) &= K - CxR(x) + \int_0^x \{ Ct + V(x-t, y) \}\ dR(t) \\ &= K - CxR(x) - \frac{C}{1 - F(y)} \left[\int_0^x R(t)dt + \int_0^x \int_y^{x+y-t} F(\xi)d\xi dR(t) \right]. \end{aligned}$$

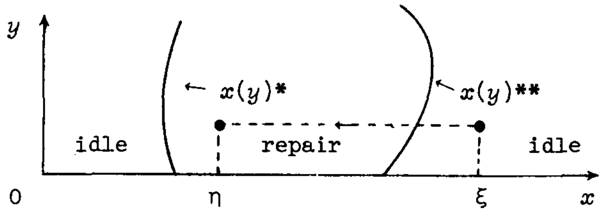
It is clear that for each $y > 0$,

$$G_y(0) = K > 0 \quad \text{and} \quad G_y(\infty) = -\infty < 0.$$

Thus, there exists at least one root x which satisfies $G_y(x) = 0$, since $G_y(x)$ is a continuous function of x for each fixed y. Note that $x(y)^* = \inf\{ x; G_y(x) = 0\}$ represents the critical value for which a repair action should be made. It is intuitively clear, and can be easily demonstrated, that the optimal region is provided by the simple form as

$$\begin{cases} \text{idle} & \text{for } 0 < x < x(y)^* \\ \text{repair} & \text{for } x > x(y)^*. \end{cases}$$

This can be verified as follows: Suppose that the optimal region is characterized by two different critical curves $x(y)^*$ and $x(y)^{**}$ ($0 < x(y)^* < x(y)^{**}$) as shown in the figure.



If the component aged y fails at $\xi > x^{**}$, the idle time policy is preferential. The age of component remains as it was y and the system is still in the state of failure until the next planned replacement point 0. After the lapse of $\xi - \eta$ (where $x^* < \eta < x^{**}$) under the idle time policy, the remaining time to the next planned replacement is given by η. At time η, the component aged y is still in the state of failure and the idle time policy has been employed ever since. On the other hand, for the remaining time η it is clear that the repair action should be made for any failed component aged y. This is a contradiction since both actions are preferential for all η and y. Thus, it is shown that the repair action should be employed for any $x > x(y)^*$. Note that the critical point depends generally on the age of component y when we consider the constant interval policy with idle time and repair. The principal advantage of constant interval policy is vanished.

In the general theory of maintenance it is shown how equation (6) may be solved in practical cases. In the next section we shall discuss some simple examples.

3. Simple Examples

(1) General Failure Distribution and Negligible Repair Time

One of the simplest examples for our problem is found in the case when the failure distribution is arbitrary and the repair time is instantaneous. In this case equation (6) is written as

$$U(x, y) = Cx + \min[\ 0;\ K - Cx + V(x, y)\ ]$$

$$= Cx + \min[\ 0;\ K - Cx + C\{\int_0^x \xi f(x+y-\xi)d\xi\} \ /\ (\ 1 - F(y)\ )\ ].$$

In the following, it is assumed that the repair is maximal. This assumption implies that the repair action is equivalent to replace the failed component with a new one. Thus, we can put $A = y = 0$ and the above equation is reduced to

$$U(x,\ y) = Cx + \min[\ 0;\ K - C\int_0^x (\ 1 - F(t)\ )dt\ ]$$

$$= Cx + C\min[\ 0;\ \frac{K}{C} - m + T_F(x)\ ],$$

where m is the mean time to failure and the transform $T_F(x)$ is defined as

$$T_F(x) = \int_x^\infty (\ t - x\ )dF(t).$$

As was shown in DeGroot[8], the transform $T_F(x)$ is a nonnegative convex and strictly decreasing function of x. Thus, the only condition necessary to ensure the existence of idle time is

$$\frac{K}{C} < m$$

The result derived in this example is exactly the same one as was shown in Woodman[11]. Under the condition that $K/C < m$, the unique critical value x^* satisfies the following equation:

$$\frac{K}{C} - m + T_F(x^*) = 0, \text{ or } \quad x^* = T_F^{-1}(m - \frac{K}{C}).$$

The solution of this equation will be found by a numerical method. The fact mentioned above specified the optimal replacement policy as follows:

$$\begin{cases} \text{idle} & \text{for } 0 \le x < x^* \\ \text{repair} & \text{for } x \ge x^*. \end{cases}$$

(2) Gamma Type Failure Distribution

Consider a Gamma type failure distribution

$$f(x) = \frac{\lambda^k}{(k-1)!} e^{-\lambda x} x^{k-1} \qquad (\ x \ge 0\ ).$$

Then equation (5) can be denoted as

$$V(x,\ y) = \frac{C\lambda^k \int_y^{x+y} e^{-\lambda x} x^{k-1} dx}{(k-1)! - \lambda^k \int_0^y e^{-\lambda x} x^{k-1}\ dx}$$

It is difficult to carry out the operation of integral explicitly except for $k = 1$ and $k = 2$.

For $k = 2$, we have

$$V(x, y) = Cx + \frac{C}{\lambda y + 1} \{ e^{-\lambda x} (x + y + \frac{2}{\lambda}) - (y + \frac{2}{\lambda})\}$$

and

$$U(x, y) = Cx + \min[0; K + \frac{C\lambda e^{-\lambda x}}{\lambda y + 1} \{ \int_0^x (t-x-y-\frac{1}{\lambda}) e^{\lambda t} R(t)dt \}].$$

For $k = 1$, the failure distribution is reduced to an exponential distribution and we have

$$V(x, y) = \frac{C}{\lambda} (\lambda x + e^{-\lambda x} - 1).$$

Thus,

$$U(x, y) = Cx + \min[0; K - Ce^{-\lambda x} \int_0^x e^{\lambda t} R(t)dt].$$

From this relationship, it is noted that the critical value x^* is given by the solution of

$$\frac{K}{C\ m} = f(x) * R(x),$$

where the symbol $*$ denotes the convolution integral. In addition to the assumption that the failure distribution is exponential, we suppose that the repair time is subject to an exponential distribution $R(t) = 1 - exp(-\mu t)$ and $\mu/\lambda = \rho > 1$. Then the above relation can be rewritten as

$$U(x, y) = Cx + \min[0; K + \frac{C\mu}{\lambda(\lambda-\mu)} (e^{-\lambda x} - e^{-\mu x}) - \frac{C}{\lambda} (1 - e^{-\mu x})].$$

Setting the second term in the bracket equal to zero, the condition for the existence of critical point x^* is given by $C > K\lambda$. To explore the critical point in detail, let us consider the following equation,

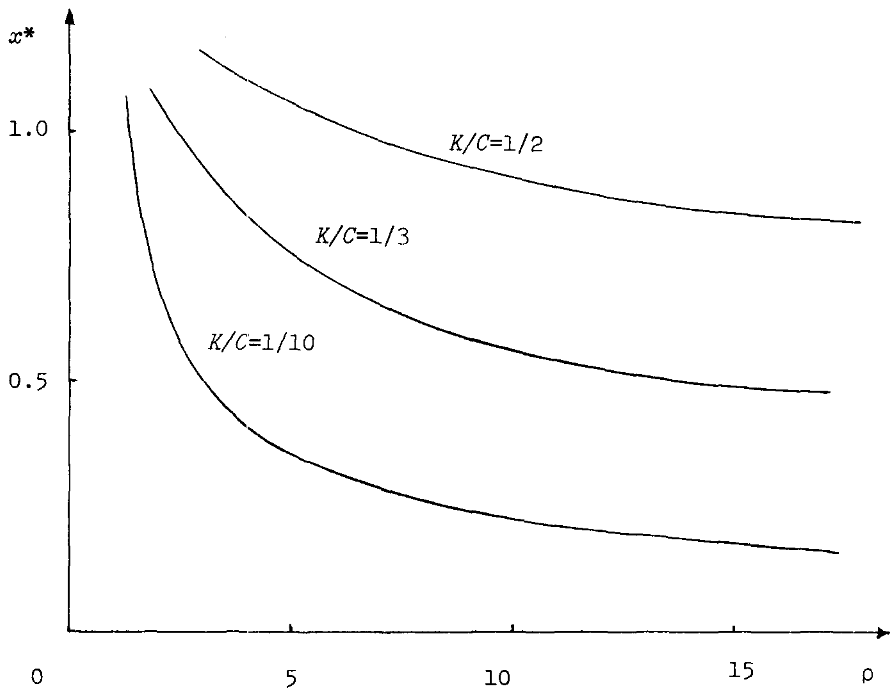
$$Z^{\rho} - \rho Z = (1 - \rho)(1 - K\lambda/C),$$

where $exp(-\lambda x) = Z$.

It is noted that the critical point x^* is the minimum positive root of this equation. For various values of parameters K/C and ρ, the root of this equation will be found by a numerical method. Table 1 provides the critical values and they are sketched in Fig. 1.

K/C ρ	1/2	1/3	1/10
0	∞	∞	∞
1	1.6780	1.0987	0.5290
2	1.2279	0.8612	0.3769
3	1.0498	0.7302	0.3146
4	0.9623	0.6577	0.2784
5	0.9088	0.6112	0.2523
6	0.8728	0.5786	0.2345
7	0.8440	0.5542	0.2206
8	0.8260	0.5361	0.2095
9	0.8077	0.5215	0.2004
10	0.7983	0.5097	0.1924
100	0.7032	0.5155	0.1154
∞	0.6988	0.4050	0.1050

(Table 1)



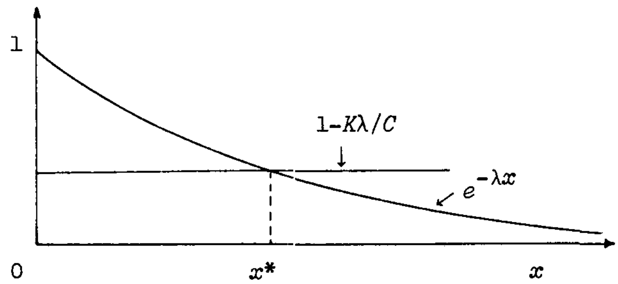
(Fig. 1)

Especially, we can easily obtain the analytical forms of this value x^* for two extreme cases $\mu = \infty$ and $\mu = 0$.

The assumption of $\mu = \infty$ shows a negligible repair time. Thus, the above equation is expressed as

$$U(x,\ y) = Cx + \frac{1}{\lambda}\min[\ 0;\ K\lambda + Ce^{-\lambda x} - C\ ].$$

Since the second term in the bracket is a decreasing function of x, there exists the unique value $x^* = (-1/\lambda)log(C-K\lambda)/C$ if $K\lambda < C$ as is explained in Fig. 2.



(Fig. 2)

On the other hand, we consider the case of $\mu = 0$. This means that the repair action never finishes in the finite horizon. Then we have

$$U(x,\ y) = Cx + \min[\ 0;\ K\ ] = Cx.$$

The result shows that the optimal policy is to be always idle for any x.

As the last example of repair time, we consider it as constant in time D. The distribution function $R(x)$ is written as

$$R(x) = \begin{cases} 0 & \text{for } 0 \le x < D \\ 1 & \text{for } x \ge D \end{cases}$$

Accordingly we have

$$U(x,\ y) = Cx + \min\left[\ 0;\ \begin{matrix} K & \text{for } 0 \le x < D \\ K - \dfrac{C}{\lambda} + \dfrac{C}{\lambda}e^{-\lambda(x-D)} & \text{for } x \ge D \end{matrix}\ \right]$$

$$= Cx + \begin{cases} 0 & \text{for } 0 \le x < D \\ \min[\ 0;\ K - \dfrac{C}{\lambda} + \dfrac{C}{\lambda} e^{-\lambda(x-D)}\ ] & \text{for } x \ge D \end{cases}$$

This equation and the following Fig. 3 show that the optimal policy is described as follows:

(i) $C > \lambda K$

$$\begin{array}{ll} \text{idle} & \text{for } 0 \le x < x^* \\ \text{repair} & \text{for } x \ge x^*\ (\ > D\ ) \end{array}$$

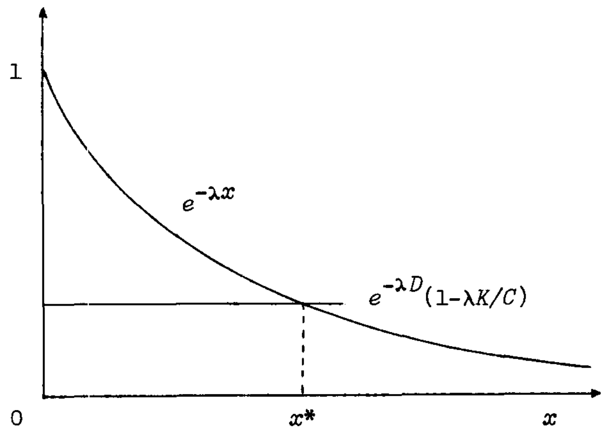
where x^* is given by

$$x^* = D - \frac{1}{\lambda}\ log(\frac{C - \lambda K}{C})$$

(ii) $C \le \lambda K$

$$\text{idle} \qquad \text{for all } x$$

since the second term is positive for all x.



(Fig. 3)

(3) Linear Failure Distribution

Under a straight line reliability function

$$F(t) = \begin{cases} \beta t & 0 \le t \le 1/\beta \\ 1 & t \ge 1/\beta \end{cases} ,$$

the failure rate is given by

$$\lambda(t) = \beta / (\ 1 - \beta t\ ) \qquad (\ 0 \le t \le 1/\beta\ ).$$

To derive an explicit expression of $V(x, y)$ and $U(x, y)$, we consider the following three cases:

Case (i) $x + y \le 1/\beta$

$$V(x, y) = \frac{C}{1-\beta y}\int_y^{x+y}(x+y-\xi)\beta\, d\xi = \frac{C\beta x^2}{2(1-\beta y)}$$

and

$$U(x, y) = Cx + \min[\,0;\ K - CxR(x) + \int_0^x\{Ct + \frac{C\beta(x-t)^2}{2(1-\beta y)}\}\, dR(t)\,].$$

Note that the condition $x + y \le 1/\beta$ suggests that the time remaining until the next planned replacement is short and the component is in the nearly new state.

Case (ii) $x + y \ge 1/\beta$ and $y \le 1/\beta$

This case means that the time remaining is long enough and the component is nearly new. Then we have

$$V(x, y) = \frac{C}{1-\beta y}\int_y^{1/\beta}(x+y-\xi)\beta\, d\xi = C\{x + \frac{1}{2}(y - \frac{1}{\beta})\}$$

and

$$U(x, y) = Cx + \min[\,0;\ K + \frac{C}{2}R(x)(y - \frac{1}{\beta}) - CR(0)\{x + \frac{1}{2}(y-\frac{1}{\beta})\}]$$

If we assume that the repair service is not finished instantaneously, then the above expression is reduced to

$$U(x, y) = Cx + \min[\,0;\ K + \frac{C}{2}R(x)(y - \frac{1}{\beta})\,].$$

Case (iii) $y \ge 1/\beta$

It is clear that $V(x, y) = \infty$. It follows that the optimal action should be always idle since the component aged y ($\ge 1/\beta$) fails with probability 1.

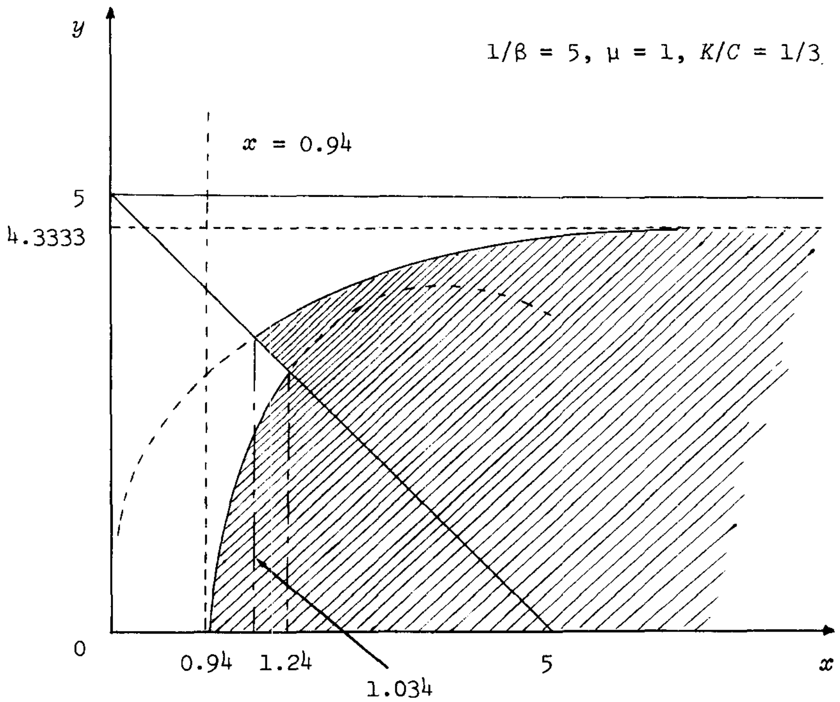
To study the optimal policy in detail, we specify the distribution of repair time $R(t)$ as follows:

(A) Exponential Distribution $R(t) = 1 - e^{-\mu t}$

From the results of three cases mentioned above, it follows that

$$U(x,\ y) = Cx + \begin{cases} \min[\ 0;\ K - Cx + \dfrac{C}{\mu}(1 - e^{-\mu x}) + \dfrac{C\beta}{1 - \beta y}\left(\dfrac{x^2}{2} - \dfrac{x}{\mu} + \dfrac{1}{\mu^2} - \dfrac{e^{-\mu x}}{\mu^2}\right)] & \text{for } x + y \le 1/\beta \\ \min[\ 0;\ K + \dfrac{C}{2}(1 - e^{-\mu x})(y - \dfrac{1}{\beta})] & \text{for } x + y \ge 1/\beta \text{ and } y \le 1/\beta \\ 0 & \text{for } y \ge 1/\beta \end{cases}$$

It is clear that the critical point x^* depends on x and y. We can find the critical point by the numerical calculation. The shaded portion in Fig. 4 illustrates the preferential region of repair service for $1/\beta = 5$, $\mu = 1$ and $K/C = 1/3$. Using Fig. 4, we can find the critical point for all pair $(x,\ y)$.



(Fig. 4)

(B) Straight Line Repair Time Distribution

Let $R(t)$ be a linear function as

$$R(t) = \begin{cases} \alpha t & 0 \le t \le 1/\alpha \\ 1 & t \ge 1/\alpha \end{cases}$$

To avoid unnecessary complications, we assume that $\alpha \ge \beta$. Then we have the following result:

$$U(x, y) = Cx + \begin{cases} \min[\,0;\ K - \dfrac{C}{2}\alpha x^2 + \dfrac{C\alpha\beta}{6(1 - \beta y)} x^3\,] & \text{for } x + y \le 1/\beta \text{ and } 0 \le x \le 1/\alpha \\ \min[\,0;\ K - Cx + \dfrac{C}{2\alpha} + \dfrac{C\beta}{2(1 - \beta y)}(x^2 - x/\alpha + 1/3\alpha^2)\,] & \text{for } x + y \le 1/\beta \text{ and } 1/\alpha \le x \\ \min[\,0;\ K + \dfrac{C\alpha}{2} x(y - 1/\beta)\,] & \text{for } x + y \ge 1/\beta,\ y \le 1/\beta \text{ and } x \ge 1/\alpha \\ \min[\,0;\ K + \dfrac{C}{2}(y - 1/\beta)\,] & \text{for } x + y \ge 1/\beta,\ y \le 1/\beta \text{ and } x \ge 1/\alpha \\ 0 & \text{for } y \ge 1/\beta \end{cases}$$

The shaded portion in Fig. 5 shows a preferential region of repair service for this example.

4. Conclusion

The present paper is concerned with a constant interval policy with repair and idle time. An optimal policy and a critical point in time to repair or to leave the failed component as it is are found by the method of dynamic programming. We show that the optimal policy depends not only on the time until the next planned replacement point but on the component age. The advantage of ordinary constant interval policy that is unnecessary to keep detailed records about age of component vanishes for our model with repair and idle time. It is difficult in general to obtain an explicit form of optimal policy for any distribution of failure and repair. A numerical calculation presents a solution to this difficult problem. For some simple examples, convenient figures which specify the critical point and the preferential region of repair action are presented. The results will be useful to solve the practical problems.

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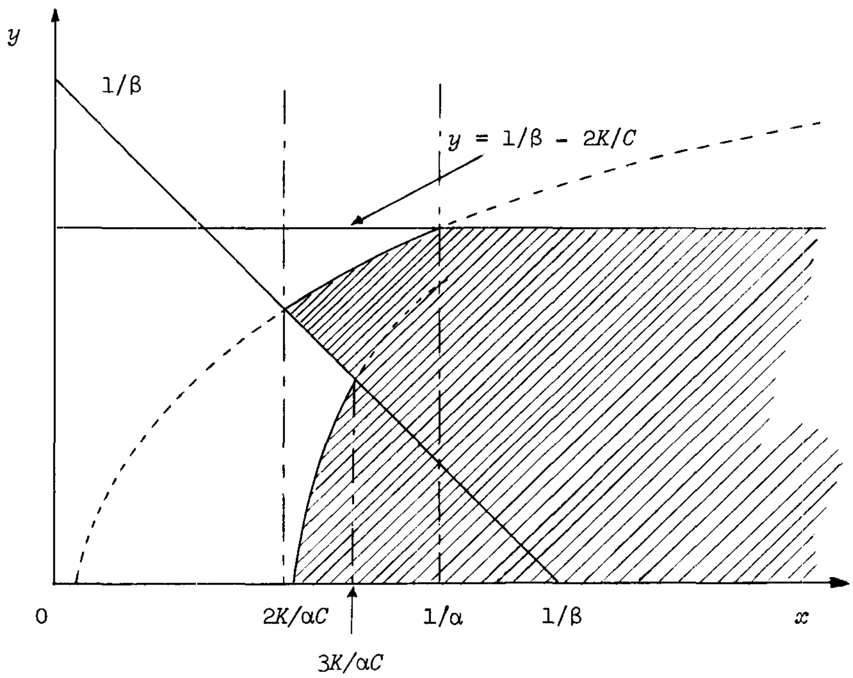

(Fig. 5)

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Yoshio TABATA: Department of Applied Physics, Faculty of Engineering, Osaka University, Suita, Osaka, 565, Japan.