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ON OPTIMAL POLICIES FOR MULTI-REPAIR-TYPE MARKOV MAINTENANCE MODELS

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Abstract This paper treats an extension of an optimal machine maintenance model with Markovian deterioration introduced by Derman. The system consists of an operating machine whose deterioration is Markovian, a finite number of identical spare machines, and several types of repair facilities where machines to be repaired are sent depending on the types of repair work required. At each period of time, a decision is made on an operating machine whether it is repaired or not, knowing its degree of deterioration, the type of repair work required if the repair decision is chosen, and the number of machines in each type of repair facility. Here, the repair time distributions, material costs, and labor cost all depend on the type of repair work required on the machine. Sufficient conditions which result in the optimality of control limit policies of some kind are obtained.

In this study a discrete time finite state Markov maintenance model with several types of repair shops is presented. Because of their wide applicability in the practical world, a number of authors have studied optimization problems for machine maintenance models when changes of states are Markovian. Their main concern has been on the structure of an optimal policy, and a simple repair rule called a control limit policy has been introduced. In 1963 Derman [1] introduced the basic maintenance model of this type. Assuming a simple cost structure and the IFR property on the transition probabilities, Derman showed the optimality of a control limit policy. Kolesar [7] extended the basic model by introducing state dependent operating costs, and Kalymon [4] further generalized the cost structure by allowing replacement costs to be stochastic. Klein [6] expanded Derman's model to include a costly inspection, or an imperfect information, which has been developed by Taylor [10] and Rosenfield [8] with emphasis on trying to find simple types of optimal policies.

The aforementioned models have the properties that the amount of time needed for the repair of a machine is one unit of time, and an unlimited supply of new spares is available. In that sense, they are replacement models rather than repair models. In 1973 Kao [5] introduced a semi-Markovian approach to the basic model. In his model, the repair time of a machine takes some random time according to its semi-Markovian nature, while the supply of new spares is kept unlimited. In this report we develop a general repair model in the sense that the repair time is a random variable, and that the supply of spare machines is limited.

1. Description of the Model

Consider a basic multi-repair-type Markov machine maintenance model. The flow of machines in the system is schematically shown in Fig. 1. There are an operating machine, S ($S \geq 1$) identical spare machines and K ($K \geq 1$) kinds of repair shops in the system. The system is observed periodically and at the beginning of each period an operating machine is classified as being in one of $I + 1$ ($I \geq 1$) states, with each state showing the degree of deterioration. 0 represents a state of a machine in its best condition, while I denotes its failure. When a machine is operating, two choices are available at the beginning of each period: to let it keep operating, or to repair it. If the

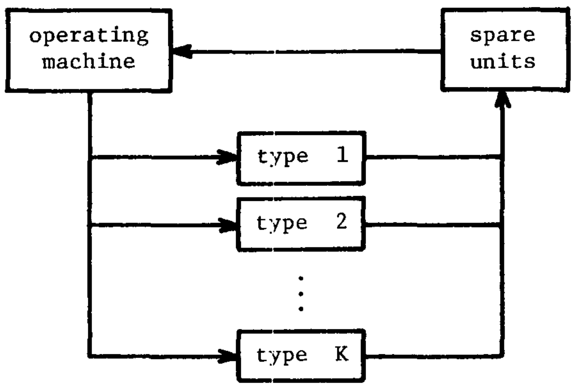

Figure 1. A Multi-Repair-Type Machine Maintenance System.

former decision is chosen, the state of the operating machine evolves from i to j in one unit of time according to the transition probability $p_{ij} \geq 0$. If the latter is selected, the machine is immediately sent to one of K repair shops depending on the repair work required, and is replaced by one of the spare machines, if any is available. The new operating machine begins to operate in its best condition. Type 1 through type K repair shops are arranged so that type 1 deals with the easiest repair work and type K the hardest. If a machine in the i-th operating condition is chosen to be repaired, its repair work determines the type of repair shop to be sent, and with probability $p_k^{(i)} \geq 0$ type k repair shop is selected. Here we assume $\sum_{k=1}^{K} p_k^{(i)} = 1$ for $0 \leq i \leq I$. A failed machine must be repaired. Furthermore, suppose the type of repair required if a repair decision is made is known to the decision maker before he makes a decision. Then the total information available to the decision maker at the beginning of each period is the condition of an operating machine, the type of repair work required if the repair decision is chosen, and the number of machines in the type k repair shop for each $1 \leq k \leq K$. Later, the case where the type of repair work required is not known to the decision maker will be considered.

If a machine in the i-th operating condition is sent to the type k $(1 \leq k \leq K)$ repair shop, the repair work starts immediately, and its repair time T_k, which is independent of i, is assumed to be a random variable having a geometric distribution with parameter q_k, i.e.,

$$\Pr\{T_k = j\} = q_k(1-q_k)^{j-1}, \qquad j = 1, 2, \ldots .$$

Geometric distribution can be considered as a discrete version of an exponential distribution, and the repair time distribution to be geometric seems suitable as a basic analysis of the problem.

If all the machines are in the repair shops, the system fails since no machine is available. In that case we must wait until one of the machines is completely repaired. A penalty cost, P, is assessed per period during the system's failure. $A(i)$ is the operating cost for a machine in the i-th operating condition, $C(i,k)$ is the material cost for repairing a machine in the type k repair shop, and $B(k)$ is the labor cost for the type k repair work for a single machine per period. The objective is to find a repair policy which minimizes the total expected α-discounted cost or the long-run expected average cost.

Before proceeding further it is useful to give an example where the above model can be applied. Consider the problem of a taxicab driver who owns several cars. He continues using one of his cars until he decides to repair it. He then begins to use another one, if any is available. We assume that the maintenance, or operating cost (excluding repair costs), of a car depends mainly on its age or the number of days after its last repair. However, the actual deterioration of the car does not necessarily coincide with its age. Among cars having the same age, some may require little repair work, while others may require substantial amounts of repair work. A car with i-th age requires type k repair work with probability $p_k^{(i)}$. If the taxicab driver is familiar with the repair job, he can get the information on the type of repair work required as well as the age of the car before he makes the decision on whether or not the car should be repaired. Otherwise he can utilize just the age of the car when he makes a decision. The repair time distribution is assumed to be geometric, and its parameter depends on the type of repair work. Also, repair costs mainly depend on the type of repair work. If all the cars are in the repair facilities, no car is available to him, and hence, he loses some amount of the expected revenue. If he is interested in the total α-discounted cost, then finding the best repair schedule is an example of the type of problem which will be treated in this paper.

2. Control Limit Policy with Respect to Operating Condition

Let

$$\mathrm{I} = \{i \mid 0 \le i \le I,\ i : \text{integer}\},$$

$$\mathrm{K} = \{k \mid 1 \le k \le K,\ k : \text{integer}\},$$

$$\mathrm{S}^m = \{(s_1,\dots,s_K) \mid \textstyle\sum_{k=1}^K s_k \le m,\ s_j \ge 0 : \text{integer } (1 \le j \le K)\},$$

$$\mathrm{S}_0^m = \{(0,1,s_1,\dots,s_K) \mid \textstyle\sum_{k=1}^K s_k = m,\ s_j \ge 0 : \text{integer } (1 \le j \le K)\}.$$

The state of this system is represented by the K+2 vector

$$X_t = (X_t^0,\ X_t^1,\ X_t^2,\ \dots\ ,\ X_t^{K+1}) = (i,\ k,\ s_1,\ \dots\ ,\ s_K),$$

where at the beginning of the t-th period there is an operating machine whose operating condition is $i \in \mathrm{I}$, type $k \in \mathrm{K}$ repair work is required if the repair decision is chosen, and s_j $(1 \le j \le K)$ machines are in the type j repair shop $((s_1,\ \dots\ ,s_K) \in \mathrm{S}^S)$. For notational convenience, we write $X_t \in \mathrm{S}_0^{S+1}$ if all the machines are in the repair facilities, or none of the machines is operating at the beginning of the t-th period. Let $q_{ss'}^{(j)}$ be the probability that s' $(s' \le s)$ machines are still in the type j repair shop at the end of the period, given s machines are in the type j $(1 \le j \le K)$ repair shop at the beginning of a period. Then

$$q^{(j)}_{s_j s'_j} = \binom{s_j}{s'_j} (1 - q_j)^{s'_j} q_j^{s_j - s'_j}, \quad 1 \le j \le K.$$

The following simplification is made. When a new machine replaces the previously operating machine, the new one starts operating in its best condition at the beginning of the next period, and the least repair work, i.e., type 1 repair is required if it is instantaneously determined to be repaired.

Let $V_\alpha(i,k,s_1,\ldots,s_K;n)$ be the minimum expected n period α-discounted cost starting from state $(i,k,s_1,\ldots,s_K)$. Then by setting $V_\alpha(i,k,s_1,\ldots,s_K;0) = 0$ for any $(i,k,s_1,\ldots,s_K) \in \mathrm{I} \times \mathrm{K} \times \mathrm{S}^S \cup \mathrm{S}_0^{S+1}$, $V_\alpha(i,k,s_1,\ldots,s_K;n)$ $(n \ge 1)$ satisfies the set of recursive equations:

$$V_\alpha(i,k,s_1,\ldots,s_K;n)$$

$$(2.1)\quad \begin{cases} = \min\{A(i) + \sum_{j=1}^{K} B(j)s_j + \alpha \sum_{i'=0}^{I} \sum_{k'=1}^{K} p_{ii'} p_{k'}^{(i')} R_\alpha(i',k',s_1,\ldots,s_K;n-1), \\ \quad C(i,k) + B(k) + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;n-1)\} \\ \qquad\qquad \text{for } (i,k,s_1,\ldots,s_K) \in \mathrm{I} \times \mathrm{K} \times \mathrm{S}^S, \\ = P + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1,\ldots,s_K;n-1) \\ \qquad\qquad \text{for } (i,k,s_1,\ldots,s_K) \in \mathrm{S}_0^{S+1}, \end{cases}$$

where

$$R_\alpha(i,k,s_1,\ldots,s_K;n) = \sum_{s'_1=0}^{s_1} \cdots \sum_{s'_K=0}^{s_K} q^{(1)}_{s_1 s'_1} \cdots q^{(K)}_{s_K s'_K} V_\alpha(i,k,s'_1,\ldots,s'_K;n).$$

Let $V_\alpha(i,k,s_1,\ldots,s_K)$ be the total expected α-discounted cost starting from state $(i,k,s_1,\ldots,s_K)$. Then it is well known that

$$V_\alpha(i,k,s_1,\ldots,s_K) = \lim_{n\to\infty} V_\alpha(i,k,s_1,\ldots,s_K;n).$$

Furthermore, the existence of a stationary policy minimizing the total expected α-discounted cost is guaranteed.

Definition. A *control limit policy with respect to operating condition* is a nonrandomized policy where there is a special operating condition i for each k $(1 \le k \le K)$, for each feasible $s = (s_1,\ldots,s_K)$, and for each period n $(n \ge 1)$, say $i_{k,s,n}$, such that for all $(i,k,s_1,\ldots,s_K)$ with $i < i_{k,s,n}$, the decision at period n is to keep an operating machine in operation, and for all $(i,k,s_1,\ldots,s_K)$ with $i \ge i_{k,s,n}$ the decision is to repair it.

Sufficient conditions for the existence of a stationary control limit policy w.r.t. operating condition which minimizes the total expected α-discounted cost of this model will be obtained through several lemmas. First the conditions which will be needed in the following discussions are given here.

1. $B(k)$ is nonnegative and nondecreasing in k for $1 \le k \le K$.
2. $C(i,k)$ is nondecreasing in k $(1 \le k \le K)$ for each i $(0 \le i \le I)$.
3. $C(i,k)$ is nondecreasing in i $(0 \le i \le I)$ for each k $(1 \le k \le K)$.
4. $A(i) - C(i,k)$ is nondecreasing in i $(0 \le i \le I)$ for each k $(1 \le k \le K)$.
5. $P \ge C(0,1)$.
6. q_k is positive and nonincreasing in k $(1 \le k \le K)$.
7. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \le i \le I-1$ where $P_i(j) = \sum_{i' \le j} p_{ii'}$, $0 \le j \le I$.
8. $P^{(i)}(\cdot) \subset P^{(i+1)}(\cdot)$ for $0 \le i \le I-1$ where $P^{(i)}(k) = \sum_{k' \le k} p_{k'}^{(i)}$, $1 \le k \le K$.

The binary relation $\subset$ is read as "is stochastically smaller than or equal to", and $F(\cdot) \subset G(\cdot)$ if and only if $F(t) \ge G(t)$ for any $t \ge 0$.

Lemma 1. For $(i,k',s_1,\ldots,s_K) \in \mathsf{I} \times \mathsf{K} \times \mathsf{S}^{S-1} \cup \mathsf{S}_0^S$,

$$R_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;n) - R_\alpha(i,k',s_1,\ldots,s_k,\ldots,s_K;n)$$
$$= (1-q_k) \sum_{s_1'=0}^{s_1} \cdots \sum_{s_k'=0}^{s_k} \cdots \sum_{s_K'=0}^{s_K} q_{s_1 s_1'}^{(1)} \cdots q_{s_K s_K'}^{(K)}$$
$$\cdot (V_\alpha(i,k',s_1',\ldots,s_k'+1,\ldots,s_K';n) - V_\alpha(i,k',s_1',\ldots,s_k',\ldots,s_K';n))$$

for $1 \le k \le K$, and for $n \ge 1$.

Proof: Proof is direct from the definition of R_α, and from the fact that for nonnegative integers m and n,

$$\binom{m}{n-1} + \binom{m}{n} = \binom{m+1}{n}. \qquad \square$$

Lemma 2. Assume conditions 1, 5 and 6 hold. Then for $n \ge 1$ and for $(i,k',s_1,\ldots,s_K) \in \mathsf{I} \times \mathsf{K} \times \mathsf{S}^{S-1} \cup \mathsf{S}_0^S$,

(a) $V_\alpha(i,k',s_1,\ldots,s_{k+1}+1,\ldots,s_K;n) \ge V_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;n)$.

(b) $V_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;n) \ge V_\alpha(i,k',s_1,\ldots,s_k,\ldots,s_K;n)$.

Proof: Proof is by mathematical induction. For $n = 1$, it is easy to check that both (a) and (b) hold for $(i,k',s_1,\ldots,s_K) \in \mathsf{I} \times \mathsf{K} \times \mathsf{S}^{S-1} \cup \mathsf{S}_0^S$, and the proof is omitted. Suppose both (a) and (b) hold for $n = m-1$ (≥ 1). We first show that (a) holds for $n = m$.

Notice that for $(i,k',s_1,\ldots,s_K) \in \mathsf{I} \times \mathsf{K} \times \mathsf{S}^{S-1} \cup \mathsf{S}_0^S$,

$$R_\alpha(i,k',s_1,\ldots,s_{k+1}+1,\ldots,s_K;m-1) - R_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;m-1)$$
$$\geq (1-q_{k+1}) \sum_{s_1'=0}^{s_1} \cdots \sum_{s_K'=0}^{s_K} q^{(1)}_{s_1 s_1'} \cdots q^{(K)}_{s_K s_K'}$$
$$\cdot (V_\alpha(i,k',s_1',\ldots,s_{k+1}'+1,\ldots,s_K';m-1) - V_\alpha(i,k',s_1',\ldots,s_k'+1,\ldots,s_K';m-1))$$

≥ 0, by Lemma 1 and inductive assumptions on both (a) and (b) for $n = m-1$.

Using the above result, for $(i,k',s_1,\ldots,s_K) \in \mathsf{S}_0^S$,

$$V_\alpha(0,1,s_1,\ldots,s_{k+1}+1,\ldots,s_K;m) - V_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;m)$$
$$= B(k+1) - B(k) + (R_\alpha(0,1,s_1,\ldots,s_{k+1}+1,\ldots,s_K;m-1)$$
$$- R_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;m-1))$$

≥ 0, from condition 1.

For $(i,k',s_1,\ldots,s_K) \in \mathsf{I} \times \mathsf{K} \times \mathsf{S}^{S-1}$, we compare the corresponding values term by term.

$$[V_\alpha(i,k',s_1,\ldots,s_{k+1}+1,\ldots,s_K;m)]_1 - [V_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;m)]_1$$
$$= B(k+1) - B(k) + \alpha \sum_{i'=0}^{I} \sum_{j=1}^{K} p_{ii'} p_j^{(i')} (R_\alpha(i',j,s_1,\ldots,s_{k+1}+1,\ldots,s_K;m-1)$$
$$- R_\alpha(i',j,s_1,\ldots,s_k+1,\ldots,s_K;m-1))$$

≥ 0,

where $[V]_i$ represents the i-th term V_i in the bracket of the right hand side of V if $V = \min\{V_1,V_2,\ldots,V_n\}$.

In a similar manner, the difference of the corresponding second terms can be easily shown to be nonnegative, yielding that (a) holds for $n = m$.

Consider (b) for $n = m$. For $(i,k',s_1,\ldots,s_K) \in \mathsf{I} \times \mathsf{K} \times \mathsf{S}^{S-1}$, we again compare the corresponding values term by term.

$$[V_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;m)]_1 - [V_\alpha(i,k',s_1,\ldots,s_k,\ldots,s_K;m)]_1$$
$$= B(k) + \alpha \sum_{i'=0}^{I} \sum_{j=1}^{K} p_{ii'} p_j^{(i')} (R_\alpha(i',j,s_1,\ldots,s_k+1,\ldots,s_K;m-1)$$
$$- R_\alpha(i',j,s_1,\ldots,s_k,\ldots,s_K;m-1))$$

≥ 0, from condition 1 and inductive assumption on (b) for $n = m-1$.

Similarly, the difference of the corresponding second terms can be shown to be nonnegative. Lastly, for $(i,k',s_1,\ldots,s_K) \in S_0^S$,

$$V_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;m) - V_\alpha(0,1,s_1,\ldots,s_k,\ldots,s_K;m)$$

$$\geq P + \sum_{j=1}^{K} B(j)s_j + B(k) + \alpha R_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;m-1)$$

$$- (C(0,1) + B(1) + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1+1,s_2,\ldots,s_K;m-1))$$

$$\geq (P - C(0,1)) + (B(k) - B(1)) \geq 0, \text{ from conditions 1 and 5.}$$

Hence, for $(i,k',s_1,\ldots,s_K) \in \mathrm{I} \times \mathrm{K} \times \mathrm{S}^{S-1} \cup \mathrm{S}_0^S$, (b) holds for $n = m$, completing the mathematical induction and yielding that both assertions (a) and (b) hold for $n \geq 1$. □

Lemma 3. Suppose conditions 1, 2, 5, and 6 hold. Then $V_\alpha(i,k,s_1,\ldots,s_K;n)$ is nondecreasing in k $(1 \leq k \leq K)$ for each fixed $(i,s_1,\ldots,s_K) \in \mathrm{I} \times \mathrm{S}^S$ and for $n \geq 1$.

Proof: Mathematical induction is applied. The assertion trivially holds for $n = 1$. Suppose it holds for $n = m-1 \geq 1$, and consider the case for $n = m$. As $[V_\alpha(i,k,s_1,\ldots,s_K;m)]_1$ for $(i,k,s_1,\ldots,s_K) \in \mathrm{I} \times \mathrm{K} \times \mathrm{S}^S$ is independent of k, it is enough to check the second term. For $(i,s_1,\ldots,s_K) \in \mathrm{I} \times \mathrm{S}^S$, and for $1 \leq k \leq K-1$,

$$[V_\alpha(i,k+1,s_1,\ldots,s_K;m)]_2 - [V_\alpha(i,k,s_1,\ldots,s_K;m)]_2$$

$$= C(i,k+1) + B(k+1) + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1,\ldots,s_{k+1}+1,\ldots,s_K;m-1)$$

$$- (C(i,k) + B(k) + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;m-1))$$

$$\geq 0, \text{ from conditions 1, 2 and Lemma 2.}$$

Hence the assertion holds for $n = m$, completing the mathematical induction and the proof of the lemma. □

Lemma 4. Suppose a function $V(i,k)$ is nondecreasing in i $(0 \leq i \leq I)$ for each fixed k $(1 \leq k \leq K)$ and nondecreasing in k $(1 \leq k \leq K)$ for each fixed i $(0 \leq i \leq I)$, and suppose conditions 7 and 8 hold. Then $\sum_{i'=0}^{I} \sum_{k=1}^{K} p_{ii'} p_k^{(i')} V(i',k)$ is nondecreasing in i for $0 \leq i \leq I$.

Proof: For $0 \le i' \le I-1$, using the conditions on $V(i',k)$ and by condition 8,

$$\sum_{k=1}^{K} p_k^{(i'+1)} V(i'+1,k) \ge \sum_{k=1}^{K} p_k^{(i'+1)} V(i',k)$$
$$\ge \sum_{k=1}^{K} p_k^{(i')} V(i',k),$$

which gives that $g(i') = \sum_{k=1}^{K} p_k^{(i')} V(i',k)$ is nondecreasing in i' $(0 \le i' \le I)$. From condition 7, this implies that $\sum_{i'=0}^{I} p_{ii'} g(i')$ is nondecreasing in i $(0 \le i \le I)$, which is what we want. □

Lemma 5. Assume conditions 1 through 8 hold. Then $V_\alpha(i,k,s_1,\ldots,s_K;n)$ is nondecreasing in i $(0 \le i \le I)$ for each fixed $(k,s_1,\ldots,s_K) \in \mathrm{K} \times \mathrm{S}^S$ and for $n \ge 1$.

Proof: Mathematical induction is again used. For $n = 1$, it is obvious that $V_\alpha(i,k,s_1,\ldots,s_K;1)$ is nondecreasing in i from conditions 3 and 4. Suppose the assertion holds for $n = m-1 \ge 1$, and consider the case for $n = m$. Then for $(k,s_1,\ldots,s_K) \in \mathrm{K} \times \mathrm{S}^S$, $[V_\alpha(i,k,s_1,\ldots,s_K;m)]_2$ is nondecreasing in i from condition 3, and as $V_\alpha(i,k,s_1,\ldots,s_K;m-1)$ is nondecreasing in i by the inductive assumption, and also is nondecreasing in k by Lemma 3, so is $R_\alpha(i,k,s_1,\ldots,s_K;m-1)$ by definition. Using Lemma 4 indicates that $\sum_{i'=0}^{I} \sum_{k'=1}^{K} p_{ii'} p_{k'}^{(i')} R_\alpha(i',k',s_1,\ldots,s_K;m-1)$ is nondecreasing in i $(0 \le i \le I)$. As $A(i)$ is nondecreasing in i $(0 \le i \le I)$ from conditions 3 and 4, so is $[V_\alpha(i,k,s_1,\ldots,s_K;m)]_1$, yielding that $V_\alpha(i,k,s_1,\ldots,s_K;m)$ is nondecreasing in i for $0 \le i \le I$, which completes the mathematical induction and the proof. □

Now using the above lemmas, we can prove the following main theorem.

Theorem 1. Assume conditions 1 through 8 hold. Then there exists a stationary control limit policy w.r.t. operating condition which minimizes the total expected α-discounted cost of the basic multi-repair-type maintenance model.

Proof: For $(i,k,s_1,\ldots,s_K) \in \mathrm{I} \times \mathrm{K} \times \mathrm{S}^S$ and for $n \ge 0$, let

$$f_{n+1}(i,k,s_1,\ldots,s_K)$$
$$= [V_\alpha(i,k,s_1,\ldots,s_K;n+1)]_1 - [V_\alpha(i,k,s_1,\ldots,s_K;n+1)]_2$$

$$= A(i) + \sum_{j=1}^{K} B(j)s_j + \sum_{i'=0}^{I} \sum_{k'=1}^{K} p_{ii'}p_{k'}^{(i')} R_\alpha(i',k',s_1,\ldots,s_K;n)$$

$$- (C(i,k) + B(k) + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;n)).$$

Now by Lemmas 3 and 5, $V_\alpha(i',k',s_1,\ldots,s_K;n)$ $(n \geq 0)$ is both nondecreasing in i' $(0 \leq i' \leq I)$ for fixed k', and nondecreasing in k' $(1 \leq k' \leq K)$ for fixed i'. So is $R_\alpha(i',k',s_1,\ldots,s_K;n)$ $(n \geq 0)$ by definition. Therefore, using Lemma 4, we have $\sum_{i'=0}^{I} \sum_{k'=1}^{K} p_{ii'}p_{k'}^{(i')} R_\alpha(i',k',s_1,\ldots,s_K;n)$ $(n \geq 0)$ is nondecreasing in i $(0 \leq i \leq I)$. The only other expression containing i is $A(i) - C(i,k)$, which is assumed to be nondecreasing in i $(0 \leq i \leq I)$, by condition 4. Hence for $n \geq 1$, $f_n(i,k,s_1,\ldots,s_K)$ is nondecreasing in i $(0 \leq i \leq I)$ for each fixed $(k,s) = (k,s_1,\ldots,s_K) \in \mathrm{K} \times \mathrm{S}^S$. That means, at the beginning of each n $(n \geq 1)$ period problem, for each $(k,s) \in \mathrm{K} \times \mathrm{S}^S$, there exists an $i_{k,s,n}$ such that $[V_\alpha(i,k,s_1,\ldots,s_K;n)]_2$ is smaller than or equal to $[V_\alpha(i,k,s_1,\ldots,s_K;n)]_1$, i.e., to repair an operating machine is optimal if and only if $i \geq i_{k,s,n}$. Thus, the existence of a control limit policy w.r.t. operating condition optimizing a finite horizon problem is guaranteed. Now using the fundamental results of Markov decision theory, the existence of a stationary control limit policy w.r.t. operating condition minimizing the infinite horizon problem can be easily obtained (see for example Wagner [11]). □

Interpretation of each condition is given now. Conditions 1, 2 and 6 characterize the fact that the types of repair works are arranged so that type 1 is the easiest and type K the hardest, since they say that the labor cost, the material cost, and the expected length of repair time, all increase as the type number of repair work increases. Condition 3 states that the worse the machine is, the more expensive its material cost is. Condition 4 requires that the operating cost must increase more than the increase in material cost for each type of repair. Condition 5 gives a simple lower bound on the penalty cost. Condition 7 is the so-called IFR property of a deteriorating system. Lastly, condition 8 indicates that the worse the state of the machine, the harder it is to repair (stochastically). All of them seem reasonable, and none of them is seriously restrictive.

Suppose all the conditions of Theorem 1 hold. Then for each $(k,s) \in \mathrm{K} \times \mathrm{S}^S$, there exists an $i_{k,s}$ such that for all (i,k,s) with $i < i_{k,s}$ to keep a machine in operation is optimal, and for all (i,k,s) with $i \geq i_{k,s}$ to repair it is optimal at any period. In addition, we can show that $i_{k,s}$ is nondecreasing in k $(1 \leq k \leq K)$ for each fixed $s \in \mathrm{S}^S$. Then one possible realization of an optimal policy for each fixed $s \in S^S$ is as shown in Fig. 2. This property is intuitively appealing since it means that the easier the required repair work is, the easier the repair action is taken. To see this, it is enough to check that $f_n(i,k,s_1,\ldots,s_K)$ is nonincreasing in k $(1 \le k \le K)$ for each $(i,s) \in I \times S^S$ and $n \ge 1$. Now if conditions 1, 5 and 6 hold, Lemma 2 holds and hence, $R_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;n)$ $(n \ge 1)$ is nondecreasing in k $(1 \le k \le K)$. Further both $B(k)$ and $C(i,k)$ are nondecreasing in k $(1 \le k \le K)$ by conditions 1 and 2. As all other terms do not contain k in their expressions, $f_n(i,k,s_1,\ldots,s_K)$ is nonincreasing in k $(1 \le k \le K)$.

The result of Theorem 1 can be easily extended to the long-run expected average cost version, and is stated as below.

Theorem 2. Suppose all the conditions in Theorem 1 hold, and furthermore, suppose that any operating machine eventually fails. Then there exists a stationary control limit policy w.r.t. operating condition minimizing the long-run expected average cost of the basic multi-repair-type maintenance model. Further, its control limit $i_{k,s}$ is nondecreasing in k $(1 \le k \le K)$ for each fixed s.

Proof: Since any operating machine eventually fails, and a failed machine must be repaired at once, and $q_k > 0$ for any k $(1 \le k \le K)$, $(i,k,s_1,\ldots,s_K) = (0,1,0,\ldots,0)$ is accessible from every other state no matter what stationary policy is employed. Hence, the existence of a stationary policy optimizing the problem is guaranteed (see Ross [9]). The application of the result for the discounted cost case to the long-run average cost can be seen in various papers (see Taylor [10], for example), and hence the rest of the proof can be omitted.□

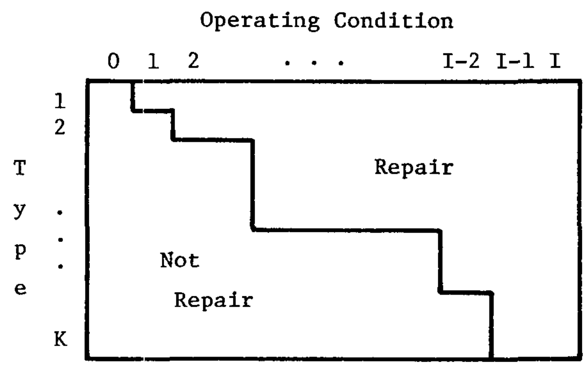

Figure 2. An Example of an Optimal Policy.

In the aforementioned model, we assumed that the type of repair work required was known to the decision maker at the time of his decision. Now we briefly discuss the case where the type of repair work required is not known to the decision maker. In this modified model, if an operating machine in the i-th operating condition is chosen to be repaired, it is randomly sent to the type k repair shop with probability $p_k^{(i)}$, and the decision maker has no knowledge of where it is to be sent at the time he makes a decision. A control limit policy for this model is defined as a policy where a machine is repaired if and only if its condition i exceeds some limit i_s for each s. Sufficient conditions under which an optimal policy is of a control limit form are to be studied. The main difference in the results is in the condition which relates the operating cost and other repair costs. The following lengthy inequality replaces condition 4.

$$A(i+1) - A(i) \geq \sum_{k=1}^{K} p_k^{(i+1)}(C(i+1,k) + B(k)) - \sum_{k=1}^{K} p_k^{(i)}(C(i,k) + B(k))$$
$$+ \sum_{k=1}^{K} \max(p_k^{(i+1)} - p_k^{(i)}, 0)(1/q_k - 1)(P + B(k)$$
$$- \min(A(0), \sum_{j=1}^{K} p_j^{(0)} C(0,j) + B(1))), \quad 0 \leq i \leq I-1.$$

The third term of the right hand side gives the effect of the penalty cost, and its value can be comparatively large. But if we can further assume that the repair time distribution is independent of the type of repair work, then the penalty term vanishes, and the property that

$$A(i) - (\sum_{k=1}^{K} p_k^{(i)}(C(i,k) + B(k)/q)) \text{ is nondecreasing in } i \ (0 \leq i \leq I)$$

can be shown to be sufficient. Other conditions are essentially the same as those in Theorem 1. See Hatoyama [3] for more detailed description on the above discussion.

3. Generalization

In this section we generalize the basic multi-repair-type maintenance model in the following manner. The operating cost of a machine depends on the type of repair work as well as its operating condition. Also, the type of repair work on a machine to be repaired depends on the type of repair work which has been required on the machine if the repair decision has been chosen one period earlier. Define

$A(i,k)$: operating cost for a machine in the i-th operating condition which requires type k repair work if it is repaired.

$p_{k'}^{(i,k)}$: probability that a machine in the i-th operating condition requires type k' repair if the repair decision is chosen given that the type k of repair has been required one period earlier.

Under this generalization, the following theorem holds:

Theorem 3. Assume conditions 1, 2, 3, 5, 6, and 7 hold. Furthermore assume

9. $A(i,k)$ is nondecreasing in k $(1 \le k \le K)$ for each fixed i $(0 \le i \le I)$.

10. $A(i,k) - C(i,k)$ is nondecreasing in i $(0 \le i \le I)$ for each fixed k $(1 \le k \le K)$.

11. $P^{(i',k)}(\cdot) \subset P^{(i',k+1)}(\cdot)$ for $1 \le k \le K-1$, $0 \le i' \le I$,

    where $P^{(i',k)}(k') = \sum_{j \le k'} p_j^{(i',k)}$

12. $P^{(i',k)}(\cdot) \subset P^{(i'+1,k)}(\cdot)$ for $0 \le i' \le I-1$, $1 \le k \le K$.

Then there exists a stationary control limit policy w.r.t. operating condition which minimizes the total expected α-discounted cost of this generalized multi-repair-type maintenance model. For the long-run expected average cost criterion, the eventual failure property on any machine is also needed.

Proof: The dynamic programming formulation of this generalized model is exactly the same as (2.1) if $A(i,k)$ and $p_{k'}^{(i',k)}$ replace $A(i)$ and $p_{k'}^{(i')}$ respectively. Lemma 2 still holds after these replacements, and Lemma 3 also holds if conditions 9 and 11 are added since thsy assure that $[V_\alpha(i,k,s_1,\ldots,s_K;n)]_1$ is nondecreasing in k in the proof. It is clear that Lemma 4 holds if condition 8 is replaced by condition 12. As in the proof of Lemma 5, we can show that $V_\alpha(i,k,s_1,\ldots,s_K;n)$ is nondecreasing in i $(0 \le i \le I)$ for each fixed $(k,s) \in \mathrm{K} \times \mathrm{S}^S$, and for $n \ge 1$, if conditions 1 through 3, 5, 6 and 9 through 12 hold, and if $A(i,k)$ is nondecreasing in i $(0 \le i \le I)$ for each k $(1 \le k \le K)$. Finally it can be shown that $f_n(i,k,s_1,\ldots,s_K)$ is nondecreasing in i for $(k,s) \in \mathrm{K} \times \mathrm{S}^S$, and for $n \ge 1$ if all the conditions hold, which yields the desired results. □

Condition 11 says that the heavier the current repair work required on a machine is, the heavier that its repair work will be in the future. Condition 12 means that the worse condition a machine is in, the more repair work required on it if it is repaired.

4. Case Where There is a Single Repairman in Each Facility

We return to the basic multi-repair-type maintenance model, where the repair work begins immediately on any machine sent to any type of repair shop. This is equivalent to saying that each repair shop has more than enough number of repairmen. In this section, we consider the other extreme case, where there is only one repairman in each type of repair facility. If a machine is to be repaired at some type of repair facility, it must wait until all the machines which have arrived earlier at the same type of repair facility are completely repaired. Machines waiting for repair form a queue in front of each type of repair facility. The system is schematically shown in Fig. 3.

Let q_k $(1 \le k \le K)$ be the probability that a machine in type k repair shop at the beginning of a period is completely repaired at the end of the period. Let $q^{(j)}_{ss'}$ be the probability that s' machines are still in the type j repair system at the end of the period given that s machines are in the type j repair system at the beginning of a period. Here the type k repair system includes the type k repair shop and the queue formed in front of the type k repair shop. Then,

$$q^{(j)}_{s_j s'_j} = \begin{cases} 1-q_j & \text{if } s'_j = s_j \text{ and } 1 \le s_j \le S+1 \\ q_j & \text{if } s'_j = s_j-1 \text{ and } 1 \le s_j \le S+1 \\ 1 & \text{if } s'_j = s_j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The dynamic programming formulation of this model is the same as (2.1) where $q^{(j)}_{s_j s'_j}$'s are not binomial any more. As we can see from the proofs of

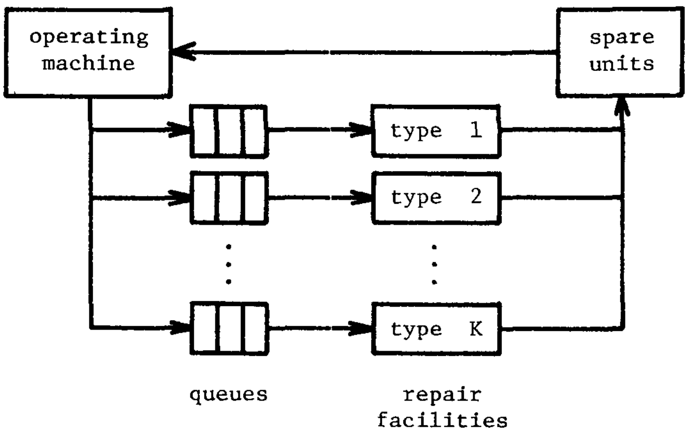

Figure 3. A Multi-Repair-Type System with One Repairman for Each Repair Facility.

Lemma 2 through Theorem 1, $q^{(j)}_{s_j,s'_j}$'s explicitly appear only in the proof of Lemma 2. Hence, if Lemma 2 can be proved under this modified model (without using Lemma 1), we can obtain the sufficient conditions for the optimality of a stationary control limit policy w.r.t. operating condition by using the result of Theorem 1. Now the definition of R_α and the simple substitution of $q^{(j)}_{s_j,s'_j}$ give the proof of Lemma 2. As a conclusion of this section, the following statement is given.

Theorem 4. If all the conditions in Theorem 1 hold, there exists a stationary control limit policy w.r.t. operating condition which minimizes the total expected α-discounted cost of the multi-repair-type maintenance model with one repairman for each repair shop. For the long-run expected average cost criterion, the eventual failure property on any machine is also needed.

5. Computational Remarks and Conclusion

As for the computational algorithms, the usual techniques such as policy improvement procedure and LP approach are applicable to compute an optimal policy for each model treated here. However, the procedure can be more efficient if "good" policies are searched iteratively among stationary control limit policies whenever possible before switching to a usual policy improvement among stationary policies. This procedure is described in [2] in detail, and is omitted here.

Models treated in this paper are the extended ones of Derman's classical model in the following sense: They are repair models rather than replacement models. They have a finite number of repair shops. They have some spare machines. Under the mild conditions, simple optimal maintenance policies called control limit policies w.r.t. operating condition are obtained.

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