

OPTIMAL INTERVAL FOR STOCHASTIC CLEARING SYSTEMS

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Abstract Stochastic clearing systems can be applied to bulk-service queue and demand-responsive public-service systems. Customers (input) arrive at a service facility and form a queue waiting to be served. When a server arrives at a service facility, waiting customers are served all together. The problem is to obtain an optimal clearing random interval T which minimizes the long-run average cost. In this paper we derive an optimal clearing random interval T and sufficient conditions under where T is characterized by optimal clearing level q , at which clearing occurs, whenever the cumulative input exceeds a critical level.

1. Introduction

We will consider stochastic clearing systems introduced by Stidham. The cumulative input to such a system is described by a nondecreasing stochastic process $\{Y(t), t \geq 0\}$; output occurs intermittently in the form of clearing operations, which instantaneously remove all the quantity in the system. The fixed number c is a clearing cost, incurred whenever the system is cleared and $g(x)$ is a variable cost per unit time, incurred when the net quantity in the system is x . The problem is to obtain an optimal clearing interval T which minimizes the long-run average cost.

Stidham [7] restricts to demand responsive systems, where clearing is triggered by the quantity of the interest exceeding a certain level. He obtains an optimal clearing level \hat{q} among the class where clearing occurs whenever the cumulative input since the last clearing instant exceeds a critical level q . The sojourn (potential) measure associated with input process is easily estimated in practice and \hat{q} can be obtained only from the sojourn measure. And clearing instance is determined by observing the local information whether the cumulative since the last clearing instance exceeds the optimal clearing level \hat{q} or not. In this paper we will show that this simple practical method is also optimal among a wider class under some

conditions.

In Section 2 we consider fundamental assumptions and theorems. Almost all mathematical inventory models have been treated in deterministic cases or stochastic cases under strong independence assumptions, for example, demand random variables to be independent. As a weak probabilistic assumption, we assume the clearing instances are regenerative points for the cumulative input. That is, the evolution of the cumulative input after a clearing instance is an independent probabilistic replica of the evolution the process beginning at time 0.

In section 3 we define three classes of clearing rules ($\tau_1 \subseteq \tau_2 \subseteq \tau_3$). For $T \in \tau_1$, T has a constant clearing level q . Let \hat{q} be the Stidham's optimal clearing level then $T(\hat{q})$ is optimal among τ_1 . And τ_2 is the class of stopping times and τ_3 is the class of nonnegative random variables. An optimal random variables \hat{T} among τ_3 will be given in Theorem 3.3. Under the natural assumption that $g(x)$ is monotone nondecreasing \hat{T} will be attained by $T(\hat{q})$ in Corollary 3.4. In this case $T(\hat{q})$ becomes an optimal stopping time.

If the cumulative input is a pure birth process, for any $g(x)$ satisfying Assumption 2 $T(\hat{q})$ is optimal among τ_2 . Especially for the input process with gamma-distributed stationary independent increments we can use the results to a generalized stochastic clearing system. And clearing parameters \hat{r} and \hat{q} in [7] offer optimal. Similar results are obtained in an optimal impulse control of diffusion processes described by Ito's formula (Bensoussan and Lions [1], Richard [4] and also Harrison and Taylor [3]). The continuation region of the optimal control is defined by levels of the diffusion process. The impulse control or clearing occurs whenever the process reaches levels given in advance.

2. Stochastic Clearing Systems

In this section we state fundamental assumptions and theorems for stochastic clearing systems according to [7]. The input to a stochastic clearing system is described by a nondecreasing, right-continuous stochastic process $\{Y(t), t \geq 0\}$; with $Y(0) = 0$, where $Y(t)$ is the cumulative input in $[0, t]$. Output occurs by intermittent clearing operation, in which the cumulative input since the last clearing instant is instantaneously removed.

Let T_1 be the first clearing instant and $T_n (n \geq 1)$ be the time between $(n - 1)$ st and n th clearings. Let $S_0 = 0, S_n = T_1 + \dots + T_n, n \geq 1$ and

$R(t) = \max \{n | S_n \leq t\}$ be the number of clearings in $[0, t]$. Then the net quantity in the system is denoted by $V(t) = Y(t) - Y(S_{R(t)})$. Since $S_{R(t)}$ is the last clearing instant until t , $V(t) = Y(t) - Y(S_{R(t)})$ is the cumulative input at time t from the last clearing instant.

Suppose that there is a clearing cost $c > 0$ incurred, independent of $Y(S_n) - Y(S_{n-1})$, whenever a clearing takes place. In addition, there is a cost per unit time, $g(x) \geq 0$, incurred when the net quantity in the system is x . Then the total cost incurred in $[0, t]$ is $cR(t) + \int_0^t g(V(s))ds$.

We must impose probabilistic conditions on a stochastic process $V(t)$ to obtain simple expressions for the long-run average cost. A natural requirement is that the clearing instants be regenerative points for $\{V(t), t \geq 0\}$. That is, the continuation of $V(t)$ beyond a clearing instant is a probabilistic replica of the whole process starting at 0 and then (T_1, T_2, \dots) is a renewal sequence.

Assumption 1. $\{V(t), t \geq 0\}$ is a regenerative process with respect to the renewal sequence (T_1, T_2, \dots) .

Next we impose a realistic condition on $g(x)$ in the following way.

Assumption 2. g is continuous and $-g$ is unimodal with mode $x_0 \in (-\infty, \infty)$.

For stochastic clearing systems with the regenerative property we have

Theorem 2.1 Provided the integrals are well defined and $ET < \infty$,

$$(2.1) \quad \lim_{t \rightarrow \infty} (cR(t) + \int_0^t g(V(s))ds)/t \\ \text{a.s.} \\ = (c + E \int_0^T g(V(s))ds)/ET.$$

The proof is given in Stidham [6] and Ross [5]. From Assumption 2 g is non-increasing on $(-\infty, X_0]$ and non-decreasing on $[x_0, \infty)$, so that these integrals are well defined. Let $C(T)$ denote the a.s. long-run average cost when we employ a random variable T as a clearing interval.

$$(2.2) \quad C(T) = (c + E \int_0^T g(V(s))ds)/ET$$

Our problem is to obtain an optimal interval T , which minimizes $C(T)$.

Denote $T(x)$ as the first entrance time of $V(t)$ into the set $[x, \infty)$ ($T(x) = \inf \{t; V(t) \geq x\}$). Then $ET(q) = W(q)$, where $W(q)$ is the sojourn measure of the set $[0, q]$ (see [6] and [7]). If $T = T(q)$ then clearing occurs whenever the net quantity exceeds the level q . Simple expressions for $C(T(q))$ are obtained in [6] and [7] as follows:

Theorem 2.2 Under the conditions of Theorem 2.1

$$(2.3) \quad \begin{aligned} C(T(q)) &= (c + \int_0^q g(x) dW(x)) / W(q) \\ &\text{a.s.} \\ &= g(q) + (c - \int_0^q W(x) dg(x)) / W(q). \end{aligned}$$

For application purposes it is feasible that $g(x)$ is a monotone non-decreasing in $x \geq 0$, when $V(t)$ is a net quantity in the system. In this case it will be proved that an optimal random variable \hat{T} among τ_3 is attained by $T(\hat{q})$ where \hat{q} is a Stidham's optimal clearing level. $-g(x)$, however, is unimodal in generalized stochastic clearing systems allowing the starting quantity in the system, m , to be different from 0. $V(t)$ can be interpreted as the cumulative input since the last clearing instant at or before t . The net quantity in the system at time t , however, is $\tilde{V}(t) = m + V(t) \quad t \geq 0$.

3. Optimal Clearing Interval

Our problem is to choose an optimal renewal sequence (T_1, T_2, \dots) , where $\{V(t), t \geq 0\}$ is a regenerative process with respect to (T_1, T_2, \dots) . Random variables $T_i (i=1, 2, \dots)$ are independent identically distributed then the problem is to obtain an optimal random variable $T (ET < \infty)$. We define three classes of decision rules. The first is $\tau_1 = \{T(q); ET(q) < \infty, q \geq 0\}$, where clearing occurs whenever the net quantity exceeds a level q . The second is $\tau_2 = \{T; \text{stopping time}, ET < \infty\}$. Let β_t be the increasing family of σ -algebras generated by $V(t)$. A random variable T is called a stopping time if for any $t \geq 0$ the event $\{T > t\}$ belongs to β_t . In other words, in order to determine whether or not the time T has occurred up to t it is sufficient to know the events in β_t . The third is $\tau_3 = \{T; \text{nonnegative random variable}, ET < \infty\}$. Let \hat{T} be an optimal random variable among τ_3 , which minimizes (2.2). From trivial relations

$$(3.1) \quad \tau_1 \subseteq \tau_2 \subseteq \tau_3$$

we have

$$(3.2) \quad C_1 \geq C_2 \geq C_3$$

where $C_i (i=1, 2, 3)$ are optimal a.s. long-run average costs among τ_i respectively.

\hat{T} will be given in Theorem 3.3. Since the decision of a clearing operation at time t depends in general on the net quantity just prior to time t , it is necessary to choose an optimal stopping time. But it seems complicate in general cases. From (3.2) C_3 is a lower bound for C_2 and C_1 is an upper bound. If $C_1 = C_3$ then $T(\hat{q})$ with an optimal clearing level \hat{q} is an optimal stopping time. \hat{q} is obtained in [7] in the following form: Since $g(x)$ is continuous in x , $\int_0^q W(x) dg(x)$ is continuous in q . Either a solution to

$$(3.3) \quad \int_0^{\hat{q}} W(x) dg(x) = c$$

exists ($x_0 \leq \hat{q}$) or $\int_0^q W(x) dg(x) < c$ for all $q \geq 0$, in which case we set $\hat{q} = \infty$. In either case, then

Theorem 3.1 $\hat{q}(\geq x_0)$ minimizes $C(T(q))$ among τ_1 .

If $\hat{q} < \infty$ then $C_1 = g(\hat{q})$.

First we treat a simple case. Suppose that $T \in \tau_3$ and there exists a clearing level q such that $ET = ET(q)$. Provided $g(x)$ be monotone non-decreasing in $x \geq 0$ then

$$(3.4) \quad \begin{aligned} E \int_0^T g(V(s)) ds - E \int_0^{T(q)} g(V(s)) ds \\ = E \int_{T(q)}^T g(V(s)) ds \geq g(q) E(T - T(q)) = 0 \end{aligned}$$

The above inequality comes from the monotonicity of $g(x)$ because $g(V(s)) \geq g(q)$ ($T(q) \leq s \leq T$) and $g(V(s)) \leq g(q)$ ($T \leq s \leq T(q)$). From (2.2) and (3.4) we have $C(T) \geq C(T(q))$. From this discussion we can intuitively conclude that $C_1 = C_2 = C_3$ when $g(x)$ is monotone nondecreasing.

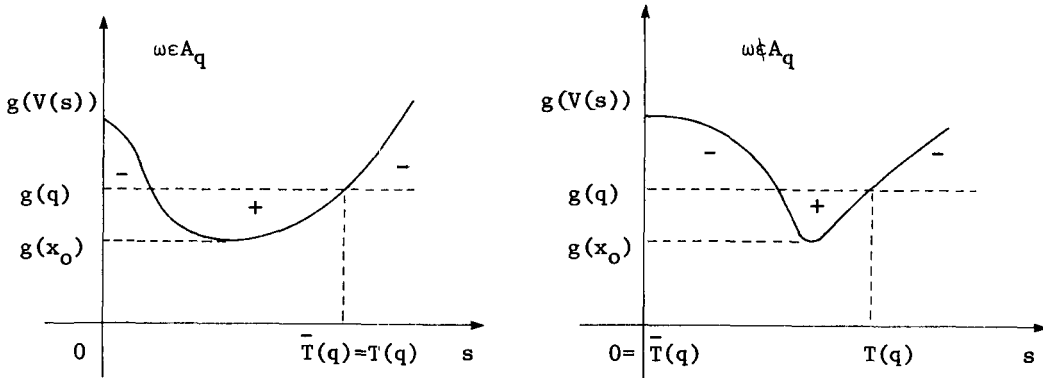
In Theorem 2.2 and 3.1, $\int_0^q W(x) dg(x)$ plays an important rule.

If $W(g) = ET(q) < \infty$ we can rewrite it the following form.

$$(3.5) \quad \begin{aligned} \int_0^q W(x) dg(x) &= W(q) g(q) - \int_0^q g(x) dW(x) \\ &= \int_0^q g(q) - g(x) dW(x) = E \int_0^q g(q) - g(x) dT(x) \\ &= E \int_0^{T(q)} g(q) - g(V(s)) ds. \end{aligned}$$

Let $x^* = \max(x_0, 0)$. For $q \geq x^*$, define a set

$$(3.6) \quad \begin{aligned} A_q &= \{\omega; \int_0^{T(q)} g(q) - g(V(s)) ds \geq 0\} \\ &= \{\omega; \int_0^q T(x) dg(x) \geq 0\}. \end{aligned}$$



Since $g(x)$ is monotone nondecreasing in $x \geq x^*$, A_q is monotone nondecreasing. Put

$$(3.7) \quad \bar{T}(q) = \begin{cases} T(q) & \omega \in A_q, \\ 0 & \omega \notin A_q \end{cases}$$

then $\bar{T}(q)$ is an optimal random variable, which maximizes $\int_0^T g(q) - g(V(s)) ds$.

Lemma 3.2 If $\hat{q} < \infty$ in (3.3) and $ET(\hat{q}) < \infty$, then there exists $\bar{T}(q^*) \in \tau_3$ such that $x^* \leq q^* \leq \hat{q}$ and

$$c = E \int_0^{\bar{T}(q^*)} (g(q^*) - g(V(s))) ds.$$

Proof. From the definition of $\bar{T}(q)$

$$D(q) \triangleq E \int_0^{\bar{T}(q)} (g(q) - g(V(s))) ds = \int_{A_q} \int_0^q T(x) dg(x) dP$$

is monotone nondecreasing in q . For $x^* \leq q \leq q' \leq \hat{q}$, $A_q \subseteq A_{q'}$, and we have

$$(3.8) \quad 0 \leq D(q') - D(q) \\ = \int_{A_{q'} - A_q} \int_0^q T(x) dg(x) dP + \int_{A_{q'}} \int_q^{q'} T(x) dg(x) dP$$

For $\omega \in A_{q'} - A_q$, $\int_0^q T(x) dg(x) \leq 0$ and the first term of the right side is nonpositive. On the other hand $\int_q^{q'} \int_{A_{q'}} T(x) dP dg(x) \leq \int_q^{q'} W(x) dg(x) \leq W(\hat{q})(g(q') - g(q))$ because $T(x) \geq 0$ and $g(x)$ is monotone nondecreasing

in $x \geq x^*$. We have $0 \leq D(q') - D(q) \leq W(\hat{q})(g(q') - g(q))$ and then $D(q)$ is continuous in $x^* \leq q \leq \hat{q}$. Since $D(x^*) = 0$ and

$$\begin{aligned} c &= \int_0^{\hat{q}} W(x)dg(x) = E \int_0^{T(\hat{q})} g(\hat{q}) - g(V(s)) ds \\ &\leq E \int_0^{\bar{T}(\hat{q})} g(\hat{q}) - g(V(s)) ds = D(\hat{q}), \end{aligned}$$

then there exists q^* such that $x^* \leq q^* \leq \hat{q}$ and $D(q^*) = c$. Moreover $E\bar{T}(q^*) \leq ET(q^*) \leq ET(\hat{q}) < \infty$. The proof is completed.

To simplify the notation we put $\hat{T} = \bar{T}(q^*)$ which will become an optimal random variables in the next theorem.

Theorem 3.3 If $\hat{q} < \infty$ and $ET(\hat{q}) < \infty$ then for any $T \in \tau_3$, $C(T) \geq C(\hat{T})$. \hat{T} is an optimal random variable among τ_3 and $C_3 = g(q^*)$.

Proof. Since

$$\begin{aligned} (3.9) \quad C(\hat{T}) &= (c + E \int_0^{\hat{T}} g(V(s)) ds) / E\hat{T} \\ &= g(q^*) + (c - E \int_0^{\hat{T}} g(q^*) - g(V(s)) ds) / E\hat{T} = g(q^*) \end{aligned}$$

then from Lemma 3.2 we have

$$\begin{aligned} (3.10) \quad C(T) - C(\hat{T}) &= (E\hat{T} - ET) / E\hat{T} \cdot ET \\ &\times (c - E \int_0^{\hat{T}} g(q^*) - g(V(s)) ds) \\ &+ \{c + E \int_0^T g(V(s)) ds - g(q^*) ET - c \\ &+ E \int_0^{\hat{T}} g(q^*) - g(V(s)) ds\} / ET \\ &= E \int_0^{\hat{T}} g(q^*) - g(V(s)) ds / ET \geq 0. \end{aligned}$$

The above inequality comes from the fact that \hat{T} an optimal random variable, which maximizes $\int_0^{\hat{T}} g(q^*) - g(V(s)) ds$. Since $C(\hat{T}) = g(q^*)$, the proof is completed.

Suppose $g(\hat{q}) \geq g(0)$. Then $g(\hat{q}) \geq g(V(s))$ ($0 \leq s \leq T(\hat{q})$) and from (3.7) $\hat{T} = T(\hat{q})$. Especially if $g(x)$ is monotone nondecreasing in $x \geq 0$, $T(\hat{q})$ is an optimal clearing interval among τ_3 . We summarize that

Corollary 3.4 Under the conditions in Theorem 3.3 if $g(\hat{q}) \geq g(0)$, then $T(\hat{q})$ is optimal among τ_3 . Especially if $g(x)$ is continuous and monotone nondecreasing in $x \geq 0$ then $T(\hat{q})$ is optimal among τ_3 .

Next we derive a sufficient condition of input process under where $T(\hat{q})$ is an optimal stopping time for any $g(x)$ satisfying Assumption 2. If $V(t)$ be a (pure) birth process (see Feller [2] p.402), then $V(t)$ has exponential

holding times from the Markovian property.

Theorem 3.5 Under the conditions of Theorem 3.3, if $V(t)$ be a pure birth process, then for any $T \in \tau_2$ $C(T) \geq C(T(\hat{q}))$. That is, $T(\hat{q})$ is an optimal stopping time.

Proof. From (3.3) and (3.5)

$$c = \int_0^{\hat{q}} W(x) dg(x) = E \int_0^{T(\hat{q})} g(\hat{q}) - g(V(s)) ds.$$

Using the same discussion of (3.9) and (3.10)

$$(3.11) \quad C(T) - C(T(\hat{q})) = E \int_T^{T(\hat{q})} g(\hat{q}) - g(V(s)) ds$$

For $T(\hat{q}) \leq T$ $g(V(s)) \geq g(\hat{q})$ ($T(\hat{q}) \leq s \leq T$), then we have the conditional expectation

$$(3.12) \quad E \left(\int_T^{T(\hat{q})} g(\hat{q}) - g(V(s)) ds \mid V(s) (0 \leq s \leq T(\hat{q})), T(\hat{q}) \leq T \right) \geq 0$$

Since $V(s)$ is a pure birth process, it is endowed with complete lack of memory. Then

$$(3.13) \quad \begin{aligned} & E \left(\int_T^{T(\hat{q})} g(\hat{q}) - g(V(s)) ds \mid V(s) (0 \leq s \leq T), T \leq T(\hat{q}) \right) \\ &= E \left(\int_T^{T(\hat{q})} g(\hat{q}) - g(V(s)) ds \mid V(T), T \leq T(\hat{q}) \right) \\ &= E \int_{V(T)}^{\hat{q}} W(x) dg(x) = \int_{V(T)}^{\hat{q}} W(x) dg(x) \geq 0. \end{aligned}$$

The last inequality comes from the fact that $\int_0^q W(x) dg(x) \leq \int_0^{\hat{q}} W(x) dg(x)$ for any $q \in [0, \hat{q}]$. Since $T \in \tau_2$, then from (3.11), (3.12) and (3.13) we have

$$\begin{aligned} & C(T) - C(T(\hat{q})) \\ &= E E \left(\int_T^{T(\hat{q})} g(\hat{q}) - g(V(s)) ds \mid V(s) (0 \leq s \leq \min(T(\hat{q}), T)) \right) \\ &\geq 0. \end{aligned}$$

The proof is completed.

In [7] generalized clearing systems are also studied, where the starting quantity m in the system is allowed to be different from 0. The net quantity in the system at time t , however, is $\tilde{V}(t) = m + V(t)$. Generalized clearing systems include as special cases most of the stationary, single product (s, S) models in the inventory literature. For a piecewise linear cost $g(x)$

optimal clearing parameters \hat{q} and \hat{r} are obtained. When $\{Y(t), t \geq 0\}$ has gamma-distributed stationary increments, parameters \hat{q} and \hat{r} are also illustrated as an example. $Y(t)$ is obtained by the limiting procedure of compound Poisson processes $\{Y_n(t)\}$ discussed in [6]. In this case we can apply Theorem 3.5 to the generalized clearing system from Theorem 3.15 in [7] as follows:

Let $g(x)$ be a piecewise linear-cost function, that is, $g(x) = -ax$, $x < 0$ and $g(x) = bx$, $x \geq 0$, where a , b are positive constants. Let $C(T, m)$ be the a.s. long-run average cost where m is a starting quantity in the system and T is an clearing interval. Our objective is to find an optimal pair (\tilde{T}, \hat{m}) that minimizes $C(T, m)$ among $T \in \tau_2$ and $m \in (-\infty, \infty)$. An optimal pair (\tilde{T}, \hat{m}) is obtained by $\tilde{T} = T(\hat{q})$ and $\hat{m} = x_0 - \hat{r}$, where optimal clearing parameters \hat{q} and \hat{r} are in [7]. The Stidham's result is also optimal stopping time for the gamma-increment process $\{Y(t), t \geq 0\}$ in the generalized stochastic clearing system.

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