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A GRAPHICAL DECOMPOSITION OF THE STOCHASTIC NETWORK

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Abstract The pivotal decomposition theorem of the reliability function is applied to the stochastic network. A graphical observation of the theorem always yields more effective result than that of algebraic aspects, that is, the well-selected pivot arc enables the resulting network to contain modules. Our theorem 1 assures that, and our algorithms will be helpful to determine the optimal pivot of the decomposition.

1. Preliminaries

A network is said to be stochastic when its arcs either function or fail with known probabilities. Hereafter, the term network will refer to a stochastic network satisfying following assumptions:

(A1) The network is an undirected network having the two specified nodes called terminals.

(A2) Only arcs are subject to failure; nodes may not fail.

(A3) All arcs are relevant, that is, there is no arc whose deletion from the network will not affect its reliability.

(A4) The network can undergo no further modular decomposition, i.e. the network contains no modules having two or more arcs.

(A5) The state of each arc is statistically independent.

(A6) A graph obtained from the network by adding an artificial arc between the terminals is planar.

We deal with only networks as described above.
The reliability of a network is the probability that there exists a path from one terminal to another. We denote the arcs by 1,2, ...,n, and its reliability (functioning probability) by p_i.

The reliability function $h(p) = h(p_1,\dots,p_n)$ is a multilinear polynomial of p_i's. The following lemma holds for h(p). (for the proof, see e.g. [1], p.21)

Lemma 1.

$$h(p) = p_i h(1_i,p) + (1-p_i)h(0_i,p) \ , \ i = 1,\dots,n \ ,$$

where

$$(1_i,p) = (p_1,\dots,p_{i-1},1,p_{i+1},\dots,p_n) \ ,$$
$$(0_i,p) = (p_1,\dots,p_{i-1},0,p_{i+1},\dots,p_n) \ .$$

A module is a subnetwork that acts as if it were just a component. If a network contains modules, we can replace each of them into single arc with the same reliabilities as that module has. These procedure are called "modular decomposition". The modular decomposition reduces the effort of calculation remarkably.

2. Graphical Representation of $h(0_i,p)$ and $h(1_i,p)$

The following two propositions are obvious but important.

Proposition 1. The network N_0^i representing $h(0_i,p)$ is isomorphic to the network obtained by deleting an arc i from the original network N.

Proposition 2. The network N_1^i representing $h(1_i,p)$ is isomorphic to the network obtained by fusing both end nodes of an arc i of the origninal network N.

By these two propositions and lemma 1, the reliability function of the given network N is calculated from those of N_0^i and N_1^i. According to the assumptions, N itself has no modules, however, the network N_0^i or N_1^i, created from N, may be decomposable. In fact, we can prove the following:

Theorem 1. For any network N, there exists a pivot arc which produces the decomposable network N_0^i or N_1^i.

Proof. Adding an arc between two terminals of the given network N, a planar graph G is constructed. By the proper transformation, G can be embedded in a plane such that every arc is drawn as a straight line segment. Therefore we can assume that G is such a graph as mentioned above.

Case 1. When G has at least one triangular region.

In this case, by fusing any two (except that those two are both terminals) of three nodes surrounding the triangular region, we can get the parallel arcs.

Hence N_1^i is decomposable.

Case 2. When G has no triangular regions.

Let e,v, and f denote the number of arcs, nodes, and regions in G, respectively. Then Euler's formula holds in e,v, and f;

(1) $f = e - v + 2$.

Since G has no triangular regions, no region in G can be bounded with fewer than four edges. Hence we would have

(2) $2e \geqq 4f$,

and, substituting for f from Euler's formula,

(3) $e \leqq 2v - 4$.

Let $d(n_i)$ denotes the degree of node n_i of the graph G, then

(4) $\sum_{i=1}^{v} d(n_i) = 2e$,

from (3) and (4),

(5) $v^{-1}\sum_{i=1}^{v} d(n_i) < 4$.

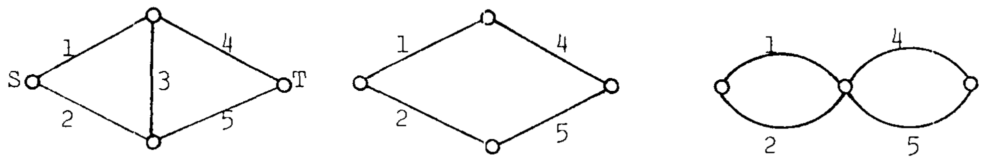
The assumptions (A3) and (A4) permit neither series arcs nor pendant vertices, hence the degree of all nodes in G cannot be less than three;

(6) $d(n_i) \geqq 3$, for all i .

The inequalities (5) and (6) show that there exists at least one node with degree three, and if we remove an arc i from this node, a module of series type is generated. (Two adjacent arcs are said to be in series if their common node is of degree two.) Consequently N_0^i is decomposable.

The following two examples illustrate our graphical decomposition procedure.

Example 1. The bridge structure.



N N_0^3 N_1^3

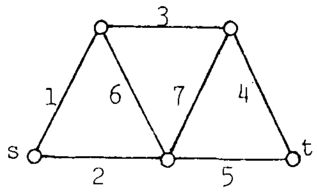
Applying lemma 1 to arc 3, the decomposable network N_0^3 and N_1^3 are obtained. The reliability functions of N_0^3 and N_1^3 are immediate;

$$h(0_3,p) = p_1p_4 + p_2p_5 - p_1p_2p_4p_5 ,$$
$$h(1_3,p) = (p_1 + p_2 - p_1p_2)(p_4 + p_5 - p_4p_5) .$$

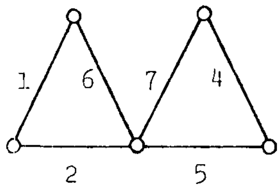
Hence the reliability function h(p) of N is

$$h(p) = (1 - p_3)h(0_3,p) + p_3h(1_3,p) .$$

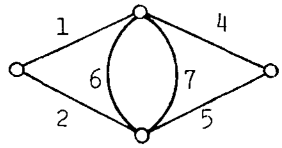
Example 2. The triangular frame.



N



N_0^3



N_1^3

Applying lemma 1 to arc 3, the decomposable networks N_0^3 and N_1^3 are obtained. N_1^3 can be reduced to the bridge structure of example 1, and the reliability functions of N_0^3 and N_1^3 are immediate;

$$h(0_3,p) = (p_1p_6 + p_2 - p_1p_2p_6)(p_4p_7 + p_5 - p_4p_5p_7) ,$$

$$h(1_3,p) = (1 - p_6 - p_7 + p_6p_7)(p_1p_4 + p_2p_5 - p_1p_2p_4p_5) + (p_6 + p_7 - p_6p_7)(p_1 + p_2 - p_1p_2)(p_4 + p_5 - p_4p_5) .$$

Hence the reliability function h(p) of N is

$$h(p) = (1 - p_3)h(0_3,p) + p_3h(1_3,p) .$$

As was shown in above two examples, our graphical method is much efficient than other known methods such as path enumeration, state enumeration, and so forth. This is mainly due to the possibility of decomposition of N_0 and N_1. Applying modular decomposition to the network reduces the computational effort especially.

3. Optimality of the Pivot Arc

In applying our graphical decomposition to a given network, it is important that by which arc as a pivot the procedure is to be done. In this section we give a criterion for selecting a pivot arc of the decomposition. The term optimal means here that the number of states in the network are minimized. To this end, a concept of complexity of a network is proposed.

The complexity c(N) of the network N is considered to be a sort of measures that indicates the amount of the computational effort required to obtain its reliability function. And, the optimal pivot is defined as an arc i which gives the minimal sum of complexities for N_0^i and N_1^i .

Birnbaum and Esary [2] formalized the concepts of module. In a wide sense, a single arc itself is a module however, we exclude this trivial case in the following discussions. We start from the definition of a level of the modules.

Let $M_1,\dots,M_p$ denote the modules in a given network N. Then the level $l(M_i)$

of the module M_i is defined recursively as follows;

(i) The whole network is a module with level zero.

(ii) If $M_i \supset M_j$ and there is no module M_k such that $M_i \supset M_k \supset M_j$, then the level of M_j is one greater than that of M_i;

$$l(M_j) = l(M_i) + 1 .$$

(iii) The two modules M_i and M_j have the same level if no inclusion relation holds between those two.

We say that M_i is higher level than M_j if the level of M_i is less than that of M_j. In the next, the complexity $c(M)$ of a module M is defined as follows:

(i) If the module M with n arcs contains no submodules, then the complexity of M is 2^n.

(ii) Let $S_1,\ldots,S_k$ denote the submodules with level $i + 1$ which are contained in the module M of level i. Then the complexity $c(M)$ of M is given by

$$c(M) = c(M') + c(S_1) + \cdots + c(S_k) ,$$

where, M' is a module obtained from M, by replacing each of M's submodules $S_1,\ldots,S_k$, respectively, by a single arc.

Now, the optimality of arc i^* can be represented as follows:

$$c(N_0^{i*}) + c(N_1^{i*}) \leqq c(N_0^{i}) + c(N_1^{i}) , \quad \text{for } i = 1,\ldots,n .$$

4. Generation of Modules

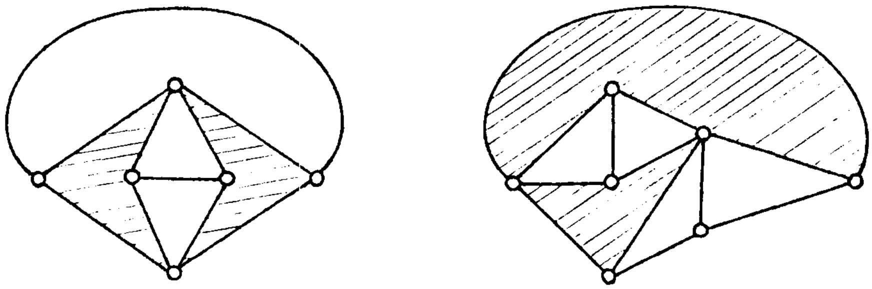
We present some results helpful to determine an optimal pivot for the graphical decomposition. As in the previous sections, we use the symbol G for the graph obtained from the network N by adding an artificial arc between two terminals of N. The following propositions 3 and 4 are from the proof of theorem 1.

Proposition 3. If there is a node with degree three in the graph G, then by deleting an arc incident to that node we can get the series-type module in N_0.

Proposition 4. If G has a triangular region then by fusing any two of the three nodes constructing that triangle we can get the parallel-type module in N_1.

In most cases we can find an optimal pivot by above propositions however, the more systematic method may be needed when the given network is complicated. We start from the discussion about some properties of the module. A knowledge of the concepts of regions is assumed. A graph theory text such as Deo [3] is recommended to those unfamiliar with these concepts.

If the network N contains a module M then M has exactly two nodes in common with M^c, the complement of M in G. In such a case M is surrounded by two regions of G as illustrated in Fig. 1. We use the term that *the regions segregate a module* for such situations.



(a) (b)

Fig. 1. The segregation of module by two regions

The reformations of a module into two regions suggest to us a procedure to choose desirable pivots for our decomposition: If a subgraph S of N is segregated by three regions of G, and if these three satisfy some conditions then we can choose an arc whose deletion or shortage makes S a module in N. These considerations are described in two algorithms.

We define two relations α and β among the regions in the network.

(D1) For the two regions R_1 and R_2, we use the notation $R_1 \alpha R_2$ if R_1 and R_2 are adjacent each other.

(D2) For the two non-adjacent regions R_1 and R_2, we use the notation $R_1 \beta R_2$ if the boundaries of R_1 and R_2 have at least one node in common.

Algorithm 1. (for the deletion of arcs)

Step 1: Construct a planar graph G from the given network N by adding an artificial arc between two terminals of N.

Step 2: Let $R_1,\ldots,R_f$ denote the regions (including the outer region) of G. From f regions of G, select all the ordered triplets of regions (R_i,R_j,R_k) which satisfy one of the following conditons;

(C1) $R_i \alpha R_j$, $R_j \alpha R_k$, $R_k \alpha R_i$.

(C2) $R_i \alpha R_j$, $R_j \alpha R_k$, $R_k \beta R_i$.

(C3) $R_i \alpha R_j$, $R_j \beta R_k$, $R_k \beta R_i$.

Step 3: If each of the selected triplet segregates a subgraph, then by deleting the arc which borders any two of the triplets a module is obtained.

Algorithm 2. (for the shortage of the arcs)

Step 1: Construct a planar graph G from the given network N by adding an artificial arc between two terminals.

Step 2: Let $R_1,\dots,R_f$ denote the regions (including the outer region) of G. From f regions of G, select all the ordered triplet of the regions (R_i,R_j,R_k) which satisfy one of the following conditions;

(C1) $R_i \,\alpha\, R_j$, $R_j \,\alpha\, R_k$, $R_k \,\alpha\, R_i$.

(C2) $R_i \,\alpha\, R_j$, $R_j \,\alpha\, R_k$, $R_k \,\beta\, R_i$.

(C3) $R_i \,\alpha\, R_j$, $R_j \,\beta\, R_k$, $R_k \,\beta\, R_i$.

(C4) $R_i \,\beta\, R_j$, $R_j \,\beta\, R_k$, $R_k \,\beta\, R_i$.

Step 3: If each of the selected triplet segregates a subgraph S, then S has just three nodes in common with its complement G - S (i.e. the vertex attachment number of S is three). And if there is an arc between the two of these three nodes, then by shortening that arc a module is obtained.

5. Discussion

The graphical decomposition is also applicable to some other problems such as maximum flow or minimum distance of the network. A slight modification of lemma 1 enables this.

Let F and c_i denote the flow from source to sink and the capacity of arc i, respectively. Note that the maximum flow is equal to the minimum of the capacities of all cut-sets. According to whether an arc i belongs to that minimal cut-set or not, the flow F can be expressed in two ways. These are concentrated as follows;

$$F(c) = \min\ (\ F(0_i,c) + c_i\ ,\ F(\infty_i,c)\ )$$

where

$$(0_i,c) = (c_1,\dots,c_{i-1},0,\dots,c_n)$$
$$(\infty_i,c) = (c_1,\dots,c_{i-1},\infty,\dots,c_n)\ .$$

Similarly, let D and d_i denote the distances between two terminals and arc i, respectively, then the following equation holds;

$$D(d) = \min\ (\ D(\infty_i,d)\ ,\ D(0_i,d) + d_i\ )\ .$$

While the graphical method has many desirable effect, it has some limitations, too. We summarize them below.

(L1) The pivot arc must be undirected.

(L2) The difficulty extremely increases when the network is non-planar.

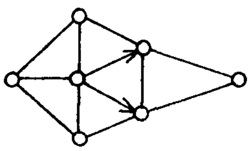
(L3) The non-graphical structures such as k-out-of-n systems[1] can not be treated.

6. Supplementation

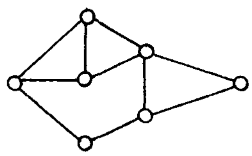
The following inequalities are obtained from the monotonicity of the reliability function $h(p)$.

$$h(0_i,p) \leqq h(p) \leqq h(1_i,p) \quad , \text{ for all } p \text{ and } i \ .$$

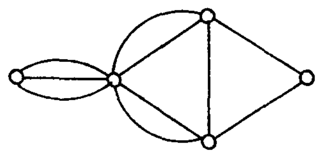
Those inequalities show that the lower and upper bounds of the reliability of N can be calculated from those of N_0^i and N_1^i , respectively. We present a numerical example using above inequalities. The example is due to Shogan[4], and some arcs in the network are directed (as arrowed in the figure). Shogan[4] also shows that his sequential bounds are always tighter than Esary-Proschan bounds. The Esary-Proschan bounds are presented in [1].



Shogan's network



lower bound's network



upper bound's network

Denote the exact reliability, the lower and upper bounds based on above inequalities, and Shogan's upper and lower bounds respectively by R, R_L, R_U, R_{SL}, and R_{SU}. The following table exhibits those values corresponding to the various reliability of arcs. For the simplicity, we let the arc's reliability $p_i=p$ for all arc i.

p	R_L	R_{SL}	R	R_{SU}	R_U
0.1	.003	.001	.005	.005	.011
0.2	.029	.011	.042	.042	.075
0.3	.101	.056	.141	.158	.206
0.4	.229	.187	.305	.350	.385
0.5	.406	.411	.506	.575	.574
0.6	.605	.654	.700	.766	.742
0.7	.786	.838	.851	.890	.868
0.8	.917	.943	.945	.957	.948
0.9	.984	.988	.988	.990	.989

Since the bounding networks are limited, our bounding are less suitable for the directed network. If the network is undirected, then our bounding will become more powerful. We are convinced that our method provides the useful tools in reliability analysis because of its intuitive advantages, its ability of decomposition, and its flexibility of usages.

References

[1] Barlow, R. and Proschan, F. : *Statistical Theory of Reliability and Life Testing* Holt, Rinehart, & Winston, New York 1975.

[2] Birnbaum, Z.W. and Esary, J.D. : "Modules of Coherent Binary Systems" *SIAM J. Appl. Math.* 13 (1965) 444-462.

[3] Deo, N. : *Graph Theory with Applications to Engineering and Computer Science* Prentice-Hall, Inc., Englewood Cliffs, N.J. 1974.

[4] Shogan, A.W. : "Sequential Bounding of the Reliability of a Stochastic Network" *Opns. Res.* 24 (1976) 1027-1044.

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