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A SEQUENTIAL ALLOCATION PROBLEM WITH TWO KINDS OF TARGETS

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Abstract A whaler has some harpoons available for catching whales and finite number of periods to go. The whales are of two kinds, type 1 and type 2; type 1 needs to be hit by the given number of harpoons in order to be caught by the whaler, whereas type 2 needs to be hit by one harpoon. Each type appears with a known probability at each period and produces its reward. When meeting the whale, he can expend, simultaneously, some harpoons of his to obtain the reward. The probability of hit is known to the whaler. The objective is to find the sequence of optimal number of harpoons which maximizes the total expected reward. We investigate the structure of optimal policy.

1. Introduction

Suppose that a whaler with specified number of harpoons voyages for given days to catch whales. The whales appear one by one at each day with a certain probability. Each whale caught by the whaler produces its reward. The reward varies with the size of the whale, etc.. Under these assumptions, how does the whaler allocate his harpoons to maximize the total expected reward?

Sequential allocation problems described above were studied by many authors. For example, Derman, Lieberman, and Ross [2] treated the case that the resource (harpoons in the above context) is a continuous quantity and the expected reward from attack depends on the amount of resource expended, not on the appearing whale. This is the case that there is only one kind of whale and the resource is regarded as a continuous quantity in Section 4. They showed that the optimal amount to be expended is a nondecreasing function of the amount of resource on hand and a nonincreasing function of the remaining time periods. Mastran and Thomas [3] dealt with the model similar to ours.

It is the case that there are infinite kind of whales in Section 4. But they did not mention the structure of optimal policy. Sakaguchi [4] considered the continuous-time version of Mastran and Thomas' model, that is, the resource is a discrete quantity, the expected reward from attack depends not only on the amount of resource expended but also on the appearing whale, and the whales appear in accordance with a Poisson process. He also showed the monotonicity of the amount of resource to be expended. Their models are reduced to the case that the whales needed to be hit by only one harpoon in order to be caught by the whaler.

In this paper, we consider the sequential allocation problem described above. We assume that the whales are of two kinds, type 1 and type 2; type 1 needs to be hit by L harpoons in order to be caught by the whaler, where L is a positive integer, whereas type 2 needs to be hit by one harpoon. We derive the structure of optimal policy for this problem. Later we extend this model to more general one and a simple numerical example is presented.

An outline of the paper is as follows: In Section 2 we derive optimality equations by a dynamic programming formulation of the problem. In Section 3 we discuss the structure of optimal policy. In Section 4 a special case is considered.

2. Model and Formulation

Suppose that a whaler with M harpoons voyages for N time periods to catch whales, where M and N are positive integers. The whales are of two kinds, type 1 and type 2; type 1 needs to be hit by L harpoons in order to be caught by the whaler, where L is a positive integer, whereas type 2 needs to be hit by one harpoon. We also assume that the type i (i=1,2) whale appears with probability r_i, $0<r_i<1$, at each period and produces its reward $x_i>0$. When meeting the whale, the whaler can expend, simultaneously, some harpoons of his, and it is assumed that the probability of hit is denoted as p, $0<p<1$. The objective is to find an optimal policy which maximizes the total expected reward by allocating the given M harpoons to the whales during the N periods.

Let

$V_n(m;i)$ = total expected reward when there are n time periods remaining, m harpoons are on hand, the whaler is meeting the type i whale, and an optimal policy is used,

$\bar{V}_n(m)$ = total expected reward when there are n time periods remaining, m harpoons are on hand, and an optimal policy is used.

By the principle of optimality [1], we have the following recursive relations:

(1) $$V_n(m;1)=\max_{j=0,\cdots,m}\{A(j,L)x_1+\bar{V}_{n-1}(m-j)\}$$

(2) $$V_n(m;2)=\max_{j=0,\cdots,m}\{(1-q^j)x_2+\bar{V}_{n-1}(m-j)\}$$

(3) $$\bar{V}_n(m)=r_1V_n(m;1)+r_2V_n(m;2)+r_3\bar{V}_{n-1}(m)$$

$$(n=1,\cdots,N \qquad m=1,\cdots,M)$$

$$V_0(\cdot;\cdot)=V_\cdot(0;\cdot)=\bar{V}_0(\cdot)=\bar{V}_\cdot(0)=0,$$

where $r_1+r_2+r_3=1$, $p+q=1$, $A(j,L)=\begin{cases}\sum_{i=L}^{j}\binom{j}{i}p^iq^{j-i} & (j\geqq L)\\ 0 & (j<L).\end{cases}$

To obtain the expression for equation (1), we note as follows: If there are n time periods remaining, m harpoons are on hand, the whaler is meeting the type 1 whale, and j harpoons are expended, then the expected reward from this attack is $A(j,L)x_1$, there are n-1 time periods remaining, and m-j harpoons are left. The maximum expected reward is $\bar{V}_{n-1}(m-j)$ when there are n-1 time periods remaining and m-j harpoons are on hand. Similarly we can obtain equation (2). Equation (3) follows from the assumption that the type i (i=1, 2) whale appears with probability r_i at each period and neither appears with probability r_3.

To find the optimal policy, we must solve the set of equations (1),(2),(3).

3. Structure of Optimal Policy

If we solve the set of optimality equations (1), (2), (3), we can find the optimal policy. Unfortunately we cannot solve them explicitly. So we investigate the structure of optimal policy.

Some properties of $V_n(m;i)$ (i=1,2) and $\bar{V}_n(m)$ are easily derived in Lemma 1 and they are used in the proof of Theorem 1.

Lemma 1. $V_n(m;1)$, $V_n(m;2)$, and $\bar{V}_n(m)$ are nondecreasing functions of m and n.

The values of j that maximize the braces of the right hand side of (1) or (2) are the optimal numbers to be expended for the type 1 whale or type 2 respectively. In order to determine the optimal policy unambiguously we define k(n,m;i) to be the smallest value of j that maximizes the braces of the right hand side of (i) (i=1,2), that is, let

$$k(n,m;1)=\min[t \mid \max_{j=0,\cdots,m}\{A(j,L)x_1+\bar{V}_{n-1}(L-j)\} =A(t,L)x_1+\bar{V}_{n-1}(m-t)],$$

$$k(n,m;2)=\min[t \mid \max_{j=0,\cdots,m}\{(1-q^j)x_2+\bar{V}_{n-1}(m-j)\} =(1-q^t)x_2+\bar{V}_{n-1}(m-t)].$$

Using these notations it is the optimal policy that allocates $k(n,m;i)$ to the type i whale when there are n time periods remaining, m harpoons are on hand, and the whaler is meeting the type i.

We investigate the structure of optimal policy according to the number of harpoons on hand, that is, (1) $m<L$, (2) $m=L$, and (3) $m>L$. But before doing this, we present the next theorem which holds for $m>0$.

Intuitively the policy that allocates any harpoons to neither the type 1 whale nor the type 2 when the whaler has some harpoons is not optimal. This conjecture is verified in the next theorem.

Theorem 1. *If $m>0$, then at least one of $k(n,m;1)$ and $k(n,m;2)$ is positive.*

Proof: Since $m>0$, we have

$$V_n(m;1)=\max\{\bar{V}_{n-1}(m),\ A_{n-1}\},$$

$$V_n(m;2)=\max\{\bar{V}_{n-1}(m),\ B_{n-1}\},$$

where

$$A_{n-1}=\max_{j=1,\cdots,m}\{A(j,L)x_1+\bar{V}_{n-1}(m-j)\},$$

$$B_{n-1}=\max_{j=1,\cdots,m}\{(1-q^j)x_2+\bar{V}_{n-1}(m-j)\}.$$

Let $C_{n-1}=\max\{A_{n-1},\ B_{n-1}\}$. Because $\bar{V}_n(\cdot)$ is a nondecreasing function of n from Lemma 1, $\{A_n\}$ and $\{B_n\}$ are nondecreasing sequences, therefore so is $\{C_n\}$. If we can show that $\bar{V}_n(m)<C_n$ for $n\geq 0$, then $\bar{V}_n(m)<A_n$ or $\bar{V}_n(m)<B_n$, so at least one of $k(n+1,m;1)$ and $k(n+1,m;2)$ is positive. The proof by induction on n is employed. For $n=0$, clearly $\bar{V}_0(m)<C_0$. Suppose this is true for some n, that is, $\bar{V}_n(m)<C_n$.

$$\begin{aligned}\bar{V}_{n+1}(m)&=r_1V_{n+1}(m;1)+r_2V_{n+1}(m;2)+r_3\bar{V}_n(m)\\&=r_1\max\{\bar{V}_n(m),\ A_n\}+r_2\max\{\bar{V}_n(m),\ B_n\}+r_3\bar{V}_n(m)\\&<(r_1+r_2+r_3)C_n\leq C_{n+1}.\end{aligned}$$

This implies it is true for $n+1$. Q.E.D.

(1) The case: $m<L$

We consider the case $m<L$, that is, the whaler has less than L harpoons. Since the type 1 whale needs to be hit by L harpoons in order to be caught by

the whaler, it is clear that he must attack only type 2. The next lemma is used in the proof of Theorem 2.

Lemma 2. $V_n(m;2)$ is a concave function of m for m<L.
This lemma follows from a standard induction argument, by making use of the optimality equation (2) and the concavity of $(1-q^j)x_2$ in j.

The following theorem presents the structure of optimal policy when the whaler has less than L harpoons.

Theorem 2. If m<L, then

(i) $k(n,m;1)=0$,

(ii) $k(n,m;2)$ is a nondecreasing function of m and a nonincreasing function of n.

(i) is obvious. The proof of (ii) is quite similar to that of Theorem 3 (ii) and (iii) in [2] and is omitted.

In words Theorem 2 suggests as follows: Suppose that the whaler has less than L harpoons. Then the less periods remaining there are the more harpoons should be expended, and the more harpoons he has the more harpoons should be expended. These results correspond to our intuition.

(2) The case: m=L

In Theorem 2, we presented the structure of optimal policy for m<L. From now on, we consider the case m=L, that is, the whaler has just L harpoons. Then it will be reasonable to attack the type 1 whale.

The following theorem shows the structure of optimal policy when he is meeting the type 2 whale.

Theorem 3. If $k(n,L;2)>0$, then $k(n,L;2)\geq k(n+1,L;2)$.

Proof: Let $j_0=k(n,L;2)>0$. Then we have $V_n(L;2)=(1-q^{j_0})x_2+\bar{V}_{n-1}(L-j_0)$. If $j_0=L$, then the theorem is obvious. Suppose $0<j_0<L$. If we can show that, for any integer c, $0\leqq c\leqq L-j_0$,

$$(1-q^{j_0+c})x_2+\bar{V}_n(L-j_0-c)\leqq(1-q^{j_0})x_2+\bar{V}_n(L-j_0),$$

the theorem will be proved. It should be noted that

$$\begin{aligned}&(1-q^{j_0})x_2+\bar{V}_n(L-j_0)-(1-q^{j_0+c})x_2-\bar{V}_n(L-j_0-c)\\ &=(1-r_2)[\{(1-q^{j_0})x_2+\bar{V}_{n-1}(L-j_0)\}-\{(1-q^{j_0+c})x_2+\bar{V}_{n-1}(L-j_0-c)\}]\\ &\quad+r_2[\{(1-q^{j_0})x_2+V_n(L-j_0;2)\}-\{(1-q^{j_0+c})x_2+V_n(L-j_0-c;2)\}].\end{aligned}\tag{4}$$

The first term in brackets is nonnegative by the definition of j_0. Next we show that the second is nonnegative. Let $j_1=k(n,L-j_0-c;2)$, then we have

$$V_n(L-j_0-c;2)=(1-q^{j_1})x_2+\bar{V}_{n-1}(L-j_0-c-j_1).$$

Since $j_1\leqq j_0$ (this is shown later),

$$(1-q^{j_0+c})x_2+V_n(L-j_0-c;2)$$
$$=(1-q^{j_0+c})x_2+(1-q^{j_1})x_2+\bar{V}_{n-1}(L-j_0-c-j_1)$$
$$\leq(1-q^{j_0})x_2+(1-q^{j_1+c})x_2+\bar{V}_{n-1}(L-j_0-c-j_1)$$
$$\leq(1-q^{j_0})x_2+V_n(L-j_0;2).$$

Hence the right-hand side of (4) is nonnegative.

[The proof of $j_1\leq j_0$]

Let $m_1=L-j_0-c$, then we have, by the definition of j_0 and c, $0\leq m_1<L$. If $j_1=0$ or 1, then $j_1\leq j_0$ is true because $j_0>0$. Suppose $j_1\geq 2$. We complete the proof by showing that, for any integer d, $0<d<j_1$,

$$(1-q^{j_1})x_2+\bar{V}_{n-1}(L-j_1)>(1-q^{j_1-d})x_2+\bar{V}_{n-1}(L-j_1+d).$$

Now, by the definition of j_1,

$$(1-q^{j_1})x_2+\bar{V}_{n-1}(m_1-j_1)>(1-q^{j_1-d})x_2+\bar{V}_{n-1}(m_1-j_1+d).$$

Rearranging the terms and using the fact that $m_1-j_1+d<L-j_1+d<L$ and $\bar{V}_{n-1}(t)$ is a concave function of t for $t<L$,

$$(1-q^{j_1})x_2-(1-q^{j_1-d})x_2>\bar{V}_{n-1}(m_1-j_1+d)-\bar{V}_{n-1}(m_1-j_1)$$
$$\geq\bar{V}_{n-1}(L-j_1+d)-\bar{V}_{n-1}(L-j_1).$$

Q.E.D.

By the above theorem, when the whaler has just L harpoons and is meeting the type 2 whale, it holds for some n's that the less periods remaining there are the more harpoons should be expended. A counterexample that $k(n,L;2)$ is not always a nonincreasing function of n is illustrated later.

Next we consider the case that the whaler is meeting the type 1 whale. For m=L, the optimality equation (1) becomes $V_n(L;1)=\max\{p^L x_1, \bar{V}_{n-1}(L)\}$. Since $\bar{V}_n(L)$ is nondecreasing in n and $\bar{V}_0(L)=0$, the following two cases are possible:

(i) $k(n,L;1)=L$ for all $n\geq 1$,

(ii) there exists a positive integer n_0 such that

$k(n,L;1)=L$, if $n<n_0$,

$k(n,L;1)=0$, if $n\geq n_0$.

Theorem 4 is concerned with the condition that such an n_0 exists. Before presenting Theorem 4, we need the following lemmas which give the asymptotic results of $\bar{V}_n(m)$ for $m\leq L$ and are used in the proof of Theorem 4.

Lemma 3. If $m<L$, then $\bar{V}_n(m)\to mpx_2$ $(n\to\infty)$ for $m<L$, $\bar{V}_n(m)<\bar{V}_{n+1}(m)$ for all $n\geq 0$ and $0<m<L$.

Proof: For $m<L$, the whaler attacks only type 2 whales. He uses the policy that he expends the harpoons one by one when meeting the type 2 whale. The total expected reward from this policy is $E[\{\min(X,m)\}px_2]$, where X is a binomial distributed random variable with parameters (n,r_2). It is noted that

since this policy is a policy of n time periods problem and $(1-q^j)x_2$ is a concave function of j,

$$E[\{\min(X,m)\}px_2]\leqq\bar{V}_n(m)\leqq mpx_2.$$

From the strong law of large numbers and Lebesgue's bounded convergence theorem, we have

$$\lim_{n\to\infty} E[\{\min(X,m)\}px_2]=E[\lim_{n\to\infty}\{\min(X,m)\}px_2]=mpx_2,$$

since $\{\min(X,m)\}px_2\leqq mpx_2$. Hence $\bar{V}_n(m)\to mpx_2$ as $n\to\infty$.

Suppose $\bar{V}_n(m)=\bar{V}_{n+1}(m)$ for some n. Then we have $V_n(m;2)=\bar{V}_{n-1}(m)$ and $k(n,m;2)=0$. This contradicts Theorem 1 and 2. Q.E.D.

Lemma 4. Let $A_{n-1}=\max_{j=1,\cdots,L}\{(1-q^j)x_2+\bar{V}_{n-1}(L-j)\}$, then $A_n\to A=Lpx_2$ $(n\to\infty)$, and $A_n<A$ for all $n\geqq 0$.

Proof: Since $\bar{V}_n(m)\to mpx_2$ as $n\to\infty$ for $m<L$, by the above lemma, we have $A=x_2\max_{j=1,\cdots,L}\{1+pL-(q^j+jp)\}$. Now, $-1=-(q^1+p)>-(q^2+2p)>\cdots>-(q^L+Lp)$. This implies $A=Lpx_2$.

Suppose $A_{n_0}=A$ for some n_0. Using Lemma 3,

$$(1-q^j)x_2+\bar{V}_{n-1}(L-j)<(1-q^j)x_2+\bar{V}_n(L-j),\ \text{if } 1\leqq j<L,$$

$$(1-q^j)x_2+\bar{V}_{n-1}(L-j)=(1-q^j)x_2+\bar{V}_n(L-j),\ \text{if } j=L,$$

then we have $A_n=(1-q^L)x_2<Lpx_2$ for all $n\geqq n_0$. This contradicts the fact that $A_n\to A$ as $n\to\infty$. Q.E.D.

Lemma 5. $\bar{V}_n(L)\to\max\{A,\ p^Lx_1\}=\max\{Lpx_2,\ p^Lx_1\}$ $(n\to\infty)$, $\bar{V}_n(L)<\max\{A,\ p^Lx_1\}$ for all $n\geqq 0$.

Proof: By the optimality equation, Theorem 1, and Lemma 4, we have

$$\bar{V}_n(L)=r_1\max\{p^Lx_1,\ \bar{V}_{n-1}(L)\}+r_2\max\{A_{n-1},\ \bar{V}_{n-1}(L)\}+r_3\bar{V}_{n-1}(L),$$

$$\bar{V}_{n-1}(L)<\max\{A_{n-1},\ p^Lx_1\}<\max\{A,\ p^Lx_1\}.$$

Since $\{\bar{V}_n(L)\}$ is a bounded nondecreasing sequence, the limit $\bar{V}_\infty$ exists.

$$(r_1+r_2)\bar{V}_\infty=r_1\max\{p^Lx_1,\ \bar{V}_\infty\}+r_2\{A,\ \bar{V}_\infty\}.$$

This implies $\bar{V}_\infty=\max\{A,\ p^Lx_1\}$. Q.E.D.

Now we present Theorem 4 which shows the necessary and sufficient condition that $k(n,L;1)=0$ for $n\geqq$some integer.

Theorem 4.

(i) If $p^Lx_1>Lpx_2$, then $k(n,L;1)=L$ for all $n\geqq 1$ and there exists a positive integer n_0 such that $n\geqq n_0$ implies $k(n,L;2)=0$.

(ii) If $p^L x_1 < Lpx_2$, then there exist positive integers n_1 and n_2 such that $n \geqq n_1$ implies $k(n,L;1)=0$ and $n \geqq n_2$ implies $k(n,L;2)=1$.

(iii) If $p^L x_1 = Lpx_2$, then $k(n,L;1)=1$ for all $n \geqq 1$.

Proof: (i) By Lemma 4 and 5, we have $\bar{V}_{n-1}(L) < p^L x_1$ for all $n \geqq 1$, and $A_{n-1} < Lpx_2 < \bar{V}_{n-1}(L)$ for sufficient large n. Hence (i) is proved.

(ii) The proof of the first part is similar to (i). To prove the latter part we note, by Theorem 1, that $k(n,L;2)>0$ for sufficient large n. Suppose that there are infinite many values of n which satisfies $k(n,L;2)=j_1 \geqq 2$. Then we have infinite many values of n which satisfies

$$\bar{V}_n(L)=r_1\bar{V}_{n-1}(L)+r_2[(1-q^{j_1})x_2+\bar{V}_{n-1}(L-j_1)]+r_3\bar{V}_{n-1}(L).$$

This contradicts the fact that $\bar{V}_n(m) \to mpx_2$ as $n \to \infty$ for $m \leqq L$. Hence (ii) is proved.

(iii) is proved similarly to (i). Q.E.D.

From the above theorem, when the whaler has just L harpoons, the optimal policy is expressed as follows:

(i) If $p^L x_1 > Lpx_2$, then all L harpoons should be expended when he is meeting the type 1 whale. No harpoon should be expended when there are sufficient many time periods remaining and he is meeting the type 2 whale.

(ii) If $p^L x_1 < Lpx_2$, then no harpoon should be expended when there are sufficient many time periods remaining and he is meeting the type 1 whale. Just one harpoon should be expended when there are sufficient many time periods remaining and he is meeting the type 2 whale.

(iii) If $p^L x_1 = Lpx_2$, then all L harpoons should be expended when he is meeting the type 1 whale.

(3) The case: m>L

The proof of Theorem 2 and 3 strongly depended on the concavity of $\bar{V}_n(\cdot)$ and this property was derived from that of $(1-q^j)x_2$. Because $A(j,L)x_1$ is not a concave function of j without the simplest case L=1, we cannot proceed for m>L by the same manner. A simple structure of optimal policy for m>L may not be expected.

Remark 1. The results of this section remain valid even when we extend $A(j,L)x_1$ and $(1-q^j)x_2$ to $R_1(j)$ and $R_2(j)$ respectively, where $R_1(j)$ is a non-decreasing function of j with the property $R_1(j)=0$ for $0 \leqq j < L$, and $R_2(j)$ is an increasing strictly concave function of j with the property $R_2(0)=0$. In this case mpx_2 $(0 \leqq m \leqq L)$ and $p^L x_1$ are replaced with $mR_2(1)$ and $R_1(L)$ respectively.

Remark 2. The following numerical example shows that $k(n,L;2)$ is not always a nonincreasing function of n and k(n,m;i) is not always a nondecreasing function of m. Let L=2, M=5, N=6, r_1=0.666, r_2=0.333, r_3=0.001, x_1=3.125, x_2=1, and p=0.5. Then the optimal policy is listed in Table 3.1 and 3.2. For example k(4,m;1) and k(2,m;2) are not nondecreasing functions of m, and k(n,2;2) is not a nonincreasing function of n. This example shows that the monotonicity of k(n,m;i) does not always hold for L$\geq$2. But it holds for L=1 as we show in the next section.

Table 3.1 k(n,m;1)

m \ n	the number of remaining periods 1	2	3	4	5	6
harpoons on hand 5	5	5	5	0	0	0
4	4	4	4	4	4	4
3	3	3	3	3	3	3
2	2	2	2	2	0	0
1	0	0	0	0	0	0

Table 3.2 k(n,m;2)

m \ n	the number of remaining periods 1	2	3	4	5	6
harpoons on hand 5	5	1	1	1	1	1
4	4	1	0	0	0	0
3	3	0	0	0	0	0
2	2	0	1	1	1	1
1	1	1	1	1	1	1

4. Special case: L=1

In this section we consider the simplest case L=1 and obtain further results about k(n,m;i). It is shown that these results can be extended to the case that the whales are of finite number of kinds. Note that this case is similar to the model discussed by Mastran and Thomas [3].

The following lemmas are used in the proof of Theorem 5. We have Lemma 6 similarly to Lemma 2.

Lemma 6. $V_n(m;i)$ (i=1,2) and $\bar{V}_n(m)$ are concave functions of m (see Theorem 3 (i) in [2]).

Lemma 7. The following inequality is satisfied:

$$\bar{V}_{n+1}(m)-\bar{V}_n(m)\leq\bar{V}_{n+1}(m+1)-\bar{V}_n(m+1) \text{ for } n\geq 0,\ m\geq 0.$$

Proof: By the optimality equation, we have

$$\bar{V}_{n+1}(m)=r_1V_{n+1}(m;1)+r_2V_{n+1}(m;2)+r_3\bar{V}_n(m),$$

$$\bar{V}_{n+1}(m+1)=r_1V_{n+1}(m+1;1)+r_2V_{n+1}(m+1;2)+r_3\bar{V}_n(m+1).$$

This yields

$$\bar{V}_{n+1}(m)-\bar{V}_n(m)=r_1[V_{n+1}(m;1)-\bar{V}_n(m)]+r_2[V_{n+1}(m;2)-\bar{V}_n(m)],$$

$$\bar{V}_{n+1}(m+1)-\bar{V}_n(m+1)=r_1[V_{n+1}(m+1;1)-\bar{V}_n(m+1)]+r_2[V_{n+1}(m+1;2)-\bar{V}_n(m+1)].$$

Hence the lemma will be proved if we can show the following inequalities:

$$V_{n+1}(m+1;i)-\bar{V}_n(m+1)\geq V_{n+1}(m;i)-\bar{V}_n(m) \quad (i=1,2).$$

Or

$$V_{n+1}(m+1;i)-V_{n+1}(m;i)\geqq\bar{V}_n(m+1)-\bar{V}_n(m) \quad (i=1,2).$$

To show this let $j_0=k(n+1,m;i)$ for $i=1,2$. Then we have $0\leqq j_0\leqq m$ and $V_{n+1}(m;i)=(1-q^{j_0})x_i+\bar{V}_n(m-j_0)$. On the other hand $V_{n+1}(m+1;i)\geqq(1-q^{j_0})x_i+\bar{V}_n(m+1-j_0)$. Therefore, using the concavity of $\bar{V}_n(\cdot)$, we have

$$V_{n+1}(m+1;i)-V_{n+1}(m;i)\geqq\bar{V}_n(m+1-j_0)-\bar{V}_n(m-j_0)\geqq\bar{V}_n(m+1)-\bar{V}_n(m).$$ Q.E.D.

We show the structure of optimal policy in Theorem 5,6, and 7. Theorem 5 and 6 are concerned with the monotonicity of $k(n,m;i)$ with respect to n,m, and i. Theorem 7 states the optimal policy when there are sufficient many time periods remaining.

Theorem 5.

(i) $k(n,m;i)$ $(i=1,2)$ is a nondecreasing function of m and $k(n,m+1;i)\leqq k(n,m;i)+1$.

(ii) $k(n,m;i)$ $(i=1,2)$ is a nonincreasing function of n. (See Proposition 2 and Theorem 3 (ii) (iii) in [2].)

Proof: (i) is proved similarly to Proposition 2 and Theorem 3 (ii) in [2]. To prove (ii), let $j_0=k(n,m;i)$ for $i=1,2$, then

(5) $$(1-q^{j_0})x_i+\bar{V}_{n-1}(m-j_0)\geqq(1-q^{j})x_i+\bar{V}_{n-1}(m-j).$$

For any integer j, $j_0\leqq j\leqq m$, we have $m-j_0\geqq m-j$. Therefore, by Lemma 7,

(6) $$\bar{V}_n(m-j_0)-\bar{V}_{n-1}(m-j_0)\geqq\bar{V}_n(m-j)-\bar{V}_{n-1}(m-j).$$

From (5) and (6), we have

$$(1-q^{j_0})x_i+\bar{V}_n(m-j_0)\geqq(1-q^{j})x_i+\bar{V}_n(m-j) \text{ for } j_0\leqq j\leqq m,\ i=1,2.$$

This implies that $k(n+1,m;i)\leqq j_0=k(n,m;i)$. Q.E.D.

This theorem says that the more harpoons the whaler has the more harpoons should be expended, and the less time periods remaining there are the more harpoons should be expended. These statements are not always true for $L\geqq2$.

Intuitively the more lucrative a detected whale is the more harpoons should be expended. This conjecture is verified in the following theorem.

Theorem 6. If $x_1>x_2$, then $k(n,m;1)\geqq k(n,m;2)$ for $n\geqq1$, $m\geqq1$.

Proof: Let $j_0=k(n,m;2)$. If $j_0=0$, then the theorem is obvious. Suppose $j_0\geqq1$. For any integer j, $0\leqq j<j_0$,

$$\{(1-q^{j_0})x_1+\bar{V}_{n-1}(m-j_0)\}-\{(1-q^{j})x_1+\bar{V}_{n-1}(m-j)\}$$

(7) $$=[\{(1-q^{j_0})x_2+\bar{V}_{n-1}(m-j_0)\}-\{(1-q^{j})x_2+\bar{V}_{n-1}(m-j)\}]$$

$$+[\{(1-q^{j_0})x_1-(1-q^{j_0})x_2\}-\{(1-q^{j})x_1-(1-q^{j})x_2\}].$$

The first bracket of the right-hand side of (7) is positive by the definition of j_0, and the second is positive from $x_1>x_2$. This implies $k(n,m;1)\geqq j_0$ $=k(n,m;2)$. Q.E.D.

The following lemma is used in the proof of Theorem 7 and is proved similarly to Lemma 3.

Lemma 8. If $x_1>x_2$, then $\bar{V}_n(m)\to mpx_1$ as $n\to\infty$.

Theorem 7. If $x_1>x_2$, then there exist positive integers n_0 and n_1 such that $n\geqq n_0$ implies $k(n,m;1)=1$ and $n\geqq n_1$ implies $k(n,m;2)=0$.

Proof: Suppose that such an n_1 does not exist. The possible values of $k(n,m;2)$ are finite, so there exists $j_0>0$ such that $k(n,m;2)=j_0$ for infinite many values of n. This implies that there are infinite many values of n which satisfies $(1-q^{j_0})x_2+\bar{V}_{n-1}(m-j_0)>(1-q^0)x_2+\bar{V}_{n-1}(m)=\bar{V}_{n-1}(m)$. This contradicts the fact that $\bar{V}_n(m)\to mpx_1$ as $n\to\infty$. We may prove the existence of n_0 by the same method. Q.E.D.

In words, when there are sufficient many time periods remaining, the whaler should expend his harpoons one by one to the most lucrative whales.

Remark. The results in this section remain valid even when the whales are of finite number of kinds and the expected reward functions are replaced with more general ones as described later. The extended problem is expressed as follows:

The whales are of I kinds, type 1, type 2, $\cdots$, and type I; type i whale appears with probability r_i, $0<r_i<1$, at each period. If the whaler expends j harpoons to the type i whale, then he obtains the expected reward $R_i(j)$ ($R_i(0)$ $=0$) from this attack. Then using the same notations, we obtain the structure of optimal policy.

(1) If $R_i(j)$ $(1\leqq i\leqq I)$ is an increasing and concave function of j, then $k(n,m;i)$ $(1\leqq i\leqq I)$ is a nondecreasing function of m and a nonincreasing function of n.

(2) If $R_i(j)$ $(1\leqq i\leqq I)$ satisfies the following relation (see [3])

$$R_i(j)-R_{i+1}(j)<R_i(j+1)-R_{i+1}(j+1) \quad (1\leqq i\leqq I-1,\ j\geqq 0),$$

then $k(n,m;i)$ is a nonincreasing function of i.

(3) If both the assumptions (1) and (2) are satisfied and $R_1(j)$ is strictly concave, then there exist positive integers n_0 and n_1 such that $n\geqq n_0$ implies $k(n,m;1)=1$ and $n\geqq n_1$ implies $k(n,m;i)=0$ for $2\leqq i\leqq I$.

5. Conclusion

In this paper, we considered the discrete-time finite horizon sequential allocation problem with discrete resource. In Section 3 we found the structure of optimal policy when the whaler has less than or equal to L harpoons. For the case of less than L harpoons on hand, the optimal number to be expended for the type 2 whale is a nonincreasing function of the remaining time periods and a nondecreasing function of the number of harpoons on hand. For the case of just L harpoons on hand, we derived the necessary and sufficient condition that the optimal number to be expended for the type 1 whale vanished, and the optimal number to be expended for the type 2 decreases as the remaining time periods increase without its zero point. We also developed that the policy which allocates any harpoons to neither the type 1 whale nor the type 2 when the whaler has some harpoons is not optimal. In Section 4 we dealt with the simplest case L=1. We proved the optimal number to be expended for the type i (i=1,2) whale is a nonincreasing function of the remaining time periods and a nondecreasing function of the number of harpoons on hand, and furthermore that the more lucrative a detected whale is the more harpoons should be expended.

It is a future problem to find the structure of optimal policy when the whaler has more than L harpoons. From the other point of view, the problem of continuous-time version is an important one to be solved near future.

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