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REVERSIBILITY OF A MULTILATERAL SEQUENTIAL GAME: PROOF OF A CONJECTURE OF SAKAGUCHI

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Abstract In this paper optimal strategies are derived for the choice by several players of a set of sequentially observed random variables. The players are permitted to have any utility functions on the chosen random variables, and any opinion on the joint distribution of the sequence of random variables, provided these utility functions and opinions are known to all palyers. A random variable is chosen if enough players want it to be chosen, in a sense made precise. Exactly r of the n available random vectors must be chosen. Once rejected, a random variable cannot later be selected; once selected, it cannot later be rejected.

It is shown that when no costs are assigned to rejection, the optimal strategies can be found by backwards induction and the order in which the players announce their decisions does not matter to any of them and does not influence the outcome, confirming a conjecture of Sakaguchi. This contrasts with the situation in which the number of rejections avaialble to each player is limited, where the order typically does matter.

1. Introduction

Suppose that K players observe a sequence of n vectors $\underset{\sim}{X}_i$, $(i=1,\ldots,n)$ of arbitrary length L, of which they will select r, $1\leq r\leq n$. Suppose that after each vector has been observed, the players, each in turn in an arbitrary order, must announce whether he wishes to accept the vector. If the set of players wishing to accept the vector is a member of a winning class W(satisfying (i) $\{1,2,\ldots,K\}\varepsilon W$, (ii) if $P\varepsilon W$ and $P'\supset P$ then $P'\varepsilon W$ and (iii) W does not depend on the order in which the players announce their preferences), then the vector is accepted as one of the r selected vectors. If the players selected m, $0\leq m\leq r-1$ of the first n-r+m random variables $\underset{\sim}{X}_i$, then they are forced to select the last r-m of them, Once rejected, a vector cannot later be selected; once selected, a vector cannot later be rejected. Each vector must be selected or rejected before the next vector is observed.

Suppose the selected vectors are $\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_r$. Before the selection has actually been made, $\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_r$ will be random variables. For any possible values $y_1, \ldots, y_r$, we let $\psi_k(y_1, \ldots, y_r)$ denote the utility function of the k^{th} player, $1 \leq k \leq K$. Both these utility functions and each player's joint distribution on $(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n)$ is assumed known to all K players. Hence each player k is assumed to make decisions in order to maximize $E_k(\psi_k(\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_r))$, where E_k denotes the expectation with respect to the joint distribution of $(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n)$ assigned by player k.

Sakaguchi [1978] in a recent article (hereafter referred to as S) considers the following two special cases: In case 1, K = L = 2, so that bivariate vectors $\underset{\sim}{X}_i = (X_{i1}, X_{i2})$ are observed. The utility functions assumed are the sum of the relevant component of the bivariate vectors: $\psi_k(y_1, \ldots, y_r) = \sum_{i=1}^{r} y_{ik}$ for k=1,2. Also W={{1,2}}, so that agreement of both parties is required for a vector to be chosen . Additionally the vectors are assumed iid from a known distribution.

In case 2, K = L = 3, so here trivariate observations $\underset{\sim}{X}_i = (X_{i1}, X_{i2}, X_{i3})$ are observed. Again the utility funcitons take the form of sums: $\psi_k(y_1, \ldots, y_r) = \sum_{i=1}^{r} y_{ik}$, k=1,2,3. In this case W={{1,2}, {1,3}, {2,3}, {1,2,3} , so that majority rule applies. Again the vectors are assumed to be iid observations from a known trivariate distribution. Sakaguchi shows that both these problems are reversible in the sense of Roth, Kadane and DeGroot [1977], (hereafter referred to as RKD), that is, none of the players has any reason to care in what order the players announce their preferences. Additionally he conjectures that reversibility holds for arbitrary K, and our purpose is to prove this in the context outlined above.

The problem discussed here differs from the case in which the number of rejections allowed to each party is fixed in advance, and is less than n-r. In this case RKD for the iid case in which $\psi_k(y_1, y_2, \ldots, y_r) = (-1)^k \prod_{i=1}^{r} y_{ik}$, k=1,2 and W={1,2}, and S for the iid case in which $\psi_k(y_1, y_2, \ldots, y_r) = \sum_{i=1}^{r} y_{ik}$, k=1,2 show that the order does in general matter. A third paper, DeGroot and Kadane [1978], (hereafter referred to as DK] shows that when the number of rejections permitted to each side is limited, the two sides assign the same joint distribution to $X_1, \ldots, X_n$ and when the utility functions satisfy

$$a_1\psi_1(\underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_r) + a_2\psi_2(\underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_r) = c$$

for all $(\underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_r)$, where $a_1 a_2 \neq 0$, the problem is reversible. This corresponds to the special case in which the interests of the two sides are either exactly aligned $(a_1a_2<0)$ or exactly opposed $(a_1a_2>0)$. Another somewhat relevant paper is by Brams and Davis [1978], although they treat simultaneous rather than sequential choice. Both the problem considered here and the problem of DK are generalizations of the problems of optimal selection discussed by Gilbert and Mosteller [1966] and DeGroot ([1970], Sec. 13.4), among others, and much earlier by Cayley [1875]. In those problems, a sequential random sample of independent and identically distributed variables Z_1, Z_2,... is to be observed without recall by a single decision maker. The decision maker must choose a stopping rule N in order to maximize $E(Z_N)$. There is a given upper bound n on the number of observations that can be taken; if the decision maker has not stopped before the value of Z_n has been observed, then he must stop and accept that value as his payoff. This problem of optimum selection is a special case of the problem in this paper in which only one observation is to be chosen (r=1), the observed vectors $X_1,\ldots, X_n$ are independent and identically distributed, there is only one player (K=1), and $Z_i=\psi_1(X_i)$ for $i=1,2,\ldots, n$.

Gilbert and Mosteller [1966] also consider the problem in which r observations must be selected out of the total of n variables to be observed sequentially, without recall, and the payoff to the decision maker is the sum of the r selected observations. This is again a special case in which K=1, and the utility function takes the special form $\psi_1(y_1,\ldots, y_r)= \sum_{i=1}^{r} \psi(y_i)$.

In section 2 of this paper, the assumptions on the random variables $\underset{\sim}{X}_1,\ldots, \underset{\sim}{X}_n$, the definition of the optimal behavior, and the backwards induction approach are discussed. Section 3 proves the principal result and discusses its consequences.

2. Backwards Induction

We assume that each player can represent his view of the process generating the sequence $\underset{\sim}{X}_1,\ldots, \underset{\sim}{X}_n$ by assigning to it a joint probability distribution. This distribution need not be the same for the players, but it is assumed that each player knows the distributions assigned to all the other players. Some important special cases of these joint distributions are of special interest:

(A) The random variables $\underset{\sim}{X}_1, \underset{\sim}{X}_2,\ldots, \underset{\sim}{X}_n$ might be independent and identically distributed with a particular known distribution agreed on by all the players.

(B) The random variables $\underset{\sim}{X}_1, \underset{\sim}{X}_2,\ldots, \underset{\sim}{X}_n$ might be independent and identically distributed with different distributions assigned by each player.

In this case, each player would believe that its own distribution is more appropriate than the distributions assigned by the other players.

(C) The exact sequence of values of $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ might be known in advance to all the players.

(D) The random variables $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$ might be exchangeable but dependent. This could be the case if each player believes that $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ form a random sample from some distribution that depends on a parameter θ whose value is unknown to all players, and each player assigns a prior distribution to θ. The players might agree on the conditional distribution of $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ given θ, but disagree on the prior distribution to be assigned to θ. In this case each player is assumed to know the priors assigned to θ by each other player. Each player will then update his own distribution for θ after each random variable X is observed, which will affect the joint distribution for the remaining X's

The assumption that the joint distributions of $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ assigned by each player is known to every player includes all these special cases and, of course, many more.

We now define what we mean by optimal strategies for the players in this game. Consider the last possible decision that could arise, when r-1 of the variables have been chosen, the n-1st random variable is being considered, and all the players except one has announced his decision. If the process has not terminated before this state is reached, then the player with this last decision to announce has a well-defined optimal decision for each possible value $\underset{\sim}{x}_{n-1}$ of $\underset{\sim}{X}_{n-1}$. Under the assumption that this last possible choice will be made optimally, the consequences of the next-to-last possible decision are known, and hence it can be made optimally also. By backwards induction, each decision can be made optimally under the assumption that all players will act optimally in all possible subsequent decisions. The optimal procedure is taken to be the one resulting from all these optimal actions of both sides.

3. Analysis of the game

Suppose that $X_1, \ldots, X_p$ have been observed, some of whom may have been accepted and the rest rejected. The history comprising these values and the decisions that were made by all the players about them is denoted h_p. The history h_p may affect the future decisions by the players in the following ways: (i) if $r' \leq r$ vectors $\underset{\sim}{X}$ have been accepted then the first r' components of the utility functions ψ_k, k=1,..., K have been determined and the utility of the remaining vectors $\underset{\sim}{X}$ may be affected. (ii) The conditional joint distribution of $\underset{\sim}{X}_{p+1},\ldots,\underset{\sim}{X}_n$ given $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_p$ now becomes relevant for each player and (iii) only r-r' vectors $\underset{\sim}{X}$ out of the remaining n-p must be selected.

We require a notation for the expected utility to each side of the optimal strategies given the history h_p. This might be denoted $E_k(\psi_k|h_p)$, but on some occasions it will be useful to emphasize particular aspects of the history h_p. Thus for example we might write $E_k(\psi_k|h_p, r-r', n-p)$ to indicate the expected utility to player k after the history h_p, where there are r-r' vectors remaining to be chosen out of the last n-p vectors $\underset{\sim}{X}$.

Suppose that some history h_p already happened, and let $\underset{\sim}{X}_{p+1}= x$ be the next vector observed. Let

$$F_k = E_k(\psi_k|\ h_p,\ x,\ r-r',\ n-p-1)$$

be the worth to player k of the situation in which x is rejected, and let

$$C_k = E_k(\psi_k|\ h_p,\ x,\ r-r'-1,\ n-p-1)$$

be the worth to player k of the situation in which x is accepted.

Note that F_k and C_k do not depend on the order in which the players in round p+1 announce their choices, but only on whether, after all the announcements, $\underset{\sim}{X}_{p+1}$ is accpeted or not. This would not be the case if a player could learn from the other player's choices, and so might be able to make a more informed choice going last than going first. However since under this model both the utility function and joint distribution of every player is known to every player, the optimal decision can be predicted with certainty and hence the announcements have no informational content.

For player k, if $F_k>C_k$ he hopes the observation $\underset{\sim}{X}_{p+1}$ will be rejected; if $F_k<C_k$ he hopes it will be accepted. If $F_k=C_k$ an ambiguous situation results in which the outcome is indifferent for player k.

The assumption on the winning class W that if $P\varepsilon W$ and $P'\supset P$ then $P'\varepsilon W$ is precisely the condition needed in this context to ensure that strategic voting in the sense of Satterwaite [1973, 1975] and Gibbard [1973] is not optimal. Thus if $F_k>C_k$, player k will announce for rejection ; if $F_k>C_k$, player k will announce for acceptance. In order not to burden our analysis and notation excessively, we assume that player k will announce for rejection only when strictly necessary, i.e. when $F_k>C_k$.

Then the set of players for acceptance is $\{k|\ F_k\leq C_k\}$. If this set is a member of the winning class W, then the observation $\underset{\sim}{X}_{p+1}$ is accepted, and otherwise not.

Note that the class W need not be symmetric in the set of players.

Thus, for instance if K=3, the class W = {{1,2}, {1,3}, {1,2,3}} obeys the axioms for W, and says that for a vector $\tilde{X}$ to be accepted, player 1 and (player 2 and/or player 3) must find it acceptable. Here player 1 is not the player who announces his choice first, but a particular player, whenever his turn to announce comes.

Since none of the objects in the condition $\{k|\ F_k \le C_k\} \varepsilon W$ depend on the order in which the choices are announced, neither does the outcome, and hence no player cares what order the decisions are announced in. Hence we have proved:

Theorem 1. Under the assumptions specified above, the order in which the players announce their decisions does not matter to any of them and does not influence the outcome.

Because whether an observation $\tilde{X}$ will be accepted can be predicted with certainty, an alternative optimal strategy is that every player will announce a willingness to accept the observation if $\{k|\ F_k \le C_k\} \varepsilon W$, and otherwise no player will. This leads to apparent unanimity, as stressed in S in the analysis of Case 1.

To say that the optimal strategies can be found by backwards induction is not to say that the necessary calculations are a trivial matter in a specific context. In S some calculations are given for Cases 1 and 2.

Finally we remark that it appears to us from this analysis that the critical assumptions needed to make reversibility obtain are 1) that the utility functions and joint distributions of every player are known to all, which assures that the optimal decision of each player can be calculated by every other player, and consequently his decision carries no information, and 2) that cost structure does not penalize a player for rejecting a vector that other players would reject for it. The first set of assumptions, but not the second, are also satisfied by the problems considered in DK.

Similarly in the case K=L, one might suppose that player k observes only the k^{th} component of $\tilde{X}$, and that his utility function depends only on $y_{1k}, \ldots, y_{rk}$. If the distributions of the players is such that the other r-1 components carry no information for the k^{th} player about future values of $\tilde{X}$ (as in cases (A), (B) and (C), but not, in general, (D), of Section 2) then again the order of play will not matter. However if this condition is not satisfied, players announcing later in the sequence gain information from the announced choices of players earlier in the sequence. Under some conditions it is possible that players announcing earlier might find it optimal to use this fact to mislead players announcing later, leading to a game of great complexity.

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References

1. Brams, S.J. and Davis, M.D.: Optimal Jury Selection: A Game-Theoretic Model of Exercise of Peremptory Challenges, to appears in *Opns. Res.*, 1978.
2. Cayley, A.: Mathematical Questions and Their Solutions. *Educational Times*, Vol. 22 (1875), pp. 18-19.
3. DeGroot, M.H.: *Optimal Statistical Decisions*, McGraw-Hill, 1970.
4. DeGroot, M.H. and Kadane, J.B.: Optimal Challenges for Selection, *Department of Statistics Technical Report* No. 143 (1978), Carnegie-Mellon University, submitted for publication.
5. Gilbard, A.: Manipulation of Voting Schemes: A General Result. *Econometrica*, Vol. 41 (1973), pp. 587-601.
6. Gilbert, J.P. and Mosteller, F.: Recognizing the Maximum of a Sequence. *J. Am. Stat. Assn.*, Vol. 61 (1966), pp. 35-73.
7. Roth, A., Kadane, J.B. and DeGroot, M.H.: Optimal Peremptory Challenges in Trials by Juries: A Bilateral Sequential Process. *Opns. Res.*, Vol. 25 (1977), pp. 901-919.
8. Sakaguchi, M.: A Bilateral Sequential Game for sums of Bivariate Random Variables, to appear in *J. Op. Res. Soc. Japan.*,]978.
9. Satterwaite, M.A.: The Existence of a Strategy Proof Voting Procedure: A Topic in Social Choice Theory, Ph. D. Dissertation, Univ. of Wisconsin, 1973.
10. Satterwaite, M.A.: Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions. *J. of Econ. Thy.*, Vol. 10 (1975), pp. 187-217.

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