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A BILATERAL SEQUENTIAL GAME FOR SUMS OF BIVARIATE RANDOM VARIABLES

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Abstract This paper explores optimal strategies for the problem of choosing several best from a set of sequentially observed bivariate random variables. For example, a couple of husband and wife to make a plan of recreations during a year, has this problem when deciding which offer (or amount of satisfaction) to accept and which to reject. Each offer on arrival is examined first by husband and, if accepted by him, then secondly by his wife. If she rejects it, the offer is rejected. Therefore the offer is "selected" only when both of husband and wife accept it. We assume that the offers are iid bivariate r.v.'s and at most n can be observed. Each offer on arrival is either selected or rejected; an offer rejected now cannot be selected later on. The objective of husband (wife) is to maximize the expected value of the sum, from his (her) standpoint, of the offers actually selected. For another example, the problem of optimal selection of r secretaries to be employed by two university professors in the same department belongs also to our problem. By using dynamic programming technique we develope the optimal procedure for this non-cooperative, sequential, bilateral game and discuss several simple examples. It is shown that it does not matter to husband (or wife) whether he (or she) decides first or second. It is also shown that it does matter to either side whether it decides first or second, if the number of rejections available to each side is given beforehand.

1. Introduction and Summary

Let (X_i, Y_i), $i = 1, \ldots, n$, be iid bivariate random variables that can be observed one by one sequentially. Players I and II are about to select a set of r, $1 \leq r \leq n$, observations. The common distribution function $H(x, y)$ of each (X_i, Y_i) is assumed to be known by both players. When a r.v. is observed, player I must firstly decide whether to accept (i.e., select) or reject the observation, and if he rejects it the next r.v. will be observed. If player I accepts the observation, then player II must nextly decide whether to accept the current observation or not, and if he rejects it the next r.v. is observed. Only when both players accept the observation, the pair of values is selected as one element of the set of r observations. Only n r.v.'s can be observed, and we shall assume that if the players have selected m, $0 \leq m \leq r-1$, observations until the final r-m r.v.'s, then they are forced to select these r-m observations of the r.v.'s.

Let (X_i', Y_i'), $i = 1, \ldots, r$, be the members of the final set of r observations, and suppose that the objective of player I's sequential decisions is to maximize the expected value, $E(\sum_{i=1}^{r} X_i')$, of the sum of the first components of the observations, they have selected. Similarly suppose that II's objective is to maximize $E(\sum_{i=1}^{r} Y_i')$.

This structure of the problem is a bilateral sequential process, in which decisions are made by each side one by one, without a simultaneous decision by the other side. Roth, Kadane and DeGroot state in a recent article [6] (hereafter referred to as RKD) that "bilateral sequential processes may be a better model for many social phenomena, such as arms races and duopoly, than traditional game theory that requires simultaneous moves by the players".

The outline of the paper is as follows. In Section 2 we put the problem in the dynamic programming formulation and derive the optimal procedure in the sense that each side maximizes its expected payoff at each stage under the assumption that both sides will continue to use these optimal strategies in all future decisions. It is shown that it does not matter to either side whether it decides first or second. In Section 3 the problem is extended to the three-person trilateral sequential game in which selection is done under the simple majority, and the optimal procedure is derived. Again we find that it does not matter to every player whether he decides first, second or third. In the final Section 4 we consider the two-person bilateral process in which the number of rejections available to each side is given beforehand. It is shown that it does matter to either side whether it decides first or second. Throughout these sections recursive relations satisfied by the critical numbers, which can determine the optimal procedure and essentially be calculated before the r.v.'s are observed, are derived. Some simple examples are given for bivariate uniform, normal, and mixed distributions.

2. Bilateral Sequential Game and the Optimal Procedure

For the problem described in the previous section, it is evident that each decision by either player can be made optimally if the player making that decision is willing to assume that both player will act "optimally" on all subsequent decisions. The optimal procedure is taken to be the one resulting

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from all these optimal choices by both players. Let $U_{n,r}$ and $V_{n,r}$ denote the expected values $E(\sum_{i=1}^{r} X'_i)$ and $E(\sum_{i=1}^{r} Y'_i)$, respectively, under the optimal procedure. Then by considering the consequences of the two possible decisions, first for I, and then, if I accepts the observation, for II, we can write

$$(2.1)\qquad U_{n,r} = E[\max\{U_{n-1,r}, (X+U_{n-1,r-1})\, \chi_{Y+V_{n-1,r-1}-V_{n-1,r} > 0} + U_{n-1,r}\, \chi_{Y+V_{n-1,r-1}-V_{n-1,r} \leq 0}\}] ,$$

where χ_E denotes the indicator function of the event E, since if I firstly accepts the observation, then nextly II, under his optimal decision,

$$\left\{\begin{matrix}\text{accepts}\\ \text{rejects}\end{matrix}\right\}, \text{ according as } Y + V_{n-1,r-1} \left\{\begin{matrix} > \\ \leq \end{matrix}\right\} V_{n-1,r} .$$

Also we can write

$$(2.2)\qquad V_{n,r} = E\, Z_{n,r}(X, Y) ,$$

if we define the r.v.

$$Z_{n,r}(X,Y) = \begin{cases} V_{n-1,r} , & \text{if I rejects,} \\ \max(Y+V_{n-1,r-1}, V_{n-1,r}) , & \text{if I accepts .} \end{cases}$$

Let

$$(2.3)\qquad \begin{aligned} \mu_{n,r} &= U_{n,r} - U_{n,r-1} \quad (r = 1,2, \ldots; U_{n,0} = 0) \\ \nu_{n,r} &= V_{n,r} - V_{n,r-1} \quad (r = 1,2, \ldots; V_{n,0} = 0) \end{aligned}$$

then

$$(2.4)\qquad U_{n,r} = \sum_{j=1}^{r} \mu_{n,j} , \quad V_{n,r} = \sum_{j=1}^{r} \nu_{n,j}$$

and hence (2.1) can be rewritten as

$$(2.5)\qquad \sum_{j=1}^{r} (\mu_{n,j}-\mu_{n-1,j}) = E[\{(X-\mu_{n-1,r})\, \chi_{Y-\nu_{n-1,r} > 0}\}^{+}] = \iint_{\substack{x>\mu_{n-1,r} \\ y>\nu_{n-1,r}}} (x-\mu_{n-1,r})\, dH(x, y) ,$$

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where z^+ denotes max (0, z). An optimal decision by I is to

$$(2.6)\qquad \left\{\begin{matrix}\text{accept}\\ \text{reject}\end{matrix}\right\} \text{ the observation, if } \left\{\begin{matrix} x > \mu_{n-1,r} \text{ and } y > \nu_{n-1,r} \\ \text{otherwise}\end{matrix}\right\}$$

This, together with (2.2) ∿ (2.4) give

$$(2.7)\qquad \sum_{j=1}^{r} (\nu_{n,j}-\nu_{n-1,j}) = \iint_{\substack{x>\mu_{n-1,r}\\ y>\nu_{n-1,r}}} (y-\nu_{n-1,r})dH(x,y) ,$$

and we find that player II, since he has to decide after I does, helplessly watch his opponent carries out his strategy and follows him if he accepts.

We need to introduce the following transformations.

$$(2.8)\qquad \begin{aligned} T_H(z, w) &\equiv \iint_{\substack{x>z\\ y>w}} (x-z)dH(x, y) , \\ S_H(z, w) &\equiv \iint_{\substack{x>z\\ y>w}} (y-w)dH(x, y) . \end{aligned}$$

These are apparently bivariate extensions of the transform

$$(2.9)\qquad T_F(z) \equiv \int_z^\infty (x-z)dF(x) ,$$

where F(·) is any univariate cdf with finite mean. The transform $T_F(z)$ is known to play an important role in optimal stopping problems. (<u>See</u> DeGroot [1; Chapter 13] and Sakaguchi [7], [9], [10], [11].) It is continuous, non-negative, convex and strictly decreasing on the set where it is positive. (DeGroot [1; Section 11.8])

With the transformations (2.8), we can rewrite (2.5) and (2.7) as

$$(2.10)\qquad \begin{aligned} \mu_{n,r} &= \mu_{n-1,r} + T_H(\mu_{n-1,r},\nu_{n-1,r}) - T_H(\mu_{n-1,r-1},\nu_{n-1,r-1}),\ 2 \leqq r \leqq n-1 \\ \mu_{n,1} &= \mu_{n-1,1} + T_H(\mu_{n-1,1},\nu_{n-1,1}) , \end{aligned}$$

and

$$(2.11)\qquad \begin{aligned} \nu_{n,r} &= \nu_{n-1,r} + S_H(\mu_{n-1,r},\nu_{n-1,r}) - S_H(\mu_{n-1,r-1},\nu_{n-1,r-1}),\ 2 \leqq r \leqq n-1 \\ \nu_{n,1} &= \nu_{n-1,1} + S_H(\mu_{n-1,1},\nu_{n-1,1}), \end{aligned}$$

respectively. We also have from (2.4)

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$$\sum_{j=1}^{n} \mu_{n,j} = U_{n,n} = nEX\,, \qquad n \geqq 1\,,$$

(2.12)

$$\sum_{j=1}^{n} \nu_{n,j} = V_{n,n} = nEY\,, \qquad n \geqq 1\,.$$

These equations (2.10) $\sim$ (2.12) define recursive relations for determining the values of $\mu_{n,r}$ and $\nu_{n,r}$. Now summarizing the above we can state:

Theorem 1. The values of the game, $U_{n,r}$ and $V_{n,r}$, are given by (2.4), where $\mu_{n,r}$ and $\nu_{n,r}$'s are determined by the recursive relations (2.10) $\sim$ (2.12). An optimal strategy for player I is given by (2.6), and when I accepts an observation an optimal decision by player II is to follow his opponent's decision (i.e., acception).

Three remarks are in order:

Remark 1. When we consider the case where player II makes the first decision on the current bivariate observation, we easily find, by the identical arguments to those used in this section, that Theorem 1 is still true, with an interchange of I and II in the second paragraph. Therefore, under the optimal strategies, it will never matter which side is required to decide first whether or not to use a rejection.

Remark 2. For the case of $r = 1$, the same result is obtained in Sakaguchi [8], [12], by a different approach. i.e., in the framework of a time-sequential non-zero-sum game with simultaneous moves by the players. In [8] the game is solved i.e., $\mu_{n,1}$ and $\nu_{n,1}$ are derived explicitly for the case when (X, Y) has uniform, normal and some other bivariate distributions.

Remark 3. We have the following conjecture: Both of $\{\mu_{n,r}\}$ and $\{\nu_{n,r}\}$ are non-decreasing in $n(\geqq r)$, for any fixed $r \geqq 1$, and non-increasing in r, $(1 \leqq r \leqq n)$, for any fixed n. However, at the time of publication of this paper we have not succeeded in proving nor disproving this conjecture.

Corollary 1. If (X, Y) is symmetrically distributed, then $\mu_{n,r} = \nu_{n,r}$ and hence $U_{n,r} = V_{n,r}$, for all n and $1 \leqq r \leqq n$.

Proof By hypothesis, X and Y have identical marginal distributions and the two conditional distributions are also identical. Therefore we can write as $H(x,y) = F(x)\, H(y|x) = F(y)\, H(x|y)$. Hence we have from (2.8)

$$T_H(z, w) = \int_{x>z} (x-z)\, dF(x) \int_{y>w} dH(y|x)$$

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$$S_H(z, w) = \int_{y>w} (y-w)\ dF(y) \int_{x>z} dH(x|y)$$

Therefore $T_H(z, w) = S_H(z, w)$, if $z = w$. Using this property and Eqs. (2.10) $\sim$ (2.12), the induction arguments on n, r give the proof of the corollary.

Example 1. Independent uniform-triangular distribution

Let $H(x, y) = xy^2$, $0 \leqq x, y \leqq 1$. Then $T_H(z, w) = \frac{1}{2}(1-z)^2(1-w^2)$ and $S_H(z, w) = \frac{1}{3}(1-z)(2+w)(1-w)^2$. For some small values of n and r, the values of $\mu_{n,r}$, $\nu_{n,r}$ $U_{n,r}$ and $V_{n,r}$ obtained form (2.4), (2.10) $\sim$ (2.12) are shown in Table 1. Now suppose that n = 10, r = 3. Observe (X_1, Y_1), and if $X_1 > 0.5766$ and $Y_1 > 0.7252$, first I and then II accept and (X_1, Y_1) is selected, and the optimal strategies for n = 9, r = 2, are followed thereafter. If (X_1, Y_1) is otherwise, I rejects the observation and the optimal strategies for n = 9, r = 3, are followed thereafter. The expected payoffs through the procedure to I and II will be 2.0020 and 2.3564, respectively.

Example 2. Independent normal distribution

Let H(x, y) have pdf

$$h(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp[-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)] , \tag{2.13}$$

where ρ, $-1 \leqq \rho \leqq 1$, is the correlation coefficient. Also let

$$\phi(x) \equiv (2\pi)^{-1/2} e^{-x^2/2} , \quad \Phi(x) \equiv \int_x^\infty \phi(t)dt , \quad \Psi(x) \equiv \phi(x) - x\Phi(x).$$

Then it can be shown (see, DeGroot [1 ; Section 11.9], Sakaguchi [7])

$$T_H(z, w) = \int_w^\infty \phi(y)dy \int_z^\infty (x-z) \frac{1}{\sqrt{1-\rho^2}} \phi(\frac{x-\rho y}{\sqrt{1-\rho^2}})\ dx$$
$$= \sqrt{1-\rho^2} \int_w^\infty \Psi(\frac{z-\rho y}{\sqrt{1-\rho^2}})\ \phi(y)\ dy .$$

If $\rho = 0$, this reduces to $T_H(z, w) = \Psi(z)\Phi(w)$. By Corollary 2 we have to compute $T_H(z, z) = \Phi(z)\Psi(z)$ only. Table 2 gives the values of $\mu_{n,r}$ and $U_{n,r}$ for some small values of n and r. We have used, for computation of the values of $\Psi(x)$, Table II of Raiffa and Schlaifer [5].

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Table 1

Results for Independent Uniform-Triangular Distribution

	r = 1	r = 2		r = 3	
	$\mu_{n,1}$ $\nu_{n,1}$	$\mu_{n,2}$ $\nu_{n,2}$	$U_{n,2}$ $V_{n,2}$	$\mu_{n,3}$ $\nu_{n,3}$	$U_{n,3}$ $V_{n,3}$
n = 1	.5000 .6667				
2	.5694 .7160	.4306 .6173	1.0000 1.3333		
3	.6146 .7475	.4857 .6536	1.1003 1.4011	.3996 .5939	1.5000 2.0000
4	.6474 .7700	.5278 .6892	1.1752 1.4592	.4414 .6264	1.6166 2.0856
5	.6727 .7872	.5610 .7129	1.2337 1.5001	.4777 .6538	1.7114 2.1539
6	.6931 .8010	.5881 .7318	1.2812 1.5328	.5084 .6764	1.7896 2.2092
7	.7099 .8123	.6106 .7475	1.3205 1.5598	.5346 .6954	1.8551 2.2552
8	.7243 .8219	.6297 .7606	1.3540 1.5825	.5570 .7114	1.9110 2.2939
9	.7366 .8301	.6463 .7719	1.3829 1.6020	.5766 .7252	1.9595 2.3272
10	.7474 .8373	.6608 .7818	1.4082 1.6191	.5938 .7373	2.0020 2.3564

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Table 2

Results for Independent Normal Distribution

	r = 1	r = 2		r = 3	
	$\mu_{n,1}$, $U_{n,1}$	$\mu_{n,2}$	$U_{n,2}$	$\mu_{n,3}$	$U_{n,3}$
n = 1	.0000				
2	.1995	-.1995	.0000		
3	.3295	-.0361	.2934	-.2934	.0000
4	.4242	.0846	.5088	-.1625	.3463
5	.4987	.1769	.6756	-.0552	.6204
6	.5597	.2521	.8118	.0323	.8441
7	.6115	.3142	.9257	.1050	1.0307
8	.6564	.3678	1.0242	.1669	1.1911
9	.6959	.4144	1.1103	.2199	1.3302
10	.7310	.4558	1.1868	.2667	1.4535

3. Trilateral Game and Reversibility.

As mentioned in Remark 1 of the previous section our bilateral time-sequential game has the property that the optimal strategy is "reversible" (by the terminology in RKD), that is, under the optimal strategies, it will never matter which side is required to decide first whether or not to use a rejection. We have a conjecture that the reversibility holds for any $k(\geq 3)$ person multilateral game, and we shall show it in this section for the case of $k = 3$.

The structure of the problem is the same as in the previous section. Let (X_i, Y_i, Z_i), $i=1, \ldots, n$, be iid trivariate random variables that can be observed one by one sequentially. Players I, II and III are about to select a set of r, $1 \leq r \leq n$, observations. The common cdf $H(x, y, z)$ of each r.v. is known to the players. Decision at any stage whether or not to use a rejection must be made first by I, and then by II and finally by III. Only when two or three players accept the observations, the triple of values is selected as one element of the set of r observations. (i.e., simple majority). Let (X_i', Y_i', Z_i'), $i = 1, \ldots, r$, be the members of the final set

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of r observations. The objectives of the players I, II and III are to maximize $E(\sum_{i=1}^{r} X'_i)$, $E(\sum_{i=1}^{r} Y'_i)$, and $E(\sum_{i=1}^{r} Z'_i)$, respectively. Let $V_i(n, r)$, i=1, 2, 3, be the expected playoff to player i, under the optimal procedure. We shall derive, in the following, the recursive relations satisfied by $V_i(n,r)$'s which will determine the optimal strategies of the players, in terms of the

$$\text{(3.1)} \qquad \begin{aligned} \mu_{n,r} &\equiv V_1(n, r) - V_1(n, r-1) , \\ \nu_{n,r} &\equiv V_2(n, r) - V_2(n, r-1) , \\ \omega_{n,r} &\equiv V_3(n, r) - V_3(n, r-1) , \end{aligned}$$

Let $(n, r \mid x, y, z)$ denote the state of the process in which the players have n r.v.'s and the obligation to select r among them, and the current observation they are facing with is (x, y, z). Also let $V_3^{rej-rej}(n, r \mid x, y, z)$ denote the conditional expected payoff to III, given that I and then II, in this order, have made their decisions rejection and rejection, respectively, in state $(n, r \mid x, y, z)$, and moreover that III can assume that all players will act "optimally" on all subsequent decisions. Let $V_3^{acc-rej}(n, r \mid x, y, z)$ and others be defined similarly. Then evidently we have

$$\text{(3.2)} \qquad \begin{cases} V_3^{rej-rej}(n, r \mid x, y, z) = V_3(n-1, r) , \\ V_3^{acc-acc}(n, r \mid x, y, z) = z + V_3(n-1, r-1) , \\ V_3^{acc-rej}(n, r \mid x, y, z) = V_3^{rej-acc}(n, r \mid x, y, z) \\ \qquad = \max\{z + V_3(n-1, r-1), V_3(n-1, r)\}, \end{cases}$$

and the optimal decision by III in succession to acc-rej or rej-acc in state $(n, r \mid x, y, z)$ is to

$$\begin{Bmatrix} \text{accept} \\ \text{reject} \end{Bmatrix} \text{ the observation, if } z \begin{Bmatrix} > \\ \leqq \end{Bmatrix} \omega_{n-1,r} .$$

Now as for player II, let $V_2^{rej}(n, r \mid x, y, z)$ and $V_2^{acc}(n, r \mid x, y, z)$ be similarly defined as in those for III. The superscripts ref. and acc. denote I's decisions already made in state $(n, r \mid x, y, z)$. Then it can be shown that

$$\text{(3.3a)} \qquad V_2^{rej}(n, r \mid x, y, z) = \max\{V_2(n-1, r) ,$$

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$$(y + V_2(n-1,\ r-1))\chi_{z+V_3(n-1,\ r-1)\ >\ V_3(n-1,\ r)}$$

$$+ V_2(n-1,\ r)\chi_{z+V_3(n-1,\ r-1)\ \leqq\ V_3(n-1,\ r)}\}$$

$$= V_2(n-1,\ r) + \{(y-\nu_{n-1,r})\chi_{z>\omega_{n-1,r}}\}^+$$

$$= \begin{cases} y + V_2(n-1,\ r-1), & \text{if } y > \nu_{n-1,\ r},\ z > \omega_{n-1,\ r} \\ V_2(n-1,\ r)\ , & \text{if otherwise} \end{cases}$$

(3.3b) $$V_2^{acc}(n,\ r \mid x,\ y,\ z) = \max\{(y+V_2(n-1,\ r-1))\chi_{z+V_3(n-1,\ r-1)\ >\ V_3(n-1,r)}$$

$$+ V_2(n-1,\ r)\chi_{z+V_3(n-1,\ r-1)\leqq V_3(n-1,\ r)},\ y + V_2(n-1,\ r-1)\}$$

$$= V_2(n-1,\ r) + \max\{(y-\nu_{n-1,r})\chi_{z>\omega_{n-1,r}}\ ,\ y - \nu_{n-1,\ r}\}$$

$$= \begin{cases} V_2(n-1,\ r)\ , & \text{if } y < \nu_{n-1,r}\ ,\ z<\omega_{n-1,\ r} \\ y + V_2(n-1,\ r-1), & \text{if otherwise} \end{cases}$$

Hereafter we shall call y, z and x to be <u>large</u> if $y > \nu_{n-1,\ r}$, $z > \omega_{n-1,r}$ and $x > \mu_{n-1,r}$, respectively. Similarly we call each r.v. <u>small</u> if the inequality sign is reversed. For present purposes we assume that the marginal distributions are continuous, so that each r.v. is either large or small. The optimal decisions for II and III, in this order, in state $(n,\ r \mid x,\ y,\ z)$, and the resulting payoffs to I can be deduced from (3.1) $\sim$ (3.3) and are summarized in Table 3.

By examining Table 3 and considering the process from the standpoint of I, we can see that the conditional expected payoff to I is

$$V_1(n,\ r \mid x,\ y,\ z) = \begin{cases} V_1(n-1,\ r)\ , & \text{if y small, z small} \\ x + V_1(n-1,\ r)\ , & \text{if y large, z large} \\ \max\{V_1(n-1,\ r),\ x+V_1(n-1,\ r-1)\}, & \text{if otherwise} \end{cases}$$

$$= V_1(n-1,\ r) + \begin{cases} 0\ , & \text{if y small, z small} \\ x - \mu_{n-1,r}\ , & \text{if y large, z large} \\ (x - \mu_{n-1,r})^+\ , & \text{if otherwise} \end{cases}$$

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Table 3

Optimal Decision by II-III, and the Resulting Payoffs to I

If I rejects, then			If I accepts, then		
y \ z	Small	Large	y \ z	Small	Large
Small	{rej - — / acc-rej} $V_1(n-1, r)$	rej- — $V_1(n-1, r)$	Small	rej-rej $V_1(n-1, r)$	{acc- — / rej-acc} $x+V_1(n-1, r-1)$
Large	{rej- — / acc-rej} $V_1(n-1, r)$	acc-acc $x+V_1(n-1, r-1)$	Large	acc- — $x+V_1(n-1, r-1)$	{acc- — / rej-acc} $x+V_1(n-1, r-1)$

and therefore

$$(3.4)\qquad V_1(n, r) = E\, V_1(n, r \mid X, Y, Z) = V_1(n-1, r) + \iiint_{R_{n-1,r}} (x - \mu_{n-1,r})\, dH(x, y, z) ,$$

where we have defined $R_{n-1,r}$ to be the set of all (x, y, z) which satisfies at least two among $x > \mu_{n-1,r}$, $y > \nu_{n-1,r}$ and $z > \omega_{n-1,r}$.

Also from Table 3 we find that the optimal decision for I is : If either y large & z small, or y small & z large holds, $\begin{Bmatrix}\text{accept}\\ \text{reject}\end{Bmatrix}$ if $x \begin{Bmatrix} > \\ < \end{Bmatrix} \mu_{n-1,r}$. In the other cases acception and rejection are indifferent for I. This decision by I, together with (3.3), give

$$V_2(n, r \mid x, y, z) = V_2(n-1, r) + \begin{cases} y - \nu_{n-1,r} , & \text{if } (x, y, z) \in R_{n-1,r} \\ 0 , & \text{if otherwise} \end{cases}$$

and hence

$$(3.5)\qquad V_2(n, r) = E\, V_2(n, r \mid X, Y, Z) = V_2(n-1, r) + \iiint_{R_{n-1,r}} (y - \nu_{n-1,r})\, dH(x, y, z) ,$$

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Combining the optimal decision by I, with those by II-III in Table 3 we obtain the optimal decisions by I, II and III in this order, which are summarized in Table 4.

Table 4

Optimal Decisions by I-II-III.

x Small			x Large		
y \ z	Small	Large	y \ z	Small	Large
Small	rej-rej- —— rej-acc-rej acc-rej-rej	rej-rej- ——	Small	*	acc-acc- —— acc-rej-acc
Large	rej-rej- —— rej-acc-rej	rej-acc-acc acc-acc- —— acc-rej-acc	Large	acc-acc- ——	**

*, ** Same as when x small

Finally we have, from (3.1), accordingly to the decisions by I-II.

$$V_3(n, r \mid x, y, z) - V_3(n-1, r) = \begin{cases} 0, & \text{if rej-rej} \\ z-\omega_{n-1,r}, & \text{if acc-acc} \\ (z-\omega_{n-1,r})^+, & \text{if otherwise} \end{cases}$$

$$= \begin{cases} z-\omega_{n-1,r}, & \text{if either (rej-acc or acc-rej) \& z large,} \\ & \text{or acc-acc ,} \\ 0, & \text{if otherwise ,} \end{cases}$$

which, by examining Table 4, becomes

$$= \begin{cases} z-\omega_{n-1,r}, & \text{if } (x, y, z) \in R_{n-1,r} \\ 0, & \text{if otherwise.} \end{cases}$$

Hence we obtain

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(3.6) $$V_3(n, r) = E\, V_3(n, r \mid X, Y, Z) = V_3(n-1, r) + \iiint_{R_{n-1,r}} (z-\omega_{n-1,r})\, dH(x, y, z)$$

Thus we have reached to state :

Theorem 2. The values of the three-person game are given by

(3.7) $$V_1(n, r) = \sum_{j=1}^{r} \mu_{n,j}, \quad V_2(n, r) = \sum_{j=1}^{r} \nu_{n,j} \text{ and } V_3(n, r) = \sum_{j=1}^{r} \omega_{n,j},$$

where $\mu_{n,r}$, $\nu_{n,r}$ and $\omega_{n,r}$'s are determined by the recursive relations

(3.8) $$\mu_{n,r} = \mu_{n-1,r} + T_1(\alpha_{n-1,r}) - T_1(\alpha_{n-1,r-1}), \qquad 2 \leqq r \leqq n-1$$

$$\mu_{n,1} = \mu_{n-1,1} + T_1(\alpha_{n-1,1}), \qquad n \geqq 2$$

(3.9) $$\nu_{n,r} = \nu_{n-1,r} + T_2(\alpha_{n-1,r}) - T_2(\alpha_{n-1,r-1}), \qquad z \leqq r \leqq n-1$$

$$\nu_{n,1} = \nu_{n-1,1} + T_2(\alpha_{n-1,1}), \qquad n \geqq 2$$

(3.10) $$\omega_{n,r} = \omega_{n-1,r} + T_3(\alpha_{n-1,r}) - T_3(\alpha_{n-1,r-1}), \qquad z \leqq r \leqq n-1$$

$$\omega_{n,1} = \omega_{n-1,1} + T_3(\alpha_{n-1,1}), \qquad n \geqq 2$$

with the boundary conditions

(3.11) $$\sum_{j=1}^{n} \mu_{n,j} = nEX, \quad \sum_{j=1}^{n} \nu_{n,j} = nEY, \quad \text{and} \quad \sum_{j=1}^{n} \omega_{n,j} = nEZ,$$

where $\alpha_{n-1,r}$ denotes the triple $(\mu_{n-1,r}, \nu_{n-1,r}, \omega_{n-1,r})$ and $T_i(\alpha_{n-1,r})$, $i = 1,2,3$, denotes the triple integral in the righthand side of Eqs. (3.4), (3.5), and (3.6), respectively. The optimal strategies for the players are given in Table 4.

Remark 4. Corollary 1 can be extended to trivariate r.v.'s. So that, if (X, Y, Z) is symmetrically distributed, then $\mu_{n,r} = \nu_{n,r} = \omega_{n,r}$, and hence $V_1(n, r) = V_2(n, r) = V_3(n, r)$, for all n and $1 \leqq r \leqq n$.

Example 3. Symmetric, independent uniform distribution.

Let $H(x, y) = xy$, $0 \leqq x,y \leqq 1$. Then, by (2.8), $T_H(s, t) = \frac{1}{2}(1-s)^2(1-t)$. Also let $H(x, y, z) = xyz$, $0 \leqq x,y,z \leqq 1$. Then

$$T_1(s, t, u) = \iiint_R (x - s)\, dx\, dy\, dz,$$

where

$$R \equiv \{(x, y, z) \mid \text{At least two among } x > s, y > t \text{ and } z > u \text{ hold}\}$$

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becomes

$$T_1(s, t, u) = (\tfrac{1}{2} - s)(1 - t)(1 - u) + \tfrac{1}{2}(1 - s)^2(t+u-2tu) .$$

By Corollary 1 and Remark 4 we need to determine $\mu_{n,r}$'s only. For some small values of n and r, the values of $\mu_{n,r}$ and the common expected payoffs to the players are obtained from (2.10) ∿ (2.12) for the two-person game, and from (3.7) ∿ (3.11) for the three-person game. The values are shown in Table 5.

Table 5 Results for Independent Uniform Distributions

	r = 1	r = 2		r = 3	
	$\mu_{n,1}$*	$\mu_{n,2}$	Exp. Payoff*	$\mu_{n,3}$	Exp. Payoff
n = 1	.5000 .5000				
2	.5625 .5625	.4375 .4375	1.0000 1.0000		
3	.6044 .5976	.4846 .5000	1.0890 1.0976	.4110 .4024	1.5000 1.5000
4	.6353 .6208	.5221 .5394	1.1574 1.1602	.4447 .4606	1.6021 1.6208
5	.6596 .6373	.5524 .5672	1.2120 1.2045	.4758 .5000	1.6878 1.7045
6	.6793 .6496	.5775 .5883	1.2568 1.2379	.5031 .5291	1.7599 1.7670
7	.6958 .6592	.5987 .6048	1.2945 1.2640	.5267 .5518	1.8218 1.8158
8	.7099 .6668	.6170 .6181	1.3269 1.2849	.5474 .5701	1.8743 1.8550
9	.7221 .6729	.6329 .6292	1.3550 1.3021	.5656 .5853	1.9206 1.8874
10	.7328 .6780	.6469 .6385	1.3797 1.3165	.5819 .5980	1.9616 1.9145

* The upper (lower) numbers relate to the two (three)-person game.

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Remark 5. This owes to a private communication by J.B. Kadane and [4]. The results on reversibility in this and previous sections can be generalized to any number of players so that the utility function of the k-th player is an arbitrary function

$$\psi_k(x'_{1k}, x'_{2k}, \ldots, x'_{rk}) , \quad k = 1, 2, \ldots, \ell ,$$

of the k-th coordinate of the r vectors $(x'_{i1}, x'_{i2}, \ldots, x'_{i\ell})$ chosen, and the observations need not be iid. Also simple majority for two and three players cases can be replaced by concept of a winning class $\mathcal{W}$, where $\{1, 2, \ldots, \ell\} \in \mathcal{W}$ and if $W \in \mathcal{W}$ and $W' \supset W$, then $W' \in \mathcal{W}$. The essential thing that makes reversibility work is the cost structure, particularly the aspect that if an observation is to be rejected it does not matter to anyone who or how many reject it. Of course generalizations of this sort make actual computations of the optimal strategies very burdensome, but the principle of backward induction still applies.

4. Bilateral Game, Each with a Given Number of Rejections.

We have shown in Section 2 the optimal strategy is "reversible" in the bilateral time-sequential game, that is, under the optimal strategies, it does not matter to either side whether it decides first or second. We have to note that this is simply because each player, especially player I, who decides first, is at any stage (except the final few stages), at all free to use a rejection. From the discussions in Section 2 we see that a complete list of the optimal decisions in state (n, r) by I-II is shown below and hence,

x \ y	Small	Large
Small*	{ rej - —— acc-rej	rej - ——
Large	{ rej - —— acc - rej	acc - acc

* In state $(n, r \mid x, y)$, x small (large) means $x < (>) \mu_{n-1,r}$ and y small (large) means $y < (>) \nu_{n-1,r}$.

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player I is indifferent between rejection and acceptance if y is small. Therefore an optimal strategy for I stated in Theorem 1 forces player II to helplessly watch his oponent carry out his strategy and follow it. On the contrary, if the players are allowed to use at most a and b rejections, respectively, the situation becomes different. Consider the circumstance, for example, where x and y are both small. I must choose his decision weighing the advantage of using his right of rejection to avoid selecting the small value of x against that of no using it and letting his opponent use his own right of rejection.

Now in this final section we shall consider the two-person bilateral process in which the number of rejections available to each side is given beforehand. It is shown that it does matter to either side whether it decides first or second. The problem in this section has a close connection with the recent work in RKD, in which the objective of one player is to maximize, and the other to minimize, the product of positive r.v.'s, and the similar result is obtained as that is discussed in the following.

For any two real numbers s and t, we define

$$(4.1)\qquad M_1(s,\ t) \equiv \iint_{\substack{x>s\\y>t}} x\ dH(x,\ y) \quad \text{and} \quad M_2(s,\ t) \equiv \iint_{\substack{x>s\\y>t}} y\ dH(x,\ y)\ .$$

It follows, from (2.8), that

$$(4.2)\qquad \begin{aligned} M_1(s,\ t) &= T_H(s,\ t) + s\ \overline{H}(s,\ t) \\ M_2(s,\ t) &= S_H(s,\ t) + t\ \overline{H}(s,\ t) \end{aligned}$$

where $$\overline{H}(s,\ t) \equiv \iint_{\substack{x>s\\y>t}} dH(x,\ y)\ .$$

Also note, from (2.8) and (2.9), that

$$M_1(s,\ t) = T_F(s) + s\ \overline{G}(s)\ ,\ \text{for any } s \text{ and any } t \leqq y_0 \equiv \inf\{y|G(y) > 0\},$$

$$M_2(s,\ t) = T_G(t) + t\ \overline{F}(t)\ ,\ \text{for any } t \text{ and any } s \leqq x_0 \equiv \inf\{x|F(x) > 0\}.$$

where F and G are the marginal cdf's of X and Y, respectively, and $\overline{F}(t) = 1-F(t)$, etc.

Let $\sigma = (j,\ a,\ b)$ denote the state of the process in which j r.v.'s remain to be selected and the players I and II have currently a and b, respectively, times of rejection still remaining. Also let

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$\alpha = (j, a-1, b)$, $\beta = (j, a, b-1)$, and $\gamma = (j-1, a, b)$.

Then these states α, β and γ represent the three possible states that can be reached from σ by the decision or decision pair rej- —— , acc-rej, and acc-acc, respectively.

Let $V_i(\sigma)$, i = 1, 2, be the value of the game for player I, and II, respectively, in state σ. Then starting from the optimality equations

$$(4.3) \qquad V_1(\sigma) = E[\max\{V_1(\alpha),\ (X + V_1(\gamma))\chi_{Y>V_2(\beta)-V_2(\gamma)} + V_1(\beta)\chi_{Y\leq V_2(\beta)-V_2(\gamma)}\}] ,$$

$$(4.4) \qquad V_2(\sigma) = E\, Z_\sigma(X, Y) ,$$

where the r.v. in (4.4) is defined by

$$Z_\sigma(X, Y) = \begin{cases} V_2(\alpha) , & \text{if I rejects} , \\ \max(Y+V_2(\gamma) ,\ V_2(\beta)) , & \text{if I accepts} , \end{cases}$$

we can obtain the following theorem, which shows that the main result in RKD goes through with a slight modification. The proof will be omitted since it so closely parallels the proof of the main result in RKD.

Theorem 3. (i) $V_i(\sigma)$, i = 1, 2, satisfy the recursive relations

$$(4.5) \qquad \begin{aligned} V_1(\sigma) &= \bar{G}(c_2)V_1(\alpha) + G(c_2)V_1(\beta) + T_H(c_1, c_2) \\ V_2(\sigma) &= (\bar{G}(c_2) - \bar{H}(c_1, c_2))V_2(\alpha) + (G(c_2) + \bar{H}(c_1, c_2))V_2(\beta) + S_H(c_1, c_2) \end{aligned}$$

where $c_1 \equiv V_1(\alpha) - V_1(\gamma)$ and $c_2 \equiv V_2(\beta) - V_2(\gamma)$. The optimal decisions by I-II are shown in Table 6 :

Table 6 Optimal Decisions by I-II

x \ y	Small	Large
Small*	acc-rej	rej- ——
Large	acc-rej	acc-acc

* x large (small) means x > (<) c_1, and y large (small) means y > (<) c_2

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(ii) The boundary conditions for (4.5) are given by :

$$(4.6)\quad \begin{cases} \text{For } j = 0,\ V_1(0, a, b) = V_2(0, a, b) = 0, \\ \text{For } a = 0,\ V_1(j, 0, b) = v_1^{**}(j, b) \text{ and } V_2(j, 0, b) = v_2^{*}(j, b), \\ \text{For } b = 0,\ V_1(j, a, 0) = v_1^{*}(j, a) \text{ and } V_2(j, a, 0) = v_2^{**}(j, a), \end{cases}$$

where the four functions $v_1^*(j, a)$ etc. are determined by

$$(4.7)\quad v_1^*(j, a) = v_1^*(j, a-1) + T_F(v_1^*(j, a-1) - v_1^*(j-1, a))$$
$$(j \geqq 1 ;\ v_1^*(0, a) \equiv 0,\ v_1^*(j, 0) = jEX)$$

$$(4.8)\quad v_2^*(j, b) = v_2^*(j, b-1) + T_G(v_2^*(j, b-1) - v_2^*(j-1, b))$$
$$(j \geqq 1 ;\ v_2^*(0, b) \equiv 0,\ v_2^*(j, 0) = jEY)$$

$$(4.9)\quad v_2^{**}(j, a) = F(d_1)v_2^{**}(j, a-1) + \overline{F}(d_1)v_2^{**}(j-1, a) + M_2(d_1, -\infty)$$
$$(j \geqq 1 ;\ v_2^{**}(0, a) \equiv 0,\ v_2^{**}(j, 0) = jEY)$$

$$(4.10)\quad v_1^{**}(j, b) = G(d_2)v_1^{**}(j, b-1) + \overline{G}(d_2)v_1^{**}(j-1, b) + M_1(-\infty, d_2)$$
$$(j \geqq 1 ;\ v_1^{**}(0, b) \equiv 0,\ v_1^{**}(j, 0) = jEX)$$

in the last two of which we have set $d_1 \equiv v_1^*(j, a-1) - v_1^*(j-1, a)$ and $d_2 \equiv v_2^*(j, b-1) - v_2^*(j-1, b)$.

Note that $v_i^*(v_i^{**})$ represents the expected value that can be attained by player i in his (his opponent's) one-sided version when only he (his opponent) has some number of rejections left.

On the contrary of the results in the previous sections (see, Remark 1), the bilateral time-sequential game in this section has the property that the optimal stategy is "irreversible", that is, under the optimal strategies, it will matter which side is required to decide first. When we consider the case where player II makes the first decision on the current bivariate observation, we can find, by the similar arguments to those used in this section, that the values of the $\tilde{V}_i(\sigma)$, i = 2, 1, to player i, satisfies, instead of (4.5),

$$(4.11)\quad \begin{aligned} \tilde{V}_2(\sigma) &= F(\tilde{c}_1)\tilde{V}_2(\alpha) + \overline{F}(\tilde{c}_1)\tilde{V}_2(\beta) + S_H(\tilde{c}_1, \tilde{c}_2) , \\ \tilde{V}_1(\sigma) &= (F(\tilde{c}_1) + \overline{H}(\tilde{c}_1, \tilde{c}_2))\tilde{V}_1(\alpha) + (\overline{F}(\tilde{c}_1) - \overline{H}(\tilde{c}_1, \tilde{c}_2))\tilde{V}_1(\beta) + T_H(\tilde{c}_1, \tilde{c}_2), \end{aligned}$$

where $\tilde{c}_1 \equiv \tilde{V}_1(\alpha) - \tilde{V}_1(\gamma)$ and $\tilde{c}_2 = \tilde{V}_2(\beta) - \tilde{V}_2(\gamma)$. The boundary conditions remain unchanged as in (4.6) ∿ (4.10) with V_i, i = 1, 2, replaced by $\tilde{V}_i$.

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Remark 6. In order to derive (4.5), we have to suppose that, for any history of the process, the relation expressed by the inequality $V_1(\alpha) \leqq V_1(\beta)$ holds. That is, player I stands advantageous if one of II's available rejections is transfered to I for his own use. Similarly, in order to derive (4.11), we have to suppose that the inequality $\tilde{V}_2(\alpha) \geqq \tilde{V}_2(\beta)$ is valid, i.e., II finds it at least as preferable to be in state α as in state β. These two assumptions seem to be reasonable, but unfortunately the conjecture that these can be proven by induction starting from (4.6) ∿ (4.10) is very roundabout to be assertained. For this point, see RKD [6 ; p.907], and a recent paper by DeGroot and Kadane [2].

Remark 7. The problem in this section is a generalization of the problems of optimal selection discussed by Gilbert and Mosteller [3 ; Sec,5c] and Sakaguchi [10 ; Sec. 2]. In these problems $X_i = Y_i$, w.p. 1, for any i, $\{X_i\}$ is iid with a common cdf F(x), and (j, a, b) = (j, n-j, 0). The restriction b = 0 means that there is a single decision maker. With σ = (r, n-r, 0), α = (r, n-r-1, 0), γ = (r-1, n-r, 0) and denoting $V_1(\sigma)$ as V(r, n-1), etc, Equations (4.3) and (4.6) become

$$V(r, n-r) = E[\max\{V(r, n-r-1) , X + V(r-1, n-r)\}] ,$$

and

$$\begin{cases} \text{For} \quad j = 0 , \quad V(0, a) = 0 \\ \text{For} \quad a = 0 , \quad V(j, 0) = j\,EX , \end{cases}$$

respectively. It is easy to show that $V(r, n-r) = \sum_{j=1}^{r} \mu_{n,j}$, and the optimal decision for I in state σ is to accept if and only if $x \geqq \mu_{n-1,r}$, where numbers $\mu_{n,r}$ are determined by

$$\mu_{n,r} = \mu_{n-1,r} + T_F(\mu_{n-1,r}) - T_F(\mu_{n-1,r-1}) , \qquad (2 \leqq r \leqq n-1) ,$$

$$\mu_{n,1} = \mu_{n-1,1} + T_F(\mu_{n-1,1}) , \qquad (n \geqq 2 ; \mu_{1,1} = EX) .$$

Example 4 One rejection each — Independent uniform distribution

Let H(x, y) = xy, $0 \leqq x,y \leqq 1$, and j = a = b = 1. Then

$$T_F(s) = T_G(s) = \frac{1}{2} (1-s)^2 , \qquad 0 \leqq s \leqq 1$$

and, for any $0 \leqq s, t \leqq 1$,

$$T_H(s, t) = \frac{1}{2} (1-s)^2(1-t) , \quad S_H(s, t) = \frac{1}{2} (1-s)(1-t)^2$$

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$$M_1(s, t) = \frac{1}{2}(1-s^2)(1-t) , \quad M_2(s, t) = \frac{1}{2}(1-s)(1-t^2) .$$

We obtain, from (4.7) $\sim$ (4.10) ,

$$v_1^*(1, 1) = v_1^*(1, 0) + T_F(v_1^*(1, 0)) = \frac{1}{2} + T_F(\frac{1}{2}) = \frac{5}{8}$$

$$v_2^*(1, 1) = v_2^*(1, 0) + T_F(v_2^*(1, 0)) = \frac{1}{2} + T_G(\frac{1}{2}) = \frac{5}{8}$$

$$v_2^{**}(1, 1) = F(\frac{1}{2})v_2^{**}(1, 0) + M_2(\frac{1}{2}, 0) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$$

$$v_1^{**}(1, 1) = G(\frac{1}{2})v_1^{**}(1, 0) + M_1(0, \frac{1}{2}) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$$

and hence, by (4.6),

$$V_1(\alpha) = V_1(1, 0, 1) = v_1^{**}(1, 1) = 1/2$$

$$V_2(\alpha) = V_2(1, 0, 1) = v_2^{*}(1, 1) = 5/8$$

$$c_1 = c_2 = \frac{1}{2}$$

$$V_1(\beta) = V_1(1, 1, 0) = v_1^{*}(1, 1) = 5/8$$

$$V_2(\beta) = V_2(1, 1, 0) = v_2^{**}(1, 1) = 1/2$$

Therefore (4.5) gives, for $\sigma = (1, 1, 1)$,

$$V_1(\sigma) = \bar{G}(\frac{1}{2})V_1(\alpha) + G(\frac{1}{2})V_1(\beta) + T_H(\frac{1}{2}, \frac{1}{2}) = 5/8$$

$$V_2(\sigma) = (\bar{G}(\frac{1}{2}) - \bar{H}(\frac{1}{2}, \frac{1}{2}))V_2(\alpha) + (G(\frac{1}{2}) + \bar{H}(\frac{1}{2}, \frac{1}{2}))V_2(\beta) + S_H(\frac{1}{2}, \frac{1}{2}) = 19/32$$

The optimal decisions at the first stage are given by Table 6 with $c_1 = c_2 = 1/2$.

Also (4.11) gives

$$\hat{V}_1(\sigma) = (F(\frac{1}{2}) + \bar{H}(\frac{1}{2}, \frac{1}{2}))V_1(\alpha) + (\bar{F}(\frac{1}{2}) - \bar{H}(\frac{1}{2}, \frac{1}{2}))V_1(\beta) + T_H(\frac{1}{2}, \frac{1}{2}) = 19/32$$

since $\hat{V}_1(\alpha) = V_1(\alpha)$ and $\hat{V}_1(\beta) = V_1(\beta)$ for $\sigma = (1, 1, 1)$. Thus I can improve his expectation by $V_1(\sigma) - \hat{V}_1(\sigma) = 5/8 - 19/32 = 1/32$ by deciding first rather than deciding second.

Example 5. One rejection each—Bivariate normal distribution.

Consider bivariate normal distribution (2.13) with $\rho = 0$. Then for any s, t

$$T_F(s) = T_G(s) = \Psi(s) ,$$

$$T_H(s, t) = \Psi(s)\,\Phi(t) , \qquad S_H(s, t) = \Phi(s)\,\Psi(t) , \text{ and}$$

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$$M(s, t) = \Phi(t)\int_s^\infty x\,\phi(x)\,dx\ , \quad M_2(s, t) = \Phi(s)\int_t^\infty y\,\phi(y)\,dy.$$

It can easily be shown that for $\sigma = (1, 1, 1)$, we have $V_1(\alpha) = V_2(\beta) = 0$, $V_2(\alpha) = V_1(\beta) = (2\pi)^{-1/2}$, and $V_1(\sigma) = (2\pi)^{-1/2} = 0.3989$, $V_2(\sigma) = \frac{3}{4}\cdot(2\pi)^{-1/2} = 0.2992$. The optimal decisions at the first stage are given by Table 6 with $c_1 = c_2 = 0$. Moreover, by symmetry, I can improve his expectation by $\frac{1}{4}\cdot(2\pi)^{-1/2} = 0.0997$ by deciding first rather than deciding second.

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