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TURNAROUND TIME EQUATIONS IN QUEUEING NETWORKS

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Abstract For Jackson type queueing networks, we consider a mean travelling time for a customer between any two stations in a closed case, or a mean sojourn time for a customer until the system departure once he arrives at each station in an open case, and show that these are given by the solutions of certain linear equations, saying turnaround time equations. These results can be extended to the more general queueing networks introduced by Baskett et al. [1].

1. Introduction

Since Jackson reported the classical papers [5], [6], many authors have concerned queueing network models. Most of these works devoted to find the equilibrium distribution for a network of queues, and the distribution could be expressed as a product form. Especially, Baskett, Chandy, Muntz, and Palacios [1] showed this for a fairly general queueing network which includes different classes of customers and general service time distributions for certain queueing disciplines. Their result can cover the classical models studied by Jackson and others.

Using these equilibrium state probabilities, methods or computational algolithms obtaining important measures are given by Buzen [2], Reiser and Kobayashi [8], and Chandy, Herzog, and Woo [3]. The most of these measures are server-oriented ones such as a mean queue length, and customer-oriented ones are a little.

In this paper, we consider a mean travelling time for a customer between any two stations in a closed case, or a mean sojourn time for a customer until the system departure once he arrives at each station in an open case, and show that these are given by the solutions of certain linear equations, saying turnaround time equations. These problems is also presented in the survey paper by Lemoine [7]. We first treat a simple closed system in the next section, and in the subsequent sections, we consider an open system and more general system introduced by Baskett et al.

2. Closed system

Consider a queueing network with fixed N customers who circulate the system and never leave there, such as repairman problems, which is called a closed system. The system consists of M stations labeled by 1,2,...,M. The service time distribution at station i is negative exponential with rate $\mu_i(n_i)$, where n_i is the queue length of station i including customers on service ($\mu_i(0)=0$). The queueing discipline at each stage is FCFS. A customer who completes the service at station i proceeds to station j instantaneously with probability $p_{i,j}$ ($\Sigma_{j=1}^{M} p_{i,j}=1, i=1,2,\ldots,M$). We assume that the transition matrix $(p_{i,j})$ is defining an ergodic Markov chain with state space {1,2,...,M}, and its stationary distibution is $\{p_1,p_2,\ldots,p_M\}$, ($\Sigma_{i=1}^{M} p_i=1,\ p_i>0\ ; i=1,2,\ldots,M$). At any time, the system state can be expressed as $S=(n_1,n_2,\ldots,n_M)$, where n_i is a nonnegative integer, $\Sigma_{i=1}^{M} n_i=N$. We write e_i for a vector which has a 1 in the i th place and other components zero, $|S|$ for the sum of n_i 's.

Let P_N be the time continuous stationary probability distribution of the closed system, then by Jackson [6], we have

$$(2.1)\quad P_N(S)= \frac{1}{G(N)} \Pi_{i=1}^{M} \Pi_{k=1}^{n_i} \frac{p_i}{\mu_i(k)} \quad ,$$

where G(N) is normalizing constant, and can be written as

$$G(N)= \Sigma_{|S|=N}\ \Pi_{i=1}^{M}\ \Pi_{k=1}^{n_i} \frac{p_i}{\mu_i(k)} \ .$$

If $n_i=0$, we assume

$$\Pi_{k=1}^{n_i} \frac{p_i}{\mu_i(k)} = 1.$$

Now, we consider the time epocks $\{t_n;\ n=\ldots,-1,0,1,\ldots, t_n \leq t_{n+1}\}$ at which a service completes at one station, and the sequences of random elements $\{X_n;\ n=\ldots,-1,0,1,\ldots\}$,$\{Y_n;\ n=\ldots,-1,0,1,\ldots\}$ which represent the system state at time t_n-0, and t_n+0 respectively. If, at time t_n, a customer completes the service at station i and proceeds to station j, and other N-1 customers are distributed as $S=(n_1,n_2,\ldots,n_m)$ where $|S|=N-1$, then we denote

$X_n=(i,S),\ Y_n=(j,S)$,

respectively. Clearly, $\{X_n\}$ and $\{Y_n\}$ are Markov chains. Let P_X and P_Y

be the stationary probability distributions of $\{X_n\}$ and $\{Y_n\}$ respectively. We have

$$P_Y(j,s) = \Sigma_{i=1}^{M} p_{i,j}\, P_X(i,s),$$

and the following lemma.

Lemma 1.

$$P_X(i,s) = P_Y(i,s) = p_i\, P_{N-1}(s),$$

where P_{N-1} is the stationary probability distribution of the closed system with N-1 customers.

Proof. Considering the rate of the service completion at station i at system state $s+e_i$, we can write

$$P_X(i,s) = \frac{1}{C}\mu_i(n_i+1)\, P_N(s+e_i),$$

where

$$C = \Sigma_{|s'|=N}\, P_N(s')\,\Sigma_{j=1}^{M}\mu_j(n_j'),\quad s'=(n_1',n_2',\ldots,n_M').$$

Then we have, using (2.1),

$$P_X(i,s) = \frac{1}{C}\,\frac{1}{G(N)}\, p_i\, \Pi_{j=1}^{M}\, \Pi_{k=1}^{n_j}\, \frac{p_j}{\mu_j(k)}$$

$$= p_i\, P_{N-1}(s)\,\frac{G(N-1)}{C\, G(N)}$$

Now, from

$$\Sigma_{i=1}^{M} p_i = 1, \text{ and } \Sigma_{|s|=N-1}\, P_{N-1}(s)=1$$

We obtain the desired result.

For $P_Y(i,s)$, we can derive it similarly using the next relations.

$$p_i = \Sigma_{j=1}^{M}\, p_j\, p_{j,i} \quad (i=1,2,\ldots,M).$$ Q.E.D.

Lemma 1 means that, when a customer completes the service at station i and he proceeds to station j, the conditional probability that the system state is s except for him is $P_{N-1}(s)$ independently of station i and j

Let λ_i and L_i be the mean arrival rate and the mean queue length (including customers on service) at station i respectively (i=1,2,...,M). From (2.1), we obtain the following expressions.

$$\lambda_i = \Sigma_{|S|=N} P_N(S) \quad (\Sigma_{j=1}^{M} \mu_j(n_j)\, p_{j,i}),$$

$$L_i = \Sigma_{|S|=N} n_i P_N(S) \ .$$

Let w_i be the mean waiting time (including service time) at station i, then, from stationarity, we have Little's formula,

$$w_i = L_i \ / \ \lambda_i.$$

For the case where

$$\mu_i(n_i) = n_i\ \mu_i\ (n_i \leq m_i), \quad = m_i\ \mu_i\ (n_i > m_i)$$

(λ_i is a positive constant, m_i is a positive integer; i=1,2,..., M), efficient algolithms obtaining λ_i 's and L_i 's are given by Buzen [2], and Chandy, Herzog and Woo [3]. An implicit form of Lemma 1 is also given in [3].

Now, we will deduce the turnaround time equations. Let $W_{i,1}$ be the mean travelling time for a customer who arrives at station i to reach the 1 st station at first. Then, we have the following theorem.

Theorem 1.

$W_{i,1}$'s (i=1,2,...,M) are given by the solutions of the linear equations for $W_{i,1}$'s,

$$(2.2) \quad W_{j,1} = w_j + \Sigma_{k=2}^{M} p_{j,k} W_{k,1} \quad (j=2,3,\ldots,M).$$

Turnaround time to the 1 st station $W_{1,1}$ can be written as

$$(2.3) \quad W_{1,1} = p_{1,1} w_1 + \Sigma_{j=2}^{M} p_{1,j} W_{j,1}.$$

Proof.

Let w(i,S) and W(i,S) be, respectively, the (conditional) mean waiting time at station i and the (conditional) mean travelling time from station i to the 1 st station of a customer who joints the queue of station i to find that the system state is S except for him, where $|S|$ = N-1. $p_i(S,S')$ be the joint probability that he finds the system state S at his arrival and S' at his service completion except for him.

Then, we have from Lemma 1,

$$(2.4) \quad \Sigma_{|s'|=N-1} P_i(s,s') = \Sigma_{|s'|=N-1} P_i(s',s) = P_{N-1}(s),$$

and

$$W(i,s) = w(i,s) + \Sigma_{|s'|=N-1} \Sigma_{j=2}^{M} \frac{P_i(s,s')}{P_{N-1}(s)} p_{i,j} W(j,s').$$

Therefore, using (2.4), we obtain

$$\begin{aligned} W_{i,1} &= \Sigma_{|s|=N-1} P_{N-1}(s) W(i,s) \\ &= \Sigma_{|s|=N-1} P_{N-1}(s) \{w(i,s) \\ &\quad + \Sigma_{|s|=N-1} \Sigma_{j=2}^{M} \frac{P_i(s,s')}{P_{M-1}(s)} p_{i,j} W(j,s')\} \\ &= w_i + \Sigma_{j=2}^{M} p_{i,j} \Sigma_{|s'|=N-1} P_{N-1}(s') W(j,s') \\ &= w_i + \Sigma_{j=2}^{M} p_{i,j} W_{j,1} . \end{aligned}$$

The equation (2.3) is obviously. Q.E.D.

It can be deduced easily that, from the ergodic assumption on the transition matrix of $(p_{i,j})$, the equation (2.2) always has unique non-negative solutions.

When the mean service time S_i at station i (i=1,2,...,M) is well defined, the total service time in $W_{i,1}$ can be obtained if we put S_i for w_i in (2.2).

3. Open system

If, in the queueing network considered in the previous section, customers arrive from an exogeneous source and leave the system after receiving several services, we say this system is open. Let the input process be Poisson with rate $\lambda(n)$ which is depend to the number of the total customers in the system n. An arriving customer proceeds to station i with probability $p_{0,i}$ (i=1,2,...,M). A customer who completes the service at station i leaves the system with probability $p_{i,0}$ or proceeds to station j with probability $p_{i,j}$ alikely to the closed system.

We also assume that the transition matrix $(p_{i,j})$ is ergodic and its stationary distribution is $\{p_0,p_1,\ldots,p_M;\ p_i > 0\ (i=0,1,\ldots,M)\}$

The system state can be expressed by an M-tuple integers as $s=(n_1,n_2,\ldots,n_M)$. Then, by Jackson [6], the stationary probability of the system state s is given by

$$P(s) = \frac{1}{G}\ \Pi_{k=0}^{|s|-1} \frac{\lambda(k)}{p_0} \cdot \Pi_{i=1}^{M}\ \Pi_{k=1}^{n_i} \frac{p_i}{\mu_i(k)} ,$$

where G is a normalizing constant, which is assumed finite.

Similarly to the previous section, we consider the processes $\{X_n\}$, $\{Y_n\}$ and their stationary distribution P_X, P_Y. Then we have

$$P_X(i,s) = P_Y(i,s) = C\ p_i\ P(s)\ \frac{\lambda(|s|)}{p_0} ,\ (i=0,1,2,\ldots,M),$$

where C is a normalizing constant, which is assumed finite.
Event $\{X_n = (0,s)\}$ means that an arrival occurs, and $\{Y_n =(0,s)\}$ means that a departure occurs.

A customer who completes the service at station i eventually leaves the system after receiving several services, and he may leave the system without proceeding to station 1, with nonzero probability. Therefore, we consider only $W_{i,0}$'s, the mean sojourn time for a customer until the system departure once he arrives at station i. We obtain the following equations similar to Theorem 1.

$$(3.1)\quad W_{i,0} = w_i + \Sigma_{j=1}\ p_{i,j}\ W_{j,0} \quad (i=1,2,\ldots,M),$$

where w_i is the mean waiting time at station i.
Each w_i can be computed by the mean arrival rate and the mean queue length at each station, using Little's formula. The total waiting time W can be obtained from the equation

$$(3.2)\quad W = \Sigma_{i=1}^{M}\ p_{0,i}\ W_{i,0}$$

However, W is also computed from the mean arrival rate to the system and the mean total queue length by using Little's formula. Therefore, in order to obtain only the total waiting time, the turnaround time equations (3.1) are not so useful.

We can say the same thing to the mean turn around time $W_{1,1}$ in the closed system treated in the previous section, which can be calculated from the mean output rate of station 1 and the mean number of customers in station 2, 3,..., and M.

4. More general system

In this section, we treat the model introduced by Baskett et al. [1]. That is, there are R different classes of customers. Customers travel through the network and change class according to transition probabilities. Thus a customer of class r who completes the service at station i will next require the service at station j in class s with a certain probability denoted $p_{i,r;j,s}$. The transition matrix $(p_{i,r;j,s})$ can be also considered as defining a Markov chain whose states are labeled by the pairs (i,r). The Markov chain is assumed to be decomposable into m ergodic subchains, $E_1, E_2, \ldots, E_m$.

Each station has one of the following properties.

1. The service discipline is FCFS; all customers have the same service time distribution which is a negative exponential.
2. Each class of customers may have a distinct service time ditribution. Each service time distribution, having a rational Laplace transform, should be expressed as a Coxian type [4]. The queueing discipline is preemptive-resume LCFS, processor sharing, or with finite servers.

For this system, Baskett et al. [1] showed that the equlibrium state probabilities can be expressed as follows;

$$P(S_1, S_2, \ldots, S_M) = (1/G)\ f_1(S_1)\ f_2(S_2)\ \cdots\ f_M(S_M),$$

where $(S_1, S_2, \ldots, S_M)$ is the state of the system, S_i represents the conditions prevailing at station i which specify each customer's class, position in queue i, and current phase in his service time. G is a normalizing constant. The form of $f_i(S_i)$ depends on the queueing discipline of station i. For this system, X_n and Y_n are expressed by the combination of (i,r) and other customers' state in the system. Using the explicit form of $P(S_1, S_2, \ldots, S_M)$ given in Baskett et al. [1], we can deduce easily that (i,r) and other customers' state in the system are stochastically independent, alike to Lemma 1.

From this, we obtain the turn around time equations for each ergodic subchain. That is, for an arbitrary m and $(j,s) \in E_m$, we have

$$W_{(i,r:j,s)} = w_{i,r} + \sum_{\substack{(k,t)\in E_m \\ (k,t)\neq(j,s)}} p_{(i,r:k,t)} W_{(k,t:j,s)}$$

for $(i,r)\in E_m, \neq (j,s)$

where $W_{(i,r:j,s)}$ is the mean travelling time for a customer from state (i,r) to state (j,s) and $w_{i,r}$ is the mean waiting time of a customer in class r at station i. $w_{i,r}$'s are also computed from the mean arrival rate and the mean queue length of customers in class r at station i, respectively. Computational algolithms obtaining these statistics are given by Reiser and Kobayashi [8].

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