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A LOWER BOUND OF AN IMPUTATION OF A GAME

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Abstract: An n-tuple is defined for each n-person monotonic characteristic function game. This n-tuple is an imputation when the sum of the components of it is equal to $v(N)$. On the boundary of the set of all monotonic games, we can obtain a condition for the n-tuple being an imputation. The n-tuple belongs to the core when it is an imputation. If the sum of the components of it exceeds $v(N)$, the kernel of the game consists only of interior points of the imputation set.

1. Introduction.

In an n-person characteristic function game, corresponding to the upper bound $b(i)$ of Milnor [4](See also Luce and Raiffa [3], ch.11), we considered a lower bound $m(i)$ in Kikuta [2]. When an imputation x belongs to a "solution" and satisfies some condition, we found in [2] that $m(i)$ is a lower bound of x_i which is the i-th component of x. While it is significant to investigate whether $m(i)$ is a lower bound or not to some solution, it often happens that the sum of $m(i)$ for all $i \in N$ exceeds $v(N)$. Then $m(i)$ cannot be a lower bound. Thus it is interesting to investigate in what case the sum of $m(i)$ equals to $v(N)$. In the present paper we consider $b(i)$ and $m(i)$ as functions on the game space. For this reason, we use $m_i(v)$, $b_i(v)$ instead of $m(i)$, $b(i)$ respectively.

2. Preliminaries.

An n-person characteristic function game with sidepayments is an ordered pair $G = (N,v)$, where $N = \{1,\ldots,n\}$ is the set of players of G and v is a non-negative-valued function (characteristic function) defined on the power set of N. We assume v satisfies

$$v(\phi) = 0,\quad v(\{i\}) = 0,\quad i = 1,\ldots,n,\quad v(N) = 1, \tag{1}$$
$$v(S) \geqq v(T) \quad \text{whenever} \quad S \supseteq T.$$

The last assumption is called the monotonicity. Then our game is a monotonic characteristic function game. Without confusion we refer to G as v. We denote by V the set of all n-person games satisfying (1). Number all of the subsets of N except N, ϕ and the one player sets. Corresponding to each $v \in V$, define a vector in R^d, $d = 2^n-n-2$, by $v = (v(S_1),\ldots,v(S_d))$. Thus we can regard v as a point in R^d. V is a convex compact set in R^d. We let X be the set of all n-tuples such that each component of it is nonnegative and the sum of all the components equals to $v(N)$. We call an element of X an imputation.

Let a game $v \in V$ be given. For S, $T \subseteq N$, define

$$\Delta_S v(T) = v(T) - v(T-S),$$

and

$$\Delta_i v(T) = v(T) - v(T-\{i\}) \quad \text{when } S = \{i\}.$$

For a player i, define

$$m_i(v) = \min_{S \in D_i} \{\Delta_i v(S)\} \tag{2}$$

and

$$b_i(v) = \max_{S \in D_i} \{\Delta_i v(S)\}, \tag{3}$$

where $D_i = \{S \subseteq N \mid |S| \geqq 2,\ S \ni i\}$ and $|S|$ is the number of players which belong to S. Let $\psi_i(v)$ be the Shapley value of a player i, that is,

$$\psi_i(v) = \sum_{S \in D_i} \gamma_n(S)\Delta_i v(S), \tag{4}$$

where $\gamma_n(S) = (|S|-1)!(n-|S|)!/n!$ (See Shapley [6]). Let $m(v)$, $b(v)$ and $\psi(v)$ be n-tuples whose i-th components are $m_i(v)$, $b_i(v)$ and $\psi_i(v)$ respectively. Note that $\psi(v)$ is an imputation. Put

$$\overline{m}(v) = \sum_{i \in N} m_i(v), \tag{5}$$

$$\overline{b}(v) = \sum_{i \in N} b_i(v), \tag{6}$$

and

$$\overline{\psi}(v) = \sum_{i \in N} \psi_i(v) = 1. \tag{7}$$

Define a game v^* by

$$v^*(S) = (|S|-1)/(n-1) \quad \text{for all } S \subseteq N,\ S \neq \phi. \tag{8}$$

We assume $n \geqq 3$ hereafter.

3. Conditions for $m(v)$ Being an Imputation and the Following Results.

Lemma 1. For any $v \in V$,

(9) $$0 \leqq \bar{m}(v) \leqq n/(n-1) \leqq \bar{b}(v) \leqq n.$$

Moreover the followings are mutually equivalent;

(i) $\bar{b}(v) = n/(n-1)$,

(ii) $\bar{m}(v) = n/(n-1)$,

(iii) $v = v^*$.

Proof: By (1),

$$0 \leqq \Delta_i v(S) \leqq 1 \quad \text{for all } S \in D_i.$$

Therefore $0 \leqq m_i(v)$ and $b_i(v) \leqq 1$. Summing with i, we have $0 \leqq \bar{m}(v)$ and $\bar{b}(v) \leqq n$.

$$\psi_i(v) \geqq \sum_{S \in D_i} \gamma_n(S) m_i(v) = ((n-1)/n) m_i(v).$$

Hence $1 = \bar{\psi}(v) \geqq (n-1)\bar{m}(v)/n$. If $1 = (n-1)\bar{m}(v)/n$, then

$$\psi_i(v) = (n-1)m_i(v)/n \quad \text{for all } i.$$

Therefore for any i, $\Delta_i v(S) = m_i(v)$ for all $S \in D_i$. In particular, when $S = \{i, j\}$, $i \neq j$, $m_j(v) = \Delta_j v(\{i, j\}) = \Delta_i v(\{i, j\}) = m_i(v)$. Hence $n/(n-1) = \bar{m}(v) = nm_i(v)$ for all i, and so $m_i(v) = 1/(n-1)$. That is, $\Delta_i v(S) = 1/(n-1)$ for all $S \in D_i$, for all i. Consequently we have $v = v^*$. In the same way, we have $\bar{b}(v) \geqq n/(n-1)$ and that (i) implies (iii).

Conversely if $v = v^*$, then it easily follows that $m_i(v) = b_i(v) = 1/(n-1)$ for all i. And so $\bar{b}(v) = \bar{m}(v) = n/(n-1)$. This completes the proof.

Define a game v_l for each $l \in N$ by

(10) $$v_l(S) = \begin{cases} 1 & \text{if } l \in S, \ |S| \geqq 2, \\ 0 & \text{if } l \notin S. \end{cases}$$

Note that $\bar{b}(v_l) = n$ for all l and $\bar{m}(v_l) = 1$ for all l.

Now, when we wish to consider $m_i(v)$ as a lower bound of x_i which is the i-th component of $x \in X$, it is necessary that $\bar{m}(v) \leqq 1$ because the sum of x_i's for all i equals to 1. For this reason, it will be significant to investigate in what case $\bar{m}(v) = 1$.

If $v \in V$ and

(11) $$v(S) = 1 \text{ or } 0 \quad \text{for any } S \subsetneq N,$$

then we call the game v a simple game. We denote by $Ex(V)$ the set of all extreme points of V, that is, the set of all simple games. For a simple game v, define

(12) $W^m(v) = \{S \subseteq N \mid v(S) = 1 \text{ and } v(T) = 0 \text{ for all } T \subsetneqq S\}.$

Theorem 1. Suppose $v \in Ex(V)$. Then $\overline{m}(v) = 1$ if and only if $v = v_l$ for some $l \in N$.

Proof: We show the necessity. Since v is a simple game, $m_i(v)$ is a non-negative integer for any i. Therefore $\overline{m}(v) = 1$ if and only if there exists a unique $l \in N$ such that $m_l(v) = 1$, and $m_i(v) = 0$ for all i such that $i \neq l$. Suppose $W^m(v) = \{S_1, \ldots, S_k\}$. If $l \notin S_j$ for some j, then $\Delta_l v(S_j \cup \{l\}) = 0$, and so $m_l(v) = 0$, contradicting $m_l(v) = 1$. Hence $l \in S_j$ for $j = 1, \ldots, k$. Moreover, for any i such that $i \neq l$, $1 = m_l(v) \leqq \Delta_l v(\{i, l\})$, so that $\{i, l\} \in W^m(v)$ for any i such that $i \neq l$. If for some j, $|S_j| \geqq 3$, then $\{i, l\} \subset S_j$ for some $i \in N$, which contradicts the minimality of S_j. Consequently we have $W^m(v) = \{\{i, l\} \mid i \neq l, i \in N\}$, which means $v = v_l$. The sufficiency has already been noted. This completes the proof.

Corollary. Suppose $v \in Ex(V)$ and $v \neq v_l$ for any $l \in N$, then $\overline{m}(v) = 0$.

Proof: By Lemma 1, $0 \leqq \overline{m}(v) \leqq n/(n-1) < 2$. Because $\overline{m}(v)$ is an integer, either $\overline{m}(v) = 0$ or 1. By Theorem 1, $\overline{m}(v) = 0$. This completes the proof.

Now, define

(13) $U = \{v \in V \mid \overline{m}(v) \geqq 1\}.$

Then $v^* \in U$. We show that v^* is an interior point of U in R^d. For each $S \subseteq N$ such that $2 \leqq |S| \leqq n-1$, define a real number ε_S as follows; $|\varepsilon_S| \leqq 1/(n(n-1))$. Define a function, $v^*+\varepsilon$, on the power set of N, by

$$(v^*+\varepsilon)(S) = \begin{cases} 1 & \text{if } S = N, \\ v^*(S) + \varepsilon_S & \text{if } 2 \leqq |S| \leqq n-1, \\ 0 & \text{if } |S| = 0 \text{ or } 1. \end{cases}$$

It is not difficult to see $v^*+\varepsilon \in V$. Choose ε_S as $|\varepsilon_S| < 1/(2n(n-1))$ for each S such that $2 \leqq |S| \leqq n-1$. When $S \ni i$ and $2 \leqq |S| \leqq n-1$,

$$\Delta_i(v^*+\varepsilon)(S) = 1/(n-1) + \varepsilon_S - \varepsilon_{S-\{i\}} \geqq 1/n.$$

$$\Delta_i(v^*+\varepsilon)(N) = 1/(n-1) - \varepsilon_{N-\{i\}} \geqq (2n-1)/(2n(n-1)).$$

Hence

$$m_i(v^*+\varepsilon) \geqq \min\{1/n, (2n-1)/(2n(n-1))\} = 1/n,$$

and we have $\overline{m}(v^*+\varepsilon) \geqq 1$, which implies $v^*+\varepsilon \in U$. Thus we find v^* is an interior point of U in R^d. It is easily seen that U is a convex set because $\overline{m}$ is a concave function on V.

Denote by $Bd(U)$, $Bd(V)$, the boundaries of U, V respectively. Note that $u \in Bd(V)$ if and only if there exist $S, T \subseteq N$ such that $u(S) = u(T)$, $S \subset T$ and

$|T| > |S| \geqq 1$.

Theorem 2. Let X_n be the convex hull of a finite set $\{v_1,\ldots,v_n\}$ of R^d, where v_l is defined by (10). Then

$$Bd(U) = \{v \in V \mid \overline{m}(v) = 1\} \supset X_n,$$

and

$$Bd(U) = \{(1-t(u))v^* + t(u)u \mid u \in Bd(V)\}, \tag{14}$$

where $t(u) = 1/(n-(n-1)\overline{m}(u))$.

Before proving the theorem, we need two lemmas.

Lemma 2. Let $u \in Bd(V)$. Then $\overline{m}(u) \leq 1$.

Proof: Suppose $u(S^*) = u(T^*)$ for some S^* and T^* such that $S^* \subset T^*$, $t^* = |T^*| > |S^*| = s^* \geqq 1$. Let $T^* - S^* = \{i_1,\ldots,i_{t^*-s^*}\}$. Then

$$m_{i_1}(u) \leqq \Delta_{i_1} u(T^*) = 0,$$

and

$$m_{i_j}(u) \leqq \Delta_{i_j} u(T^* - \{i_1,\ldots,i_{j-1}\}) = 0 \quad \text{for } j = 2,\ldots,t^*-s^*.$$

Let $N - T^* = \{k_1,\ldots,k_{n-t^*}\}$, and $S^* = \{l_1,\ldots,l_{s^*}\}$. Then

$$m_{k_1}(u) \leqq \Delta_{k_1} u(N),$$

$$m_{k_j}(u) \leqq \Delta_{k_j} u(N - \{k_1,\ldots,k_{j-1}\}) \quad \text{for } j = 2,\ldots,n-t^*,$$

$$m_{l_1}(u) \leqq \Delta_{l_1} u(T^*),$$

and

$$m_{l_j}(u) \leqq \Delta_{l_j} u(T^* - \{l_1,\ldots,l_{j-1}\}) \quad \text{for } j = 2,\ldots,s^*.$$

Hence

$$\begin{aligned}
\overline{m}(u) &= \sum_{i \in T^*-S^*} m_i(u) + \sum_{i \in N-T^*} m_i(u) + \sum_{i \in S^*} m_i(u) \\
&\leqq 0 + \Delta_{k_1} u(N) + \sum_{j=2}^{n-t^*} \Delta_{k_j} u(N - \{k_1,\ldots,k_{j-1}\}) \\
&\quad + \Delta_{l_1} u(T^*) + \sum_{j-2}^{s^*} \Delta_{l_j} u(T^* - \{l_1,\ldots,l_{j-1}\}) \\
&= u(N) - u(T^*) + u(T^*) - u(T^*-S^*) \\
&= u(N) - u(T^*-S^*) \leqq 1.
\end{aligned}$$

This completes the proof.

Remark. The converse of Lemma 2 is not true. For instance, there exists $u \notin Bd(V)$ such that $0 < u(S) < 1/n$ for all $S; |S| = 2$. Then $m_i(u) < 1/n$ for any i and so $\overline{m}(u) < 1$.

Denote by $Int(V)$ the set of all interior points of V. Define a function on $I \times Bd(V)$ by

(15) $$w(t,u) = (1-t)t^* + tu \quad \text{for } t \in I,\ u \in Bd(V).$$

Here I is the unit interval. Note that $w(t,u)$ belongs to V for each fixed (t,u). The following Lemma 3 is an elementary result in convex set theory and we omit the proof (Note the Corollary 2 at page 21 of Nikaido [5]).

Lemma 3. Suppose $v \in Int(V)-\{v^*\}$. Then there exists a unique $(t,u) \in I \times Bd(V)$ such that $v = w(t,u)$.

Proof of Theorem 2: By the continuity of function $\overline{m}$, v belongs to $Bd(U)$ if and only if $\overline{m}(v) = 1$. Now, suppose $\overline{m}(v) = 1$. Then $v \in Bd(V)$ or $v \in Int(V)$. If $v \in Bd(V)$, $v = w(1,v)$. Let $v \in Int(V)$. By Lemma 3, there exists a unique (t,u) such that $v = w(t,u)$. Because $\overline{m}(v) = 1$,

$$1 = \overline{m}((1-t)v^* + tu) = (1-t)n/(n-1) + t\overline{m}(u),$$

so that $t = t(u) = 1/(n-(n-1)\overline{m}(u))$. Note that $\overline{m}(u) \leqq 1$ by Lemma 2.

Inversely, when $v = (1-t(u))v^* + t(u)u$, it easily follows that $\overline{m}(v) = 1$. Suppose $v \in X_n$. We can express v uniquely as

$$v = \sum_{i=1}^{n} x_i v_i, \quad x = (x_1,\dots,x_n) \in X.$$

From the concavity of $\overline{m}$,

$$\overline{m}(v) \geqq \sum_{i=1}^{n} x_i \overline{m}(v_i) = \sum_{i=1}^{n} x_i = 1.$$

Moreover

$$m_i(v) \leqq \Delta_i v(N) = x_i \quad \text{for all } i.$$

Therefore

$$\overline{m}(v) = \sum_{i=1}^{n} m_i(v) \leqq \sum_{i=1}^{n} x_i = 1.$$

We have $\overline{m}(v) = 1$, which implies $v \in Bd(U)$. This completes the proof.

Lemma 4. Assume $v(S^*) = 1$ for some S^* such that $2 \leqq |S^*| \leqq n-1$. Then for $\overline{m}(v) = 1$, it is necessary and sufficient that

$$v(T) = \sum_{i \in T \cap S^*} m_i(v) \quad \text{for all } T \text{ such that } T \not\subseteq S^*.$$

Proof: For $i \in N-S^*$,

$$m_i(v) \leqq \Delta_i v(S^* \cup \{i\}) = 0.$$

We have

(16) $$\overline{m}(v) = \sum_{i \in S^*} m_i(v).$$

Suppose $\overline{m}(v) = 1$ and $T \cap S^* = \{i_1,\dots,i_l\}$ for $T \not\subseteq S^*$. Then $m_{i_1}(v) \leqq \Delta_{i_1} v(T)$ and $m_{i_j}(v) \leqq \Delta_{i_j} v(T-\{i_1,\dots,i_{j-1}\})$ for $j = 2,\dots,l$. Hence

(17) $$\sum_{i\in S^*\cap T} m_i(v) \leqq \Delta_{S^*}v(T) \leqq v(T).$$

Let $S^*-T = \{j_1,\dots,j_k\}$. Then $m_{j_1}(v) \leqq \Delta_{j_1} v(T\cup S^*)$ and $m_{j_p}(v) \leqq \Delta_{j_p} v((T\cup S^*)-\{j_1,\dots,j_{p-1}\})$ for $p = 2,\dots,k$. We have

$$\sum_{i\in S^*-T} m_i(v) \leqq \Delta_{S^*-T}v(T\cup S^*) = 1 - v(T),$$

that is,

(18) $$v(T) \leqq \sum_{i\in T\cap S^*} m_i(v),$$

since $\overline{m}(v) = 1$ and (16). By (17) and (18), we have

(19) $$v(T) = \sum_{i\in T\cap S^*} m_i(v) \text{ for } T \text{ such that } T\nsubseteq S^*.$$

Conversely, suppose $v(T) = \sum_{i\in T\cap S^*} m_i(v)$ for all T such that $T\nsubseteq S^*$. Then $1 = v(N) = \sum_{i\in S^*} m_i(v) = \overline{m}(v)$. This completes the proof.

Lemma 5. Assume $v(S^*) = 0$ for some S^* such that $2 \leqq |S^*| \leqq n-1$. Then for $\overline{m}(v) = 1$, it is necessary and sufficient that

$$v(T) = \sum_{i\in T\cap(N-S^*)} m_i(v) \text{ for all } T \text{ such that } T\nsubseteq N-S^*.$$

Proof: For $i\in S^*$, $m_i(v) \leqq \Delta_i v(S^*) = 0$. We have

$$\overline{m}(v) = \sum_{i\in N-S^*} m_i(v).$$

Suppose $\overline{m}(v) = 1$. Put $T^* = (N-S^*)\cup\{i_0\}$ for $i_0\in S^*$ and let $N-S^* = \{i_1,\dots,i_{n-s^*}\}$. Then $m_{i_1}(v) \leqq \Delta_{i_1} v(T^*)$ and $m_{i_j}(v) \leqq \Delta_{i_j} v(T^*-\{i_1,\dots,i_{j-1}\})$ for $j = 2,\dots,n-s^*$. Therefore $1 = \sum_{i\in N-S^*} m_i(v) \leqq v(T^*)$ and we have $v(T^*) = 1$. By $n-1 \geqq |T^*| \geqq 2$, Lemma 4, and $m_{i_0}(v) = 0$,

$$v(T) = \sum_{i\in T\cap T^*} m_i(v) = \sum_{i\in T\cap(N-S^*)} m_i(v) \quad \text{for all } T\nsubseteq T^*.$$

Put $T^{**} = (N-S^*)\cup\{j_0\}$ for $j_0\in S^*$ and $j_0 \neq i_0$. Note that $|S^*| \geqq 2$. In the same way as above we have

$$v(T) = \sum_{i\in T\cap(N-S^*)} m_i(v) \quad \text{for all } T\nsubseteq T^{**}.$$

In particular, if $T\nsubseteq N-S^*$ but $T\subseteq T^*$, then $T\nsubseteq T^{**}$. Consequently, we have

$$v(T) = \sum_{i\in T\cap(N-S^*)} m_i(v) \quad \text{for all } T\nsubseteq N-S^*.$$

Conversely suppose $v(T) = \sum_{i\in T\cap(N-S^*)} m_i(v)$ for T such that $T\nsubseteq N-S^*$. Then $1 =$

$v(N) = \sum_{i \in N-S^*} m_i(v) = \overline{m}(v)$. This completes the proof.

Now, suppose $v \in Bd(V)$. Assume $v(S^*) = 0$ for some S^*, or $v(T^*) = 1$ for some T^*. Thus we can apply Lemma 4 and Lemma 5 to obtain a necessary and sufficient condition for $\overline{m}(v) = 1$. Put

$$L(v) = \{S \subset N \mid v(S) = 0 \text{ and } |S| \geqq 2\}$$

and

$$W(v) = \{S \subset N \mid v(S) = 1 \text{ and } |S| \leqq n-1\}.$$

Theorem 3. Suppose $v \in Bd(V)$. Assume $L(v) \neq \phi$ and $W(v) \neq \phi$. Put $S^0 = \bigcap_{S \in L(v)} (N-S)$ and $T^0 = \bigcap_{S \in W(v)} S$ and $S^* = S^0 \cap T^0$. Then $\overline{m}(v) = 1$ if and only if

$$v(T) = \sum_{i \in S^* \cap T} m_i(v) \text{ for all } T \text{ such that } T \not\subseteq S^*.$$

Proof: By Lemma 4 and Lemma 5, we have $m_i(v) = 0$ when $i \notin S^*$. And we have $\overline{m}(v) = \sum_{i \in S^*} m_i(v)$. Suppose $\overline{m}(v) = 1$ and $T \not\subseteq S^*$. Then either $T \not\subseteq S^0$ or $T \not\subseteq T^0$. If $T \not\subseteq S^0$, there exists S in $L(v)$ such that $T \not\subseteq N-S$. From Lemma 5, it follows $v(T) = \sum_{i \in T \cap (N-S)} m_i(v) = \sum_{i \in T \cap S^*} m_i(v)$. When $T \not\subseteq T^0$, there exists S in $W(v)$ such that $T \not\subseteq S$. By Lemma 4, $v(T) = \sum_{i \in T \cap S} m_i(v) = \sum_{i \in T \cap S^*} m_i(v)$. The converse is easily seen because $1 = v(N) = \sum_{i \in N \cap S^*} m_i(v) = \sum_{i \in S^*} m_i(v) = \overline{m}(v)$. This completes the proof.

When $\overline{m}(v) = 1$, $m(v)$ is an imputation. In this case, it seems to be interesting to investigate whether $m(v)$ belongs to some "solution". Define the core of v by

(20) $$C(v) = \{x \in X \mid \sum_{i \in S} x_i \geqq v(S) \text{ for any } S \subseteq N\}.$$

Theorem 4. Suppose $\overline{m}(v) = 1$. Then

$$m(v) \in C(v).$$

Proof: For any S such that $2 \leqq |S| \leqq n-1$, let $N-S = \{i_1, \ldots, i_{n-s}\}$ and $s = |S|$. Then

$$m_{i_j}(v) \leqq \Delta_{i_j} v(S \cup \{i_1, \ldots, i_j\}) \text{ for } j = 1, \ldots, n-s.$$

Therefore

$$\sum_{i \in N-S} m_i(v) \leqq \sum_{j=1}^{n-s} \Delta_{i_j} v(S \cup \{i_1, \ldots, i_j\})$$
$$= \Delta_{N-S} v(N) = 1 - v(S).$$

Since $\sum_{i \in N-S} m_i(v) = 1 - \sum_{i \in S} m_i(v)$, we have

$$\sum_{i \in S} m_i(v) \geqq v(S).$$

This completes the proof.

Well, fix a game v. Let $x \in X$ and $S \subseteq N$. We define the excess of S with respect to x by

(21) $$e_v(S,x) = v(S) - \sum_{i \in S} x_i,$$

and the maximum surplus of a player k against a player l, $k \neq l$, with respect to x by

(22) $$s_{kl}(v,x) = \max_{S \in T_{kl}} e_v(S,x),$$

where $T_{kl} = \{S \subseteq N | k \in S \text{ and } l \notin S\}$. We define the kernel [1] of v by

(23) $$K(v) = \{x \in X | (s_{kl}(v,x) - s_{lk}(v,x))x_l \leqq 0 \text{ for all } k,l \in N, k \neq l\}.$$

Theorem 5. Suppose $v \in Int(U)$ and $x \in K(v)$. Then

$$x_i > 0 \text{ for } i = 1,\ldots,n.$$

Proof: Since $v \in Int(U)$, $\overline{m}(v) > 1$. Thus there exists $i_0 \in N$ such that $x_{i_0} < m_{i_0}(v)$. Assume $x_l = 0$ for some $l \in N$. Then by theorem 1 of Kikuta [2], $x_i \geqq m_i(v)$ for $i = 1,\ldots,n$, which is a contradiction. Therefore $x_i > 0$ for $i = 1,\ldots,n$. This completes the proof.

Theorem 6. Suppose $v \in Ex(V)$. Then for any $x \in K(v)$,

(24) $$x_i \geqq m_i(v) \text{ for } i = 1,\ldots,n.$$

Proof: It is clear when $\overline{m}(v) = 0$. Suppose $\overline{m}(v) = 1$. By Theorem 1, $v = v_i$ for some $i \in N$. By definition of v_i, all players except i are dummies, hence $x_j = 0$ for all $x \in K(v)$ and all $j \neq i$. Consequently $K(v) = \{e_i\}$ where the j-th component of e_i is δ_{ij}, which is the Kronecker's delta. On the other hand $m(v) = e_i$. This completes the proof.

4. A Concluding Remark.

Define a function on $I \times Bd(V)$ by

(25) $$p(t,v) = \overline{m}(tv + (1-t)v^*)/\overline{b}(tv + (1-t)v^*).$$

By Lemma 1, $0 \leqq p(t,v) \leqq 1$. By the definitions of functions $\overline{m}$ and $\overline{b}$, $p(t,v)$ is continuous on $I \times Bd(V)$. Moreover,

$$m_i(tv + (1-t)v^*)$$
$$= \min_{S \in D_i} \{\Delta_i(tv + (1-t)v^*)(S)\}$$

$$= \min_{S \in D_i} \{\Delta_i v(S) + (1-t)\Delta_i v^*(S)\}$$

$$= 1/(n-1) - t\{1/(n-1) - m_i(v)\}.$$

Summing up with i,

$$\overline{m}(tv + (1-t)v^*) = n/(n-1) - t\{n/(n-1) - \overline{m}(v)\}.$$

Similarly, we have

$$\overline{b}(tv + (1-t)v^*) = n/(n-1) + t\{\overline{b}(v) - n/(n-1)\}.$$

By Lemma 1, $\overline{m}(tv + (1-t)v^*)$ is decreasing and $\overline{b}(tv + (1-t)v^*)$ is increasing in t. Hence $p(t,v)$ is decreasing in t. Put $v_t = tv + (1-t)v^*$. $\Delta_i v_t(S)(S \ni i)$ represents the marginal contribution of a player i when he enters into $S-\{i\}$ in a game v_t. If $\Delta_i v_t(S)$ has little variation when S varies in D_i, then $b_i(v_t) - m_i(v_t)$ will be small, and will be large if $\Delta_i v_t(S)$ has much variation. Thus

$$\overline{b}(v_t) - \overline{m}(v_t) = \sum_{i=1}^{n} \{b_i(v_t) - m_i(v_t)\}$$

will be large if, as a whole, $\Delta_i v_t(S)$ $(i = 1,\ldots,n)$ has much variation. Dividing $\overline{b}(v_t) - \overline{m}(v_t)$ by $\overline{b}(v_t)$ for normalization, we have

$$(\overline{b}(v_t) - \overline{m}(v_t))/\overline{b}(v_t) = 1 - p(t,v).$$

Moreover, it seems to be interesting to consider some function $q(t,v) = q(x_1,\ldots,x_n)$ where $x_i = m_i(tv + (1-t)v^*)/b_i(tv + (1-t)v^*)$, $i = 1,\ldots,n$, and $(t,v) \in I \times Bd(V)$.

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