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GENERALIZED FACTOR THEOREM AND AN ALGORITHM FOR FINDING A FACTOR

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Abstract Let F be a subset of the edge-set of a given graph. The edge-set F (or the edge-subgraph consisting of the edges in F and their end-vertices) is referred to as a (p, q)-factor of the graph if the number of the edges in F incident to each of the vertices is between two non-negative integers $p(v)$ and $q(v)$ preassigned to the vertex. In this paper we present a necessary and sufficient condition for a given graph to have a (p, q)-factor. Tutte's condition for the existence of an f-factor is derived from our condition only by substitution.

1. Existence Theorem of a (p,q)-Factor

Tutte [8,9,10,11,12] presented a necessary and sufficient condition for an f-factor to exist in a given graph, where f-factor is a subset of the edge-set (or equivalently an edge-subgraph consisting of the edges in the subset and their end-vertices) the number of whose edges incident to each of the vertices is equal to the specified number $f(v)$. He gave a partially constructive proof in which we could see the germ of Edmonds' maximum matching algorithm (See [1,2,3,4]). Lovász [7] generalized the f-factor problem and considered the problem of finding a subset of the edge-set the number of whose edges incident to each of the vertices is a member of the set $H(v)$ of integers preassigned to the vertex.

In this paper we consider the case where $H(v)$ is an interval $[p(v), q(v)]$ and present not only a necessary and sufficient condition for the existence of the required edge-set but also an algorithm for finding one.

Every graph G in this paper is a finite, undirected and connected graph without self-loops. Let $<G>$ and $[G]$ be the vertex-set and the edge-set of G, respectively. For a pair of subsets V and W of $<G>$, let $D(V,W)$ be the set of edges which have one end-vertex in V and the other in W. We abbreviate $D(V,<G>-V)$ and $D(\{v\},<G>-\{v\})$ to $D(V)$ and $D(v)$, respectively, where v is a vertex. $D^E(\cdot,\cdot)$ indicates the set $D(\cdot,\cdot)\cap E$ for a subset E of $[G]$. We define $d(\cdot,\cdot)=|D(\cdot,\cdot)|$, $d^E(\cdot,\cdot)=|D^E(\cdot,\cdot)|$, where $|X|$ implies the number of elements contained in the set X. Since a graph G is characterized by its vertex-set $<G>$, edge-set $[G]$ and the mapping D of its incidence relation, we write $G=(<G>,[G],D)$. d is called the mapping of degrees.

Let p and q be mappings from $<G>$ into the set of nonnegative integers such that $p(v)\leqq q(v)$ for each vertex v. A subset F of $[G]$ (or equivalently the edge-subgraph consisting of the edges in F and their end-vertices) is called a (p,q)-factor if $p(v)\leqq d^F(v)\leqq q(v)$ and is called a q-subfactor if $d^F(v)\leqq q(v)$ for each vertex v of G.

For a subset T of $<G>$ we denote by $G(T)$ the subgraph of G obtained by suppressing the vertices in T and all edges of G having either one or both end-vertices in T. For a pair of disjoint subsets S and T of $<G>$, we define

$$K(S,T) = \left\{ <H> \;\middle|\; \begin{array}{l} H \text{ is a connected component of } G(S\cup T) \text{ such that} \\ \text{(i)} \quad p(v)=q(v) \text{ for every vertex } v \text{ of } H, \text{ and} \\ \text{(ii)} \quad \sum_{v\in<H>} p(v)+d(<H>,S)\equiv 1 \pmod 2. \end{array} \right\}$$

$$r(S,T)=|K(S,T)|+\sum_{v\in S}\{p(v)-d^{[G(T)]}(v)\}.$$

If $p(v)=q(v)$ for every vertex, $K(S,T)$ and $r(S,T)$ are the same as $K(G(T),S)$ and $r(G(T),S)$, respectively in Tutte [10].

Theorem 1.

A graph G has a (p,q)-factor if and only if

(1) $\quad \sum_{v\in T} q(v) \geqq r(S,T)$

for every pair of disjoint subsets S and T of $<G>$.

If $p(v)=0$ and $q(v)\geqq d(v)$ for every vertex v of G, any subset of $[G]$ is a (p,q)-factor. In this case, since $r(S,T)$ is nonpositive for any pair of disjoint subsets S and T of $\langle G\rangle$, the condition (1) is satisfied. If $p(v)>d(v)$ for some vertex, it is trivial that no (p,q)-factors exist. If we let T be empty and S be the set of vertices with $p(v)>d(v)$, then the condition (1) is violated, where the summation taken over an empty set is assumed to be zero.

The necessity of Theorem 1 can be seen in the similar way as in Tutte [10].

Proof of the Necessity of Theorem 1.

Let F be a (p,q)-factor. For an arbitrary subset T of $\langle G\rangle$, $B=F\cap[G(T)]$ is a q-subfactor of $G(T)$. If $\langle H\rangle\in K(S,T)$ for a subset S of $\langle G\rangle-T$, then from the definition

(2) $$\sum_{v\in\langle H\rangle} p(v)+d(\langle H\rangle,S)\equiv 1 \pmod 2.$$

Letting $R=[G(T)]-B$, then it is trivial that

(3) $$\sum_{v\in\langle H\rangle} d^B(v)+d(\langle H\rangle,S)-d^R(\langle H\rangle,S)=\sum_{v\in\langle H\rangle} d^B(v)+d^B(\langle H\rangle,S)\equiv 0 \pmod 2,$$

for the expression on the right is equal to twice the number of edges in B having one end-vertex in $\langle H\rangle$. Let

$$K^*(S,T)=\left\{\langle H\rangle \,\middle|\, \begin{array}{l}\langle H\rangle\in K(S,T) \text{ and } p(v)=d^B(v)\\ \text{for each vertex of } H.\end{array}\right\},$$

then from (2) and (3), we get that

(4) $$d^R(\langle H\rangle,S)\equiv 1 \pmod 2,$$

for any member $\langle H\rangle$ of $K^*(S,T)$. Let A be the set of vertices of $G(T)$ such that $d^B(v)\leqq p(v)$, then

(5) $$\sum_{v\in A}\{p(v)-d^B(v)\}\geqq \sum_{\langle H\rangle\in K(S,T)}\ \sum_{v\in\langle H\rangle}\{p(v)-d^B(v)\}+\sum_{v\in S\cap A}\{p(v)-d^B(v)\}$$

(by the fact that B is a q-subfactor of $G(T)$ and that $p(v)=q(v)$ for v in $<H>$)

$$\geqq \sum_{<H>\varepsilon K(S,T)} \sum_{v\varepsilon <H>} \{p(v)-d^B(v)\}+ \sum_{v\varepsilon S} \{p(v)-d^B(v)\}$$

(by the fact that $p(v)-d^B(v)<0$ for any vertex not in A)

$$\geqq \sum_{<H>\varepsilon K(S,T)-K^*(S,T)} \sum_{v\varepsilon <H>} \{p(v)-d^B(v)\}+ \sum_{v\varepsilon S} \{p(v)-d^B(v)\}$$

(by the definition of $K^*(S,T)$)

$$\geqq |K(S,T)-K^*(S,T)|+ \sum_{<H>\varepsilon K^*(S,T)} d^R(<H>,S)$$

$$+ \sum_{v\varepsilon S} p(v)$$

$$-\{ \sum_{v\varepsilon S} d^B(v)+ \sum_{<H>\varepsilon K^*(S,T)} d^R(<H>,S)\}.$$

Using (4), we get that

(6) $$|K(S,T)-K^*(S,T)|+ \sum_{<H>\varepsilon K^*(S,T)} d^R(<H>,S) \geqq |K(S,T)|.$$

Since $\sum_{v\varepsilon S} d^{[G(T)]}(v)\geqq\sum_{v\varepsilon S} d^B(v)+ \sum_{<H>\varepsilon K^*(S,T)} d^R(<H>,S)$, then by (5) and (6) we have that

(7) $$\sum_{v\varepsilon A} \{p(v)-d^B(v)\}\geqq|K(S,T)|+ \sum_{v\varepsilon S} \{p(v)-d^{[G(T)]}(v)\}=r(S,T).$$

On the other hand

(8) $$\sum_{v\varepsilon T} q(v)\geqq \sum_{v\varepsilon A} \{p(v)-d^B(v)\}$$

must hold by the existence of a (p,q)-factor. Then by (7) and (8), we obtain the required relation

$$\sum_{v\varepsilon T} q(v)\geqq r(S,T).$$

Q.E.D.

2. Algorithm for Finding a (p,q)-Factor

To prove the sufficiency of Theorem 1, we propose an algorithm for finding a (p,q)-factor, which generates either a (p,q)-factor or a pair of disjoint subsets of $<G>$ violating the condition of Theorem 1, but not both.

Let V be an arbitrary subset of $<G>$. Shrinking V implies to generate the new graph $G'=(<G'>,[G'],D')$ from $G=(<G>,[G],D)$ such that

(i) $<G'>=(<G>-V)\cup\{u\}$,

(ii) $[G']=[G]-D(V,V)$,

(iii) $D'(v)=D(v)$ for every vertex in $<G>-V$, and $D'(u)=D(V)$,

where u is a new vertex not in $<G>$. We write $G'=G/V$, $u=V/V$ and u is referred to as a pseudovertex, abbreviated to Ps. We use $<u>$ to denote the vertex-set V, and even if u is not a Ps, we also use $<u>$ to denote the set of the single vertex u. Let $G_{i+1}=G_i/V_i$ $(i=0,1,\dots)$ be a sequence of graphs generated by sequential shrinkings, where V_i is a subset of $<G_i>$. We inductively define the shrinking level $s(u)$ for vertices of G_i as follows:

(i) $s(u)=0$ if $u\varepsilon<G_0>$,

(ii) $s(u)=1+\max_{v\varepsilon<u>} s(v)$ if $u\not\varepsilon<G_0>$,

and the subset $<<u>>$ of $<G_0>$ corresponding to the vertex u as follows:

(i) $<<u>>=<u>$ if $s(u)\leqq 1$,

(ii) $<<u>>=\bigcup_{v\varepsilon<u>} <<v>>$ if $s(u)\geqq 2$.

Let B be a q-subfactor of G and $R=[G]-B$. We suppose that the edges in B are colored blue and R red. We call a vertex with $d^B(v)<p(v)$ a deficient vertex and a vertex with $p(v)\leqq d^B(v)\leqq q(v)$ a sufficient vertex. A sequence $\{v_0,e_0,\dots,v_i,e_i,v_{i+1},\dots,e_{n-1},v_n\}$ of vertices and edges such that for $i,j=0,1,\dots,n-1$, v_i and v_{i+1} are both end-vertices of e_i and $e_i\neq e_j$ whenever $i\neq j$ is called a path. An alternating path is a path whose edges are alternately blue and red. An augmenting path is an alternating path with the deficient initial vertex v_0 and the red initial edge e_0 satisfying one of next conditions:

(i) $v_n=v_0$, e_{n-1} is red and $q(v_n)-d^B(v_n)\geqq 2$,

(ii) $v_n\neq v_0$, e_{n-1} is red and $q(v_n)-d^B(v_n)\geqq 1$,

(iii) $v_n \neq v_0$, e_{n-1} is blue and $d^B(v_n)-p(v_n) \geq 1$.

The augmentation of B with an augmenting path P implies the replacement of B by $B'=(B-[P]\cap B)\cup([P]\cap R)$. It is clear from the definitions that B' is also a q-subfactor and that $d^B(v_0) < d^{B'}(v_0)$.

In the following algorithm we assign outer or inner labels to the vertices and construct trees (not necessarily spanning). For convenience the edges of the tree and the edges not of the tree are called arcs and fronds, respectively and the vertices not of the tree points. When a cycle consisting of a frond and some arcs is formed and satisfies some conditions, it is shrunk into a new *Ps*. It is noteworthy that we assign no label to a *Ps* since a *Ps* plays double the parts of outer and inner vertices. The vertex-set shrunk into a *Ps* corresponds to a member of $K(S,T)$, which is referred to as a bicursal unit (Tutte [10]). To avoid redundant expressions we shall use the symbol $\emptyset$ (I) to denote an outer (inner) vertex as well as the label itself and $(v)e(w)$ to denote an edge connecting vertices v and w, for example $(\emptyset)$*frond*(I) implies a frond connecting an outer vertex with an inner vertex.

The Algorithm

Step 0. Choose an initial q-subfactor B of G (B may be empty).

Step 1. Set $G_0=G$, $i=0$ and color the edge-set B blue and the rest red. Discard the current tree and vertex labels.

Step 2. If the deficient vertices exist, go to Step 3. Otherwise, terminate since a (p,q)-factor is obtained.

Step 3. Choose one of the deficient vertices as r and label it $\emptyset$. Set T_0 be the tree of G_0 consisting of the single vertex r. $v=r$ and go to Step 4.

Step 4. If there is $(\emptyset)$*red frond(point)*, (I)*blue frond(point)* or (Ps)*frond(point)*, then choose one as e and go to Step 5. Otherwise, terminate since G has no (p,q)-factor (see Lemma 13).

Step 5. Let v be the end-point of e. Add T_i the frond e and the point v and let T_i be the resultant tree. Go to Step 6.

Step 6. If e is red, assign label I to v and go to Step 7. If blue, assign label $\emptyset$ to v and go to Step 8.

Step 7. If $d^B(v)<q(v)$, then go to Step 20. Otherwise, go to Step 9.

Step 8. If $d^B(v)>p(v)$, then go to Step 20. Otherwise, go to Step 9.

Step 9. If v is $\emptyset$, then go to Step 10, if I, to Step 11, and if Ps, to Step 12.

Step 10. If there is *(v)red frond(∅ or Ps)*, then choose one as g and go to Step 15. Otherwise, go to Step 4.

Step 11. If there is *(v)blue frond(I or Ps)*, then choose one as g and go to Step 15. Otherwise, go to Step 4.

Step 12. If there is *(v)arc(Ps)*, then choose one as g and go to Step 13. Otherwise, go to Step 14.

Step 13. $G_{i+1}=G_i/\{x,y\}$, $T_{i+1}=T_i/\{x,y\}$, $v=\{x,y\}/\{x,y\}$ and $i=i+1$ where x and y are the both end-vertices of g in G_i. Go to Step 14.

Step 14. If there is *(v)red frond(∅)*, *(v)blue frond(I)* or *(v)frond(Ps)*, then choose one as g and go to Step 15. Otherwise, go to Step 4.

Step 15. If the cycle C_i consisting of the frond g and the arcs has the vertex r, then go to Step 16. Otherwise, go to Step 17.

Step 16. If $d^B(r)\leqq q(r)-2$, then go to Step 20. Otherwise, go to Step 17.

Step 17. If C_i has a deficient $\emptyset$ other than r, then choose one as v and go to Step 20. Otherwise, go to Step 18.

Step 18. If C_i has a vertex v with $p(v)<q(v)$ other than r, then go to Step 20. Otherwise, go to Step 19.

Step 19. $G_{i+1}=G_i/\langle C_i\rangle$, $T_{i+1}=T_i/\langle C_i\rangle$, $v=\langle C_i\rangle/\langle C_i\rangle$, $i=i+1$ and go to Step 9.

Step 20. Since there is an augmenting path P in G from r either to r itself or to v, augment the current q-subfactor with P and go to Step 1.

The algorithm starts with $G_0=G$ and an initial q-subfactor and generates a sequence of graphs G_i and T_i, which is followed by an augmentation to make a new q-subfactor. And the algorithm makes a restart with the new q-subfactor. The following argument is concerned with one of the above sequences of graphs.

Since it is clear that the subgraph T_i of G_i is a connected, acyclic subgraph, we called it a tree in the algorithm and we shall refer to Step 5 as growing tree step. The cycle C_i of Step 15 is uniquely determined by the property of T_i. We use r^* to denote the vertex r itself or the Ps corresponding to the vertex-set containing r, and we refer to r^* as the root-(pseudo-) vertex of the tree.

Let v be a vertex of T_i other than the root-vertex r^*. We call the unique path from r^* to v on T_i the tree-path to v and write it as $P_i(v)$.

The final edge of $P_i(v)$ is called the entrance-edge of v and is written as $e_i(v)$. A path $P=\{v_0,e_0,..,e_{i-1},v_i,e_i,...,e_{n-1},v_n\}$ is called a pseudo-alternating path if the colors of successive edges e_{i-1} and e_i are distnct in the case where the vertex v_i is a non-Ps (a vertex that is not a Ps).

The following three lemmas are trivial from the algorithm and the definitions.

Lemma 1.

For any vertex v of T_i other than the root-vertex r^*,

(i) $e_i(v)$ is blue if v is $\emptyset$, and

(ii) $e_i(v)$ is red if v is I.

Lemma 2.

Each arc is the entrance-edge of one of its end-vertices.

Lemma 3.

A tree-path $P_i(v)$ is a pseudo-alternating path.

Let v^* be the vertex of the cycle C_i of Step 15 such that

$$|[P_i(v^*)]| = \min_{v\in\langle C_i\rangle} |[P_i(v)]|$$

Since T_i is acyclic, v^* is uniquely determined. We write it as $b(C_i)$ and call it the base-vertex of the cycle C_i. If $\langle C_i\rangle$ contains the root-vertex r^*, we set $b(C_i)=r^*$.

Lemma 4.

The cycle C_i of Step 15 is a pseudo-alternating path from $b(C_i)$ to $b(C_i)$ itself. Moreover if $b(C_i)$ is a non-Ps,

(i) two edges of C_i incident to $b(C_i)$ are red if $b(C_i)=r$, and

(ii) two edges of C_i incident to $b(C_i)$ are of the same color which is different from the color of $e_i(b(C_i))$ if $b(C_i)\neq r$.

proof. Let g be the frond in $[C_i]-[T_i]$ and x and y be its two end-vertices. By Lemma 3, $P_i(x)$ and $P_i(y)$ are pseudo-alternating paths. Suppose that x is a non-Ps, then $e_i(x)$ is blue if x is $\emptyset$ and red if I. Since the edge g must satisfy one of the conditions of Steps 10, 11, and 12, the color of g differs from that of $e_i(x)$. Since the same argument holds with y, C_i is a pseudo-alternating path from $b(C_i)$ to $b(C_i)$ itself.

Suppose that $b(C_i)$ is a non-Ps other than the root-vertex r. Let e be an edge of C_i incident to $b(C_i)$. If e is an arc, then the color of e is different from that of $e_i(b(C_i))$, by Lemma 3. If e is a frond, then e is in $[C_i]-[T_i]$, that is the edge g aforementioned. Then the above argument shows that the color of e differs from that of $e_i(b(C_i))$. Thus two edges of C_i incident to $b(C_i)$ are of the same color which differs from that of $e_i(b(C_i))$.

In the case where $b(C_i)=r$, since the root-vertex r is $\emptyset$, two edges of C_i incident to $b(C_i)$ are red from the algorithm and the above argument.

Q.E.D.

Corollary 1.

Let z be an arbitrary non-Ps of C_i other than $b(C_i)$. If $b(C_i)=r$, there are two distinct pseudo-alternating paths from r to z such that

(i) both initial edges are red, and

(ii) the colors of the final edges are distinct.

Otherwise, there are two distinct pseudo-alternating paths to z such that

(i) both paths have $e_i(b(C_i))$ as the initial edges, and

(ii) the colors of the final edges are distinct.

Lemma 5.

Let u be a Ps of G_i other than r^* and e be an arbitrary edge incident to u other than the entrance-edge $e_i(u)$. Then in G there is an alternating path whose initial edge is $e_i(u)$ and final edge is e.

proof. We shall verify the assertion by the induction over the shrinking level $s(u)$.

(i) the case where $s(u)=1$. Since there is no Ps in $\langle u\rangle$, u is a Ps shrunk in Step 19 and can be written as $\langle C_j\rangle/\langle C_j\rangle$ for some j less than i. Let z be the vertex of C_j to which e is incident. Then in the case where $z\neq b(C_j)$ there is in G an alternating path whose initial edge is $e_j(b(C_j))$ and final edge is e irrespective of the color of e by Corollary 1. In the case where $z=b(C_j)$, the path consisting of $e_j(b(C_j))$ and e is an alternating path when their colors are distinct, and the path consisting of $e_j(b(C_j))$, C_j and e is an alternating

path by Lemma 4 when their colors are identical. Since $e_i(u)=e_j(b(C_j))$, there is the required alternating path in G.
(ii) the case where $s(u)=k$. When u is a Ps shrunk in Step 13, that is $u=\{x,y\}/\{x,y\}$, where x and y are both end-vertices of the edge g in G_j for some j less than i, then x and y are Ps's whose shrinking levels are both less than k. $e_i(u)$ is the entrance-edge of one of the two Ps's, say x. Since g is an arc, then by Lemma 2, g is the entrance-edge of y. Therefore if we apply the inductive hypothesis to x and y, it is seen that there is the required alternating path in G. Consider the case where u is a Ps shrunk in Step 19, that is $u=\langle C_j\rangle/\langle C_j\rangle$ for some j less than i. If the vertex z of C_j to which e is incident is a Ps, then the path on T_j whose initial edge is $e_j(b(C_j))$ and final edge is $e_j(z)$ makes a pseudo-alternating path together with the edge e. Let us consider the case where the vertex z is a non-Ps. If $z\neq b(C_j)$, then by Corollary 1 there is a pseudo-alternating path whose initial edge is $e_j(b(C_j))$ and final edge is e irrespective of the color of e. If $z=b(C_j)$, $e_j(b(C_j))$ and e make an alternating path when their colors are distinct, and $e_j(b(C_j))$, C_j and e make a pseudo-alternating path when their colors are identical. Since the entrance-edge of every Ps except the initial and final vertices of the above pseudo-alternating paths is contained in the paths, there is in G an alternating path whose initial edge is $e_j(b(C_j))$ and final edge is e by the inductive hypothesis. Since $e_i(u)=e_j(b(C_j))$, the lemma follows.

Q.E.D.

Lemma 6.

Let us consider the case where r^* is the root-Ps. Let e be an arbitrary edge incident to r^*. Then in G there is an alternating path from the root-vertex r whose initial edge is red and final edge is e.

proof. This lemma can be verified in the same way as Lemma 5. Q.E.D.

By Lemmas 1, 3, 4, 5, 6 and Corollary 1, we obtain Lemma 7.

Lemma 7.

Augmentation can be made in Step 20.

Since shrinking decreases the number of vertices of the graph and growing tree increases the number of vertices of the tree, both operations are not repeated infinitely. Thus after finite number of iterations of shrinking and growing tree, augmentation is made unless the algorithm terminates. For a q-subfactor B let

$$def(B)=\Sigma\{p(v)-d^B(v)\}$$

where the summation is taken over all deficient vertices. If B is augmented to $B'=(B-[P]\cap B)\cup([P]\cap R)$, then $def(B')$ is less than $def(B)$ by one or two. Therefore the algorithm terminates after finite number of iterations.

3. Tutte's Tree

Let us consider the case where the algorithm terminates at Step 4 without obtaining a (p,q)-factor. Let G_n and T_n be the final graph and the final tree, which is referred to as Tutte's tree. Let D_n and d_n be the mappings of the incidence relation and of degrees of G_n. Let S be the set of $\emptyset$'s, T the set of I's, U the set of Ps's and V the set of the other vertices of G_n.

Lemma 8.

(i) $D_n(T,T)\cap B$, (ii) $D_n(S,S)\cap R$, and (iii) $D_n(U,U)$ are empty.

proof. Suppose there were a blue edge e_b connecting two vertices v and w in T. If e_b were an arc, e_b should be the entrance-edge of either v or w, which is contrary to Lemma 1. Then e_b is a frond, and it forms a cycle with the arcs (see Step 11). Then either augmentation or shrinking could be made (see Steps 15-20). This is contrary to the definition of G_n.

It is verified in the same way that $D_n(S,S)\cap R$ is empty.

Suppose that there were an edge e connecting two Ps's. If e were an arc, then its both end-vertices could be shrunk (see Step 13). Otherwise, since e forms a cycle with the arcs (see Step 14), either argmentation or shrinking could be made (see Steps 15-20). This contradicts the definition of G_n.
the definition of G_n. Q.E.D.

Lemma 9.

(i) $D_n(T,V)\cap B$, (ii) $D_n(S,V)\cap R$, and (iii) $D_n(U,V)$ are empty.

proof. Since the vertices in V are all points, we could grow tree T_n if such edges existed (see Steps 4 and 5). This is contrary to the definition of T_n.

Q.E.D.

Let u be a *Ps* other than r^*. We call u a blue or a red *Ps* after the color of its entrance-edge.

Lemma 10.

(i) Let u_b and u_r be a blue and a red *Ps* of G_n, respectively. The end-vertex of $e_n(u_b)$ opposite to u_b is in T, and that of $e_n(u_r)$ opposite to u_r is in S.

(ii) Let e be an edge incident to a blue or red *Ps* u of G_n other than its entrance-edge $e_n(u)$. The end-vertex of e opposite to u is in S if e is blue and in T if red.

(iii) (ii) is true for any edge incident to the root-*Ps* r^*.

proof. We shall prove the lemma for a blue *Ps* u_b becuase other cases can be verified in the same way.

For each edge incident to u_b, its opposite end-vertex to u_b is in either S or T by Lemmas 8 and 9. Let v be the end-vertex of $e_n(u_b)$ opposite to u_b and suppose that v were in S. Since there is no blue arc incident to the root-vertex r, v is not the root vertex. Then by Lemma 1, there is a blue entrance-edge $e_n(v)$. Since the tree-path $P_n(u_b)$ to u_b contains $e_n(v)$, this is contrary to Lemma 3. Therefore we get that v is in T.

Let e_b be an arbitrary blue edge incident to u_b other than $e_n(u_b)$. Let w_b be the end-vertex of e_b opposite to u_b, and suppose that w_b were in T. If we further suppose that e_b were a frond, augmentation or shrinking could be made (see Steps 15-20), which is a contradiction. Then e_b is an arc. Therefore by Lemma 2 we obtain that $e_b=e_n(w_b)$, which is contrary to Lemma 1.

Let e_r be an arbitrary red edge incident to u_b, and w_r be the end-vertex of e_r opposite to u_b. Suppose that w_r were in S. By the same argument we have that e_r is an arc, then $e_r=e_n(w_r)$. This is also contrary to Lemma 1.

Q.E.D.

By Lemmas 8, 9 and 10, the final graph G_n is sketched as shown in the figure.

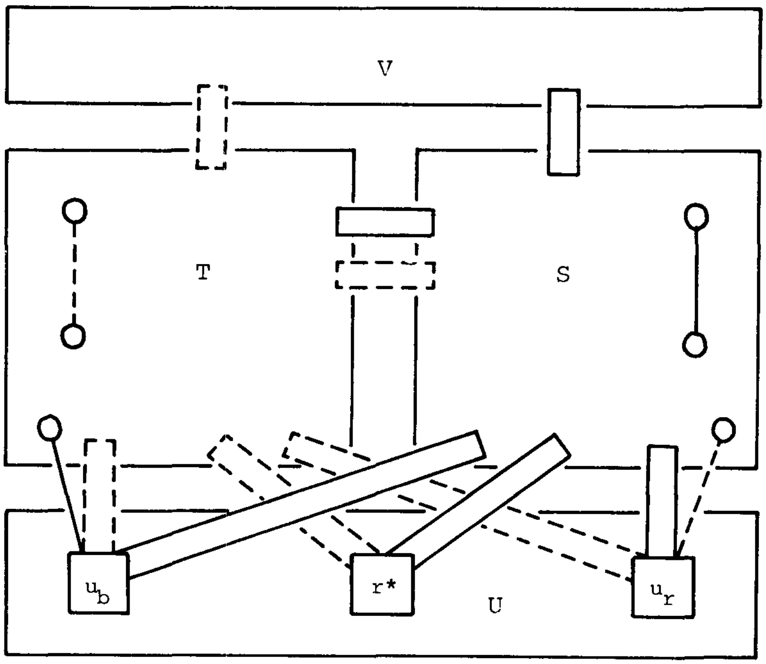

Fig. final graph G_n

—— a blue edge ▭ several blue edges

- - - - a red edge ⬚ several red edges

○ a non-*Ps* □ a *Ps*

<u>Lemma 11.</u>

Let B be the current q-subfactor of G_i and u be a blue or a red *Ps* of G_i. For every vertex v in $<<u>>$, $p(v)=q(v)=d^B(v)$. If r^* is the root-*Ps* of G_i, then $p(v)=q(v)=d^B(v)$ for every vertex in $<<r^*>>$ except the root-vertex r with $p(r)=q(r)=d^B(r)+1$.

proof. We shall prove the lemma by induction over the shrinking level $s(u)$.
(i) the case where $s(u)=1$. Since no vertices in $<u>$ are *Ps*'s, $u=<C_j>/<C_j>$. If there were deficient vertices or vertices with $p(v)<q(v)$ in $<C_j>$, augmentation could be made (see Steps 18-20). This contradicts the fact that $<C_j>$, has been shrunk.
(ii) the case where $s(u)=k$. By the same argument it is seen that $p(v)=q(v)$ $=d^B(v)$ holds for all non-*Ps*'s in $<u>$. Let v be a *Ps* in $<u>$, then v is not the

root-*Ps* for u is not the root-*Ps*. Then by the inductive hypothesis $p(w)=q(w)=d^B(w)$ for every vertex w in $<<v>>$. Thus the assertion follows.

For the root-*Ps* r^* it is verified in the same way that $p(v)=q(v)=d^B(v)$ for every vertex in $<<r^*>>$ except the root-vertex r itself. By the definition the root-vertex r is deficient, that is $d^B(r)<p(r)$. On the other hand $q(r)-d^B(r)$ cannot exceed unity, otherwise augmentation could be made (see Step 16). Thus $1\leqq p(r)-d^B(r)\leqq q(r)-d^B(r)\leqq 1$, that is $p(r)=q(r)=d^B(r)+1$.

Q.E.D.

Lemma 12.

If u is in U, then $<<u>>$ is a member of $K(S,T)$.

proof. Since u is an isolated vertex of $G_n(S\cup T)$ by Lemmas 8 and 9 (see the figure), $<<u>>$ is the vertex-set of a connected component of $G(S\cup T)$. By Lemma 11, $p(v)=q(v)$ for every vertex in $<<u>>$. The equality

$$\sum_{v\in<<u>>} d^B(v)+d^B(<<u>>)\equiv 0 \pmod 2$$

holds for the expression on the left is equal to twice the number of edges in B having one end-vertex in $<<u>>$. Since $d^B(<<u>>)=d_n^B(u)$, we get that

$$(9)\qquad \sum_{v\in<<u>>} d^B(v)+d_n^B(u)\equiv 0 \pmod 2.$$

Let u_b, u_r and r^* be a blue *Ps*, a red *Ps*, and the root-*Ps*, respectively, then by Lemma 10 (see the figure),

$$d_n(u_b,S)=d_n^B(u_b)-1,$$

$$(10)\qquad d_n(u_r,S)=d_n^B(u_r)+1,$$

$$d_n(r^*,S)=d_n^B(r^*).$$

When a *Ps* u is a blue or a red *Ps*, by (9), (10), Lemma 11 and the fact that $d(<<u>>,S)=d_n(u,S)$,

$$\sum_{v\in\langle\langle u\rangle\rangle} p(v)+d(\langle\langle u\rangle\rangle,S)=\sum_{v\in\langle\langle u\rangle\rangle} p(v)+d_n(u,S)$$

$$=\sum_{v\in\langle\langle u\rangle\rangle} d^B(v)+d_n^B(u)\pm 1\equiv 1 \quad (mod\ 2).$$

Thus $\langle\langle u\rangle\rangle$ is a member of $K(S.T)$

By Lemma 11, we obtain that

$$\sum_{v\in\langle\langle r^*\rangle\rangle} p(v)=\sum_{v\in\langle\langle r^*\rangle\rangle} d^B(v)+1.$$

Hence by (9), (10), and the fact that $d(\langle\langle r^*\rangle\rangle,S)=d_n(r^*,S)$,

$$\sum_{v\in\langle\langle r^*\rangle\rangle} p(v)+d(\langle\langle r^*\rangle\rangle,S)=\sum_{v\in\langle\langle r^*\rangle\rangle} d^B(v)+1+d_n(r^*,S)$$

$$=\sum_{v\in\langle\langle r^*\rangle\rangle} d^B(v)+d_n^B(r^*)+1\equiv 1 \quad (mod\ 2).$$

Thus $\langle\langle r^*\rangle\rangle$ is also a member of $K(S,T)$. Q.E.D.

Lemma 13.

The pair of subsets S and T of $\langle G\rangle$ violates the condition of Theorem 1.

proof. S and T are trivially disjoint. Let k_b and k_r be the number of blue and red Ps's of G_n, respectively. Since $q(v)=d^B(v)=d_n^B(v)$ for every vertex in T (see Step 7),

$$\sum_{v\in T} q(v)=\sum_{v\in T} d_n^B(v)$$

By lemmas 8, 9, and 10 (see the figure),

$$\sum_{v\in T} d_n^B(v)=k_b+d_n^B(T,S)$$

$$=k_b+d_n^B(S)-\{d_n^B(S,V)+d_n^B(S,U)\}$$

$$=k_b+k_r+d_n^B(S)-\{d_n^B(S,V)+d_n^B(S,U)+k_r\}$$

$$=k_b+k_r+d_n^B(S)-\{d_n^B(S,V)+d_n(S,U)\}$$

$$=k_b+k_r+d_n^B(S)-d_n^{[L]}(S) \ ,$$

where $L=G_n(T)$.

Let m be the number of edges in $D_n(S,S)$. Since every edge in $D_n(S,S)$ is blue by Lemma 8,

$$\sum_{v\in S} d_n^B(v)=2m+d_n^B(S),$$

$$\sum_{v\in S} d_n^{[L]}(v)=2m+d_n^{[L]}(S).$$

Then, since $d_n^B(v)=d^B(v)$ and $d_n^{[L]}(v)=d^{[G(T)]}(v)$ for every vertex v in S,

$$\text{(11)} \quad \sum_{v\in T} q(v)=k_b+k_r+\sum_{v\in S}\{d_n^B(v)-d_n^{[L]}(v)\}$$

$$=k_b+k_r+\sum_{v\in S}\{d^B(v)-d^{[G(T)]}(v)\}.$$

The vertex r is contained either in S or in the vertex-set $\langle\langle r^*\rangle\rangle$ corresponding to the root-Ps r^*. Let us first consider the case where the vertex r is in S. For every vertex v in S, $d^B(v)\leqq p(v)$ (see Step 8), and $d^B(r)<p(r)$. Then

$$\text{(12)} \quad \sum_{v\in S}\{d^B(v)-d^{[G(T)]}(v)\}<\sum_{v\in S}\{p(v)-d^{[G(T)]}(v)\}.$$

Therefore, by Lemma 12, (12), and (11),

$$r(S,T)=|K(S,T)|+\sum_{v\in S}\{p(v)-d^{[G(T)]}(v)\}$$

$$\geqq k_b+k_r+\sum_{v\in S}\{p(v)-d^{[G(T)]}(v)\}$$

$$>k_b+k_r+\sum_{v\in S}\{d^B(v)-d^{[G(T)]}(v)\}$$

$$=\sum_{v\in T} q(v).$$

Consider the case where r is contained in $\langle\langle r^*\rangle\rangle$. Since $d^B(v)\leqq p(v)$ for every vertex in S and by Lemma 12,

$$k_b+k_r+1\leqq|K(S,T)|,$$

we have that

$$r(S,T)>\sum_{v\in T} q(v)$$

Thus the lemma follows. Q.E.D.

Proof of the Sufficiency of Theorem 1.

If the condition of Theorem 1 holds for every pair of disjoint subsets of $\langle G\rangle$, the algorithm never generates Tutte's tree. Thus by the finiteness of the algorithm the sufficiency of Theorem 1 is verified. Q.E.D.

Tutte's existence condition of an f-factor [10,11,12] is obtained only by substituting $f(v)$ for both $p(v)$ and $q(v)$ of the condition of Theorem 1 and the definitions of $K(S,T)$ and $r(S,T)$.

Corollary 2. ([10] Theorem XIV)

A graph G has an f-factor if and only if

$$\sum_{v\in T} f(v)\geqq r'(S,T)$$

for every pair of disjoint subsets S and T of $\langle G\rangle$, where

$$K'(S,T)=\left\{\langle H\rangle \;\middle|\; \begin{array}{l} H \text{ is a connected component of } G(S\cup T) \text{ such that} \\ \sum_{v\in\langle H\rangle} f(v)+d(\langle H\rangle,S)\equiv 1 \quad (mod\ 2). \end{array}\right\}$$

$$r'(S,T)=|K'(S,T)|+\sum_{v\in S}\{f(v)-d^{[G(T)]}(v)\}.$$

Theorem 1, together with the algorithm, reveals the fact that the difficulty of the factor problem is attributed to the existence of odd cycles consisting of vertices with $p(v)=q(v)$. Therefore the condition for the existence of (p,q)-factors is simplified as for the graphs having no such cycles. In fact observing the definition of $K(S,T)$ leads to Corollary 3.

Corollary 3.

Let $p(v)$ be less than $q(v)$ for every vertex of G. Then the graph G has a (p,q)-factor if and only if

$$\sum_{v\in T} q(v) \geqq \sum_{v\in S} \{p(v)-d^{[G(T)]}(v)\}$$

for every pair of disjoint subsets S and T of $\langle G\rangle$.

Corollary 4.

Let G be a bipartite graph with the vertex-set $\langle G\rangle=V_1\cup V_2$. G has a (p,q)-factor if and only if

$$\sum_{v\in T} q(v) \geqq \sum_{v\in S} \{p(v)-d^{[G(T)]}(v)\}$$

for every pair of subsets $S\subseteq V_i$ and $T\subseteq V_j$ where $i,j=1,2$ and $i\neq j$.

proof. The assertion is immediate if we notice that (i) shrinking is not made, and that (ii) once a vertex in V_i is labeled $\emptyset$ (I), no vertices in V_i can be I $(\emptyset)$ for $i=1,2$. Q.E.D.

Koren [6] presented a distinct discussion on the (p,q)-factor problem and provided an equivalent condition to Corollary 3 (Theorem 4.1, see also Ford and Fulkerson [5] p.77) by means of the network flow theory.

Corollary 4 is the condition under which there is a feasible flow in a given bipartite network each of which arcs has unit capacity.

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