# CHARACTERIZATION OF THE SATISFACTORY DECISION PRINCIPLE

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*Abstract*    This paper discusses essential significances of the satisfactory decision criterion in an axiomatical way. For decision problems under undertainty the traditional decision criteria, for instance, the max-min, the regret and the Laplace criterion are well known. The satisfactory decision criterion is another type of principle towards decision problems under uncertainty. The traditional decision criteria requires first to arrange all alternatives in a linear order and then to subset the first elements as its solution. On the other hand, the satisfactory decision criterion is one that first arranges alternative in a linear order and second devides the linear order into two parts and third selects the "good" part as its solution, which is called a satisfactory subset. In this sense the satisfactory decision criterion is much simpler than the traditional ones. It is introduced by Simon to explain a realistic decision behaviour of human being and is formulated by Mesarovic as one of the most important decision criteria for control theory.

First we investigate properties of a satisfactory subset and find that a subset of the set of alternatives is a satisfactory subset if and only if it is an algebriac closed set. Second we axiomatize the satisfactory decision criterion by decomposing the resultant properties and make its "degree of simpleness" clear. Finally by comparing our axiom system with the axion systems of the traditional decision criteria obtained by Milnor, we find that the max-min and the regret criterion can be seen as special cases of the satisfactory decision criterion, that is, as the satisfactory decision criteria with special types of aspiration levels. The Laplace and the Hurwitz, however, can not be seen as the same as the other decision criteria including the satisfactory decision criterion.

In this paper we do not assume any structure in the sets of alternatives nor of uncertainties. Strong results under such more specified conditions will be presented later.

## 1. Introduction

In this paper a characterization of the satisfactory decision criterion is presented. Decision makings are traditionally classified into three categories; decision making under certainty, under risk and uncertainty. Each category is associated with specific decision criteria. For decision making

under certainty optimization criterion is used.  For decision making under
risk optimization with respect to expectation is used.  Decision making under
uncertainty has four criteria, that is, the max-min criterion, the regret
principle, the Hurwitz α-criterion and the Laplace criterion.  Meaning of
optimization or optimization with respect to expectation is clear and hence
its problem is to find good algorithms to realize it.  As for criteria for
uncertainty their meanings are not so simple and sometimes their usage becomes
quite controversial.  It is, therefore, important to extract their essential
meanings as well as to find their realization algorithm.   Milnor [3] studied
these four criteria and succeeded in axiomatizing them.  His results revealed
to some extent real properties of the criteria.

The satisfactory decision criterion is another type of decision criterion
for decision making under uncertainty.  It was introduced by Simon [5] to
explain a realistic behaviour of human being and is now accepted as are the
traditional decision criteria in the field of organization theory and psychology.
Recently it has attracted attention of engineers and is formulated by Mesarovic
[2] as one of the most important decision criteria for control theory.

In spite of its popularity no work has been done to reveal an essential
significance of the satisfactory criterion in the same manner as Milnor's.
This paper is trying to accomplish this theme.

If optimization is used as a decision criterion, a given set of alterna-
tives  $M$  should be linearly ordered and the best elements of it selected as
decision.  Contrary to optimization, as pointed out by Simon, the satisfactory
decision criterion is just to divide a set of alternatives $M$ into two parts;
a satisfactory subset  $M'$  and an unsatisfactory subset   $M - M'$.  This is
simple enough to be applicable to most cases.  If we can get sufficient infor-
mation, optimization is certainly superior to the satisfactory criterion.
In many practical cases, however, we are forced to make a decision with poor
information, in other words, under the bounded rationality, and then opti-
mization is quite powerless.  As we face more complicated problems, the situ-
ation becomes more severe.  This is the reason why the satisfactory criterion
has recently attracted more and more attention.


2.  Definitions and Notations

In this section basic concepts and definitions are given.


2.1.  Decision Problem under Uncertainty

Let  $\mathcal{M}$ , $\mathcal{U}$ and  $R$  be the set of all alternatives, the set of all un-

certainties and the set of real numbers, respectively. Then the set of all
decision problems under uncertainty is defined by

$$\mathscr{D} = \mathscr{P}(\mathcal{M}) \times \mathscr{P}(\mathcal{U}) \times R^{\mathcal{M} \times \mathcal{U}}$$

where $\mathscr{P}(\mathcal{M})$ is the power set of $\mathcal{M}$ , $\mathscr{P}(\mathcal{U})$ the power set of $\mathcal{U}$ , while
$R^{\mathcal{M} \times \mathcal{U}}$ is the set of all functions of the form

$$\bar{g} : \mathcal{M} \times \mathcal{U} \to R .$$

A decision problem under uncertainty $D \in \mathscr{D}$ is, then, given by a
triplet

$$D = ( M , U, g )$$

where $M \in \mathscr{P}(\mathcal{M})$ is called a set of alternatives, $U \in \mathscr{P}(\mathcal{U})$ a set of
uncertainties or a set of states of nature, while $g : M \times U \to R$ is called
a performance function. The performance function $g$ is given by the re-
striction of some $\bar{g} : \mathcal{M} \times \mathcal{U} \to R$ , that is, $g = \bar{g} \mid M \times U$.

It should be noticed that in general a set of alternatives $M$ and a set
of uncertainties $U$ of $D$ are different from $\mathcal{M}$ and $\mathcal{U}$ . It is because
a decision maker may not identify all the alternatives and all the uncertain-
ties when he is forced to make a decision.

In this paper $g$ is always assumed a performance function, and so a
decision maker is supposed to try to find $m \in M$ which maximizes $g( m, u)$
for each fixed $u \in U$ .

When $D = ( M, U, g )$ is given and if $U$ is not singleton, the decision
maker can not select an alternative in an "absolutely rational" way. He has to
use a decision criterion ( e.g. the max-min, the regret and so on ) in order
to make a "rational" decision under such a situation. In this paper a
decision criterion $c$ is represented by a mapping

$$c : \mathscr{D} \to \mathscr{P}(\mathcal{M} \times \mathcal{M}) ,$$

where $c(D) \in \mathscr{P}(\mathcal{M} \times \mathcal{M})$ is a partially ordered preference relation denoted by
$\succcurlyeq_c$ . In a general preference relation two indifferent alternatives are not
always equal and in this sense it is not correct to call $\succcurlyeq_c$ a partial order,
but to simplipfy the notation $\succcurlyeq_c$ will be treated as a partial order. $c(D)$
$((M, \succcurlyeq_c ))$ is called as relation (relational) structure (on $M$ ) with
respect to $c$.

In the sequel we will consider an order structure $\succcurlyeq_{M'}$ of a set $M'$ which
is generated by adding a member $m$ to or by extracting from a set $M$ with
order $\succcurlyeq_M$ . In this case we always assume the following natural relations
hold :

If $M' = M \cup \{ m \}$ , then $m' \succcurlyeq_M m''$ implies $m' \succcurlyeq_{M'} m''$.

If $M' = M - \{ m \}$ , then $m' \succcurlyeq_{M'} m''$ implies $m' \succcurlyeq_M m''$.

$$S_\tau(D) = \{m \in M \mid (\forall u \in U)(g(m, u) \geq \tau(u))\}.$$

When a linear order has only maximal and minimal elements, it is called binary structure.  A structure with respect to the satisfactory criterion is clearly binary.


## 2.3.  Max-min criterion

The max-min decision criterion, one of the typical criteria for uncertainty, is defined as follows :

Definition 2.3.

For a given decision problem $D = ( M, U, g )$, find $\hat{m} \in M$ such that

$$inf_{u \in U} \, g(\hat{m}, u) = sup_{m \in M} \, inf_{u \in U} \, g(m, u).$$

$\hat{m}$ is called a solution with respect to the max-min criterion.  Though, strictly speaking, this criterion should be called sup-inf criterion, we will call it max-min as usual.  For the max-min criterion, a linear order $\geqslant_m$ is introduced as

$$m \geqslant_m m' \quad \longleftrightarrow \quad inf_{u \in U} \, g(m, u) \geq inf_{u \in U} \, g(m', u).$$

It is obvious from the definition that the solution set $S_m(D)$ of $D$ under the max-min criterion is defined by

$$S_m(D) = \{m \in M \mid (\nabla m' \in M)(m' \geqslant_m m \to m \geqslant_m m')\}.$$


## 2.3.  Regret principle

The regret principle is defined by :

Definition 2.4.

For a given decision problem $D = ( M, U, g )$, find $\hat{m} \in M$ such that

$$inf_{u \in U} \, g(\hat{m}, u) = sup_{m \in M} \, inf_{u \in U}(g(m, u) - sup_{m \in M} \, g(m, u)).$$

Such an $\hat{m}$ is called a solution with respect to the regret decision principle.

For the regret principle a linear order $\geqslant_r$ is introduced by

$$m \geqslant_r m' \quad \longleftrightarrow \quad r_m \geq r_{m'},$$

where $\quad r_m = inf_{u \in U} \, (g(m, u) - sup_{m \in M} \, g(m, u))$

and $\quad r_{m'} = inf_{u \in U} \, (g(m', u) - sup_{m \in M} \, g(m, u))$ , respectively.

$g( m, u) - \underset{m \in M}{sup} \, g( m, u )$ is called regret of opportunity loss and represents the difference in the relative value that would result if $u$ were true state of nature and if $m$ were adopted instead of the best strategy under $u$  .
Obviously

$g(m, u) - sup_{m \in M} \, g(m, u) \leq 0.$

$r_m$ shows the least regret of $m$ with respect to $u$ . The solution set $S_r(D)$ of $D$ under the regret principle is obviously given by

$S_r(D) = \{m \in M \mid (\forall m' \in M)(m' \succcurlyeq_r m \to m \succcurlyeq_r m')\}.$

## 2.5. Hurwitz α - criterion

The Hurwitz α -criterion is defined by :

Definition 2.5.

For a given problem $D = (M, U, g)$, find $\hat{m} \in M$ such that

$inf_{u \in U} \, g(\hat{m}, u) = sup_{m \in M}\{\alpha \, sup_{u \in U} \, g(m, u) + (1-\alpha) \, inf_{u \in U} \, g(m, u)\}.$

Such an $\hat{m}$ is called a solution with respect to the Hurwitz α - criterion. For the Hurwitz α - criterion, a linear order $\succcurlyeq_h$ on $M$ is introduced by

$m \succcurlyeq_h m' \longleftrightarrow h_m \geq h_{m'}$

where $h_m = \alpha \, sup_{u \in U} \, g(m, u) + (1-\alpha) \, inf_{u \in U} \, g(m, u)$

and $h_{m'} = \alpha \, sup_{u \in U} \, g(m', u) + (1-\alpha) \, inf_{u \in U} \, g(m', u)$, respectively.

α is a real number between 0 and 1 reflecting "degree of optimism". α = 0 gives the pessimistic max-min ( 2.3 ), and α = 1 gives an optimistic behaviour occasionally called max-max. The solution set $S_h(D)$ of $D$ under the Hurwitz α - criterion is given by

$S_h(D) = \{m \in M \mid (\forall m' \in M)(m' \succcurlyeq_h m \to m \succcurlyeq_h m')\}.$

## 2.6. Laplace criterion

When the number of uncertainty is finite, say n, the Laplace criterion is defined by :

Definition 2.6.

For a given problem $D = (M, U, g)$, where $U = \{u_1, u_2, \cdots, u_n\}$, find $\hat{m} \in M$ such that

$inf_{u \in U} \, g(\hat{m}, u) = sup_{m \in M} \, \frac{1}{n} \sum_{i=1}^{n} g(m, u_i).$

Such an $\hat{m}$ is called a solution with respect to the Laplace criterion. For the Laplece criterion a linear order $\succcurlyeq_L$ on $M$ is introduced

by $m \succcurlyeq_L m' \longleftrightarrow L_m \geq L_{m'}$ ,

where $L_m = \frac{1}{n} \sum g(m, u_i)$ ,

and   $L_{m'} = \frac{1}{n} \sum g(m, u_i)$ , respectively.

The solution set $S_L(D)$ of $D$ under the Laplace criterion is, then, given by

$$S_L(D) = \{m \in M \mid (\forall m' \in M)(m' \succcurlyeq_L m \to m \succcurlyeq_L m')\}.$$

## 3. Axiomatization of the Satisfactory Decision Criterion

### 3.1. Properties of a satisfactory subset

In this section we will discuss some properties of a satisfactory subset as preparation of axiomatization in Section 3.2. Our problem in this section is : Given a decision problem $D = ( M, U, g )$ and given a subset $M' \subset M$ , find necessary and sufficient conditions of $M'$ for existence of some aspiration level $\tau : U \to R$ such that $M'$ is a satisfactory subset of $M$ with respect to $\tau$ .

If $M' \subset M$ is arbitrary, a desirable aspiration level does not necessarily exist as the following simple example shows.

Example 3.1.

Suppose $M = \{m_1, m_2, m_3\}$ and $U = \{u_1, u_2, u_3\}$ and a pay-off matrix is given as below.

| $g$ | $m_1$ | $m_2$ | $m_3$ |
|-----|-------|-------|-------|
| $u_1$ | 2 | 3 | 4 |
| $u_2$ | 1 | 2 | 3 |
| $u_3$ | 3 | 1 | 5 |

Table 3-1

For this problem there is no aspiration level $\tau$ such that $M' = \{m_1, m_2\}$ is a satisfactory subset of $M$ with respect to $\tau$ . In fact, since any aspiration $\tau$ that makes $M'$ a satisfactory subset of $M$ must satisfy relations

$\tau(u_1) \leqq 2$ ,    $\tau(u_2) \leqq 1$ and $\tau(u_3) \leqq 1$,

$\{m_3\}$ is also satisfactory with respect to $\tau$.

In order to present results in a concise way, we introduce the following definition.

Given a decision criterion $c$ , the solution function

$$S_c : \quad \mathscr{D} \quad \rightarrow \quad \mathscr{P}(\hbar)$$

is defined by $S_c(D) = \{ m \in M \mid (\forall m \in M)(m' \succcurlyeq_c m \rightarrow m \succcurlyeq_c m') \}$.

$S_c(D)$ is the set of maximal elements which are the most desirable with respect to $c$ and is called the solution set of $(D, c)$ or the solution set of $D$ under $c$ .

In the latter part of this section we will give brief definitions of decision criteria to be used in the paper.

## 2.2.  Satisfactory criterion

The satisfactory criterion proposed by Simon was formulated by Mesarovic. In his definition the existence of an aspiration level $\tau : U \rightarrow R$ is essential. The satisfactory decision criterion is defined by :
Definition 2.1.

For a given decision problem $D = (M, U, g)$ and for a given aspiration level $\tau : U \rightarrow R$ , find $\hat{m} \in M$ such that

$$(\forall u \in U)( g(\hat{m}, u) \geqq \tau(u)).$$

Such an $\hat{m}$ is called a satisfactory alternative with respect to $\tau$ .

The definition says that $m$ is satisfactory if the performance by $m$ is "better" than the aspitation level for any $u \in U$ .
Definition 2.2.

For a given decision problem $D = (M, U, g)$ a subset $M'$ of $M$ is called a satisfactory subset of $M$ with respect to $\tau$

if $(\forall m \in M')(\forall u \in U)(g(m, u) \geqq \tau(u))$ and
$(\forall m \notin M')(\exists u \in U)(g(m, u) < \tau(u)).$

The definition says that if all the alternatives of $M'$ are satisfactory and all the alternatives of $M - M'$ are unsatisfactory with respect to the aspiration level $\tau$ , then $M'$ is a satisfactory subset of $M$ .

It should be noticed in the definition that if both $m \in M$ and $m' \in M$ belong to $M'$ or $M - M'$ , then $m$ and $m'$ are indifferent and that if $m \in M'$ and $m' \in M - M'$ then $m$ is strictly preferred to $m'$. Therefore, the relation structure with respect to the satisfactory decision criterion is clearly a linear order which has only maximal and minimal elements.

Let $\succcurlyeq_\tau$ be the satisfactory order relation by an aspitation level $\tau$ such that

$$(m \succcurlyeq_\tau m') \;\&\; not(m' \succcurlyeq_\tau m) \longleftrightarrow (m \in M') \;\&\; (m' \in M - M') .$$

The solution set $S_\tau(D)$ of $D$ under the satisfactory criterion is defined by

Definition 3.2.

    For a subset $M'$ of $M$ a function

$$\tau_{M'} : U \to R$$

is defined by

$$\tau_{M'}(u) = \inf_{m \in M'} g(m, u).$$

    $\tau_{M'}$ is called fundamental aspiration level for $M'$.

Propositon 3.3.

    For any subset $M'$ of $M$ the following three statements are equivalent.

(1)    $M'$ is a satisfactory subset of $M$.

(2)    $M'$ is a satisfactory subset with respect to the fundamental aspiration
       level $\tau_{M'}$ .

(3)    There exist $U' \subset U$ and $\tau' : U' \to R$ such that
       $(\forall m \in M)(\forall u \in U')(g(m, u) \geq \tau'(u))$ and
       $(\forall m \notin M')(\exists u \in U')(g(m, u) < \tau'(u)).$

    Proof : Refer to Appendix 1.

    Now let us consider meaning of Proposition 3.3.   According to (2) of
Proposition 3.3, whenever $M'$ is a satisfactory subset, $M'$ is also a satis-
factory subset with respect to the fundamental aspiration level $\tau_{M'}$ .
Intuitively, $\tau_{M'}$ can be viewed as the "worst" performance by the alternatives
of $M'$ as shown by the fat line in Fig. 3-1 where the horizontal and vertical
axes represent $u$ and real numbers, respectively.   It should be noticed in
Fig. 3-1 that although $m'$ is not preferred to any alternative constituting
$\tau_{M'}$ ( e.g. $m_1$ ), in order to be a satisfactory subset of $M$ $M'$ must contain $m'$.
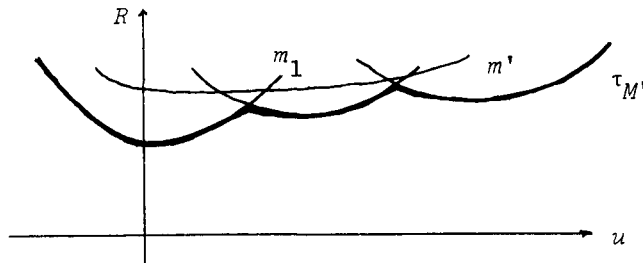


Figure 3-I

    (3) of Proposition 3.3 shows that we can determine whether or not a subset
$M'$ of $M$ is a satisfactory subset by calculating the performance only for a
subset $U' \subset U$ .   If there exists an aspiration level $\tau'$ which makes $M'$ satis-
factory for $U'$ , then there also exists an aspiration level $\tau$ which makes $M'$
satisfactory for $U$.

Axiom A-1

There exists a linear order $\succeq_l$ on $M$.

Axiom A-2

$(M, \succeq_l)$ is a binary structure. That is, any alternative is either maximal or minimal with respect to $\succeq_l$. Let $D = (M, U, g)$ be a decision problem. The solution set $S(D)$ of $D$ is the set of all maximal alternatives, that is,

$$S(D) = \{m \in M \mid (\forall m' \in M)(m' \succeq_l m \to m \succeq_l m')\}.$$

Axiom A-3

Let $D = (M, U, g)$ be a decision problem ;

then $S(D) \supset S(M - \{m\}, U, g')$

for any $m \notin S(D)$, where $g' = g \mid (M - \{m\}) \times U$.

This says that even if a nonsolution $m$ of a problem $(M, U, g)$ becomes unavailable, that is, $(M, U, g)$ changes to $(M - \{m\}, U, g')$, every solution of $(M, U, g)$ still remains as a solution of $(M - \{m\}, U, g')$.

Axiom A-4

Let $D = (M, U, g)$ be a decision problem ;

$m \in S(M \cup \{m\}, U, \overline{g}')$, if $(\forall u \in U)(\exists m' \in S(D))(g(m, u) \geq \overline{g}'(m', u))$,

where $\overline{g}' = \overline{g} \mid (M \cup \{m\}) \times U$ and $m \notin M$.

This means that if the decision maker finds a new alternative $m$ satisfying a special condition, then it must be a solution of $(M \cup \{m\}, U, \overline{g}')$.

Note that no statement about an aspiration level appears explicitly throughout the axiom system. Axiom A-1 and A-2 are straightforward requirements from the concept of the satisfactory decison making and can not be controversial. Axiom A-3 is a restatement of Assumption P-2 which is, we believe, not artifitial.

Proposition 3.4.

The satisfactory decision criterion under Assumption P-1, P-2, P-3 and P-4 satisfies Axiom A-1, A-2, A-3 and A-4.

Proof : Refer to Appendix 2.

On the other hand, it is also true that a decision behaviour satisfying Axiom A-1, A-2, A-3 and A-4 is a decision behaviour by the satisfactory decision criterion under Assumption P-1, P-2, P-3 and P-4.

## 3.2. Axiomatization of the Satisfactory Decision Criterion

In this section we will axiomatize the satisfactory decision criterion using the results obtained in Section 3.1.   As pointed out in Section 2, the existence of an aspiration level  $\tau : U \to R$  is essential for the satisfactory decision criterion.   In real situations an aspiration level is usually sensitive to change of a set of alternatives and/or of a set of uncertainties. As a matter of fact, it is a realistic assumption that if the number of the available alternatives increases, aspiration level of a decision maker goes up because he can find a satisfactory alternative more easily.   Furthermore, human being usually modifies his aspiration level in accordance with his information, his experiences and learnings.   (Refer to the learning theory in psychology [6])    In this paper, however, we will not discuss a dynamic situation like a learning process, but *consider a situation where the variation of the aspiration level is none or very small even if the set of alternatives and/or the set of uncertainties change.*   For instance, if a set of alternatives is quite large, we can assume that change of the number of alternatives does not affect the aspiration level so much.

The followings are the assumption of  $\tau$ :

Assumption P-1

The aspiration level  $\tau$  is determined uniquely for a decision maker once a problem is given.

Assumption P-2

When the set of alternatives decreases, the aspiration level does not go up.   That is, let  $\tau$  and  $\tau'$  be aspiration levels for  $( M, U, g )$  and  $( M', U, g )$ , respectively.   Then if  $M \supset M'$ ,  $(\forall u \in U)(\tau(u) \geq \tau'(u))$  holds.

Assumption P-3

When the set of alternatives increases, the aspiration level does not go down.   Furtheremore, if an added alternative  $m$  is satisfactory with respect to  $\tau$  ,  $m$  will not become unsatisfactory with respect to a new aspiration level  $\tau'$  generated by this increase, that is, if  $(\forall u \in U)(g(m, u) \geq \tau(u))$ , then   $(\forall u \in U)(g(m, u) \geq \tau'(u))$ .

Assumption P-4

The aspiration level may move as the change of the set of uncertainties. We assume no conditions for uncertainty.

The satisfactory decision criterion under Assumption P-1, P-2, P-3  and P-4 is axiomatized as follows.

Proposition 3.5.

    If a decision behaviour satisfies Axiom A-1, A-2, A-3 and A-4, then it is a decision behaviour by the satisfactory decision criterion under Assumption P-1, P-2, P-3 and P-4.

    Proof : Refer to Appendix 3.

    Combining Proposition 3.4 and Proposition 3.5. we obtain the following necessary and sufficient condition .

Proposition 3.6.

    A satisfactory decision behaviour satisfying Assumption P-1, P-2, P-3 and P-4 is equivalent to Axiom A-1, A-2, A-3 and A-4.

    As we mentioned above, Axiom A-4 may be a little bit too structured to be called an axiom. In order to analize A-4 we introduce the following two definitions.

Definition 3.7.

    For each order structure $( M , \succcurlyeq_l )$ and for each $m \in M$ an equivalence class $E(m)$ of $m$ is defined by

$$E(m) = \{m' \in M \mid (m \succcurlyeq_l m') \ \& \ (m' \succcurlyeq_l m)\}.$$

    For any $S \subset M$ and $g : M \times U \to R$ let us define $\Lambda S : U \to R$ such that $\Lambda S(u) = \inf\limits_{m \in S} g(m, u)$.

Definition 3.8.

    A set of alternatives $M$ is called satisfying $\Lambda$ - closure property with respect to $g : M \times U \to R$ if for any $S \subset M$ there exists $m \in M$ such that $\Lambda S = g(m, -)$. In particular, when $g$ is fixed, we omit the phrase "with respect to $g$". Furthermore , if $\Lambda S = g(m, -)$, we can practically identify $\Lambda S$ by $m$ and so we will represent $\Lambda S$ by $m$ , that is, we will use the expression $g(\Lambda S, -)$ for $g(m, -)$.

    It should be noticed that if $M$ does not have the $\Lambda$-closure property then $\Lambda S$ is imaginary and does not necessarily belong to $M$.

    In terms of the above definitions, the axiom system is restructured by the following.

Axiom B-1

    There exists a linear order $\succcurlyeq_l$ on $M$.

Axiom B-2

    $( M , \succcurlyeq_l )$ is a binary structure. That is, any alternative is either maximal or minimal with respect to $\succcurlyeq_l$ . Let $D = (M, U, g)$ be a decision problem. The solution set $S(D)$ of $D$ is the set of all maximal alternatives,

that is,

$$S(D) = \{m \in M \mid (\forall m' \in M)(m' \succeq_l m \to m \succeq_l m')\}.$$

Axiom B-3

Let $D = (M, U, g)$ be a decision problem.    Then

$$S(D) \subset S(M - \{m\}, U, g')$$

for any $m \notin S(D)$, where $g' = g \mid (M - \{m\}) \times U$.

Axiom B-4

If          $(\forall u \in U)(g(m, u) \geq g(m', u))$    holds, then $m \succeq_l m'$.

Axiom B-5

For any $m \in M$ and for any subset $S$ of $E(m)$, $\wedge S \succeq_l m$.

Proposition 3.9.

Suppose $M$ has the $\wedge$ - closure property.

If the axiom system A-1, A-2, A-3 and A-4 ( or A-axiom system) hold, then the axiom system B-1, B-2, B-3, B-4 and B-5 ( or B-axiom system ) hold.

Proof :  Refer to Appendix 4.

Proposition 3.10.

Suppose $M$ has the $\wedge$ - closure property.

If B-axiom system holds, then A-axiom system holds.

Proof :  Refer to Appendix 5.

Axiom B-4 means that if an alternative $m$ is superior to another alternative $m'$ uniformly with respect to uncertainty $u$ , $m$ should  be preferred to $m'$ in a decision.  This condition should be always satisfied for any rational decision making.  Axiom B-5, which is derived from ( 2 ) of Proposition 3.3, roughly implies that an alternative insensitive to a change of uncertainty is preferred to a sensitive alternative even if the latter is quite desirable for some specific uncertainty.  This axiom, therefore, reveals that the satisfactory decision criterion is conservative.  Each axiom of B-axiom system has an intutively clear conceptual meaning, while Axiom A-4 is conceptually complicated.  In this sense B-axiom system is superior to A-axiom system.  However, as Proposition 3.9  and Proposition 3.10 say, B-axiom system is meaningful only if $M$ satisfies the  $\wedge$-closure property.  In many practical cases we can expect that the property should hold though we can easily construct a counterexample.

Axiom B-5 is essential for the satisfactory decision criterion.   Since Axiom B-4 should hold for any rational decision making and since Axiom B-1, B-2 and B-3 are natural requirements of the satisfactory decision making and since the satisfactory decision making is considered the simpliest one, we are apt to expect that Axiom B-1, B-2, B-3 and B-4 characterize the satisfactory decision

criterion. But as the following example shows, this expectation is not true. In other words, Axiom B-5 is essential for the satisfactory decision criterion and cannot be derived from Axiom B-1, B-2, B-3 and B-4.

Example 3. 11.

Let $M = \{m_1, m_2, m_3\}$ and $U = \{u_1, u_2, u_3\}$ and a pay off matrix is given as below.

| $g$ | $m_1$ | $m_2$ | $m_3$ |
|-----|-------|-------|-------|
| $u_1$ | 2 | 3 | 2 |
| $u_2$ | 1 | 2 | 2 |
| $u_3$ | 3 | 2 | 2 |

Table 3-II

If we introduce a linear order $\succeq_l$ such that $(m_1 \succeq_l m_2)$, $(m_2 \succeq_l m_1)$ and $(m_2 \succeq_l m_3)$, then $\succeq_l$ does not contradict Axiom B-1, B-2, B-3 and B-4. However, $M' = \{m_1, m_2\}$ can not be a satisfactory subset.

## 4. Comparison between Traditional and Satisfactory Criteria

### 4.1. Traditional criteria and satisfactory decision making

The satisfactory criterion appears quite different from the traditional decision criteria such as the max-min, the regret, the Hurwitz $\alpha$ and the Laplace decision criterion since another information, an aspiration level, is necessary. In this section we compare the satisfactory criterion with the traditional criteria.

It is clear that an order structure of each traditional decision criterion is finer than that of the satisfactory decision criterion, and that the traditional criteria do not necessarily satisfy Axiom A-2.

As for the max-min criterion and the regret principle, we have next proposition.

Proposition 4.1.

The max-min criterion and the regret principle satisfy Axiom A-1, A-3 and A-4.

Proof : Refer to Appendix 6.

Proposition 4.1, however, is not true for the Hurwitz $\alpha$ and the Laplace

criterion as the following simple examples show.

Example 4.2.  Hurwitz  α-Criterion

　　Let  $M = \{ m_1 , m_2 , m_3 \}$  and  $U = \{ u_1 , u_2 , u_3 \}$  and a pay-off matrix is given as below.

| $g$ | $m_1$ | $m_2$ | $m_3$ | $m$ |
|-----|-------|-------|-------|-----|
| $u_1$ | 2 | 0 | 4 | 0 |
| $u_2$ | 1 | 2 | 3 | 2 |
| $u_3$ | 3 | 8 | 5 | 5 |

Table 4-I

We will show that Axiom A-4 is violated for  $\alpha = \dfrac{1}{2}$ .

　　Indeed, since  $h_{m_1} = 2$ ,  $h_{m_2} = 4$  and  $h_{m_3} = 4$ ,  $S_h ( D ) = \{ m_2 , m_3 \}$ .

Suppose  $m$  is a new alternative.  Then since  $g( m , u_1 ) = g( m_2 , u_1 )$ ,

$g( m , u_2 ) = g( m_2 , u_2 )$  and  $g( m , u_3 ) = g( m_3 , u_3 )$  the assumption of

Axiom A-4

$$(\forall u \in U)(\exists m' \in S_h(D))(g (m, u) \geqq g (m', u))$$

is satisfied.  But  $h_m = \dfrac{1}{2} ( 0 + 5 ) = \dfrac{5}{2} < 4$   shows that  $m$   never belongs

to  $S_h(M \cup \{m\}, U, g')$ .

Example 4.3.  Laplace Critrion

　　Let  $M$  and  $U$  be the same as in Example 4.2 and a pay-off matrix is given as below

| $g$ | $m_1$ | $m_2$ | $m_3$ | $m$ |
|-----|-------|-------|-------|-----|
| $u_1$ | 2 | 0 | 4 | 0 |
| $u_2$ | 1 | 0 | 3 | 0 |
| $u_3$ | 3 | 12 | 5 | 5 |

Table 4-II

We will show that Axiom A-4 is violated.

Since  $L_{m_1} = 2, L_{m_2} = 4$  and  $L_{m_3} = 4$ ,  $S(D) = \{m_2 , m_3\}$ .          Suppose a new

alternative  $m$  is attached.   Then since  $g( m , u_1 ) = g( m_2 , u_1 ), g( m , u_2)$

$= g( m_2 , u_2 )$  and   $g( m , u_3 ) = g( m_3 , u_3 )$   the assumption of Axiom A-4 is

satisfied.    However,    $L_m = \dfrac{5}{3} < 4$    show that $m$ never belongs to $S_L(M \cup \{m\}, U, \bar{g}')$.

As shown above, we can conclude that the max-min and the regret principle are essentially  the same as the satisfactory criterion except the fineness of the structures.  A decision making by the max-min criterion is first to arrange alternatives in linear order by $\succ_m$ and then to select the first elements as its solution.   On the other hand, according to Proposition 4.1 a decision making by the max-min criterion satisfying Axiom A-2 is one that first arranges alternatives in linear order by $\succ_m$ second divides the linear order into two parts and third selects the first part as its solution.    The regret principle satisfying Axiom A-2 is similarly defined.    Then :

Corollary 4.4.

A decision making by the max-min criterion satisfying A-2 is a satisfactory decision making for some aspiration level  $\tau$ .

Corollary 4.5.

A decision making by the regret principle satisfying A-2 is a satisfactory decision making for some aspiration level  $\tau$ .

## 4.2.  Aspiration levels for traditional criteria

As Corollary 4.4 and Corollary 4.5 show, the max-min and the regret criterion are, in some sense, special cases of the satisfactory criterion, that is, they are satisfactory criteria with special types of aspiration levels. In this section, we will investigate what types of aspiration levels are satisfied for the max-min criterion and the regret principle.   Before the discussion we present types of aspiration levels.

The simplest aspiration level is of the form

$$\tau(u) = c,$$

where  $c$ is constant, and is called constant aspiration level.   This kind of aspiration level is the most often used.

Next, an aspiration of the form

$$\tau(u) = \sup_{m \in M} g(m, u) - c,$$

where  $c$ is a nonnegative constant, is called least opportunity loss aspitation level.   It is obvious from the definition that a least opportunity loss aspiration level is deeply related to the regret decision criterion.

Besides these two types there is another kind of aspiration level of the form,

$$\tau(u) = g(\hat{m}, u),$$

where $\hat{m} \in M$ is a current alternative. The satisfactory criterion with this type of aspiration level aims at improvement of the present situation. Though it plays an important role in the theory of multi-level system [1], we will not discuss it here.

As defined in Section 2, the ordering $\succcurlyeq_m$ of the max-min criterion is given by

$$m \succcurlyeq_m m' \longleftrightarrow \inf_{u \in U} g(m, u) \geq \inf_{u \in U} g(m', u).$$

Then the usual order topology for $M$ can be induced by $\succcurlyeq_m$. In this section, the closedness of $M' \subset M$ with respect to $\succcurlyeq_m$ means $M'$ is closed with respect to the order topology induced by $\succcurlyeq_m$. The relationship between the satisfactory criterion and the max-min decision criterion is, then, given by the next proposition.

Proposition 4.6.

Let $h : M \to M$ be the identity map, and $(M, \succcurlyeq)$ some binary structure. If $h$ is an order homomorphism from $(M, \succcurlyeq_m)$ into $(M, \succcurlyeq)$, that is, $m \succcurlyeq_m m' \to m \succcurlyeq m'$ and if $M' = \{m \in M \mid (\forall m' \in M)(m' \succcurlyeq m \to m \succcurlyeq m')\}$ is closed with respect to $\succcurlyeq_m$, then $(M, \succcurlyeq)$ is the satisfactory structure with a constant aspiration level.

Conversely, if $(M, \succcurlyeq)$ is the satisfactory structure with a constant aspiration level, then $h$ is an order homomorphism.

Proof : Refer to Appendix 7.

Proposition 4.6 shows that the satisfactory criterion with a constant aspiration level is essentially equivalent to the max-min decision criterion. The difference is, as we mentioned in Section 3, only that the structure of the max-min decision criterion is generally not a binary structure so that it is finer than that of the satisfactory criterion. We should emphasize that the satisfactory criterion with a constant aspiration level is as pessimistic a criterion as the max-min.

The ordering $\succcurlyeq_r$ under the regret decision criterion is given by

$$m \succcurlyeq_r m' \longleftrightarrow r_m \geq r_{m'},$$

where $r_m = \inf_{u \in U} (g(m, u) - \sup_{m \in M} g(m, u))$

and $r_{m'} = \inf_{u \in U} (g(m', u) - \sup_{m \in M} g(m, u))$, respectively.

In the same manner as the max-min decision criterion, we obtain the next proposition.

Proposition 4.7.

Let $h : M \to M$ be the identity map and $( M , \geqslant )$ some binary structure. If $h$ is an order homomorphism from $( M , \geqslant_r )$ into $( M , \geqslant )$, that is, $m \geqslant_r m' \to m \geqslant m'$ and if $M' = \{ m \in M \mid ( \forall m' \in M )( m' \geqslant m \to m \geqslant m' ) \}$ is closed with respec to $\geqslant_r$, then $( M , \geqslant )$ is the satisfactory structure with a least opportunity loss aspiration level. Conversely, if $( M , \geqslant )$ is the satisfactory structure with a least opportunity loss aspiration level, then $h$ is an order homomorphism.

Proof : Refer to Appendix 8.

In the same manner as for the max-min decision criterion, we can conclude that the satisfactory criterion with a least opportunity loss aspitation level is viewed essentially equivalent to the regret decision criterion.

## 5. Conclusion

We have studied the satisfactory criterion in an axiomatical way and clarified an essential property of it. The followings are the summary of some of our results :

(1)    The axiom system we proposed in this paper indicates a "degree of simpleness" of the satisfactory criterion : Though the satisfactory criterion is considered quite simple and rational, it is not so simple as decribed by only the axioms B-1, B-2, B-3 and B-4; essential is B-5.

(2)    The max-min and the regret principle can be considered a type of the satisfactory criterion with a special aspiration level.

Although we have not assumed any structure of $M$ and $U$, we can give strong results by introducing detailed structure, for instance, topology into them. The following is one of the results.

Suppose that

(1)  $M$  is a compact subspace of a linear metric space,

(2)  $U$  is a compact metric space and

(3)  $g( - , u ) : M \to R$  is strictly concave for each $u \in U$.

Then a subset $M'$ of $M$ is a satisfactory subset for some continuous aspiration level if and only if

(a)  $M'$  is closed ,

(b)  $M'$  has nonempty interior and

(c)   $\partial M' = Bd\ (\ M'\ ) \neq \phi$  ,

where  $\partial M'$  is defined by

$$\partial M' = \{\ m\ \varepsilon\ M\ |\ (\ \exists\ u\ \varepsilon\ U\ )(\ g(\ m\ ,\ u\ ) = \tau_{M'}(\ u\ )\ )\}$$

and  $Bd(\ M'\ )$  denotes the boundary of  $M'$.

These results will be reported later.

## Appendices

1.  Proof of proposition 3.3 ;

(1) → (2) :  Since $M'$ is a satisfactory subset with respect to some aspiration level  $\tau : U \to R$ ,  $(\ \forall\ m\ \varepsilon\ M'\ )(\forall\ u\ \varepsilon\ U\ )(\ g(\ m\ ,\ u\ ) \geqq \tau\ (\ u\ ))$ and  $(\ \forall\ m\ \not\varepsilon\ M'\ )(\ \exists\ u\ \varepsilon\ U\ )(\ g(\ m\ ,\ u\ ) < \tau\ (\ u\ )\ )$  hold.   From the definition of  $\tau_{M'}$ ,  we have  $(\ \forall\ m\ \varepsilon\ M'\ )(\forall\ u\ \varepsilon\ U\ )(\ g(\ m\ ,\ u\ ) \geqq \tau_{M'}(u)\ )$  and  $(\forall\ u\ \varepsilon\ U\ )$  $(\tau_{M'}(\ u\ ) \geqq \tau(\ u\ ))$  implies  $(\ \forall\ m\ \not\varepsilon\ M'\ )(\ \exists\ u\ \varepsilon\ U\ )(\ g(\ m\ ,\ u\ ) < \tau_{M'}(\ u\ ))$.

(2) → (3) :  By setting  $U' = U$  and  $\tau' = \tau_{M'}$  we obtain the result.

(3) → (1) :  We show that  $M'$  is a satisfactory subset with respect to  $\tau^* :$  $U \to R$    such that $\tau^*(\ u\ ) = \tau'(\ u\ )$, if   $u\ \varepsilon\ U'\ ; = \inf_{m\ \varepsilon\ M'}\ g(\ m\ ,\ u\ )$,

otherwise.    From (3),  $(\forall\ m\ \varepsilon\ M)(\forall\ u\ \varepsilon\ U')(g(m, u \geqq \tau^*(u))$  holds. From the definition of $\tau^*$, we have  $(\forall\ m\ \varepsilon\ M')(\forall\ u\ \varepsilon\ U - U')(g(m, u) \geqq \inf_{m\ \varepsilon\ M}\ g(m, u)$  $= \tau^*(u))$. On the other hand, because  $(\forall\ m\ \not\varepsilon\ M')(\exists\ u\ \varepsilon\ U')(g(m, u) < \tau^*(u))$ from (3) and $U' \subset U$,  $(\forall\ m\ \not\varepsilon\ M')(\exists\ u\ \varepsilon\ U)(g(m, u) < \tau^*(u))$.

Q. E. D.

2.  Proof of Proposition 3.4 ;

Axiom A-1 and Axiom A-2 :  Let  $D = (M, U, g)$    be a decision problem and  $\tau : U \to R$    an aspiration level and  $S_\tau(D)$    a satisfactory subset of  $M$, i.e.,

$$S_\tau = \{\ m\ \varepsilon\ M\ |(\forall\ u\ \varepsilon\ U)(g(m, u) \geqq \tau(u)\}.$$

Define an order relation  $\geqslant$  on $M$ by

$$m \geqslant m' \longleftrightarrow [m\ \varepsilon\ S_\tau(D)\ \&\ m'\ \varepsilon\ S_\tau(D)\ ]$$
$$\vee\ [m\ \not\varepsilon\ S_\tau(D)\ \&\ m'\ \not\varepsilon\ S_\tau(D)]$$
$$\vee\ [m\ \varepsilon\ S_\tau(D)\ \&\ m'\ \not\varepsilon\ S_\tau(D)\ ].$$

then  $\geqslant$  is clearly a linear order with only maximal and minimal elements.

Axiom A-3 : For any  $m° \not\varepsilon\ S_\tau(D)$    let us consider a problem

$D' = (M - \{m^\circ\}, U, g')$, where $g' = g | (M - \{m^\circ\}) \times U$.

Because of Assumption P-2, the new aspiration level $\tau'$ for $D'$ is not higher than $\tau$, so that

$$(\forall u \in U)(g(m, u) \geqq \tau'(u)) \text{ for any } m \in S_\tau(D).$$

Therefore, $S_\tau(D) \subset S_{\tau'}(D)$, which means $S(D) \subset S(M - \{m^\circ\}, U, g')$ since we have $S_\tau(D) = S(D)$.

Axiom A-4 : For any $m \notin M$ such that

$$(\forall u \in U)(\exists m' \in M')(\overline{g}'(m, u) \geq g(m', u)),$$

we find that $(\forall u \in U)(\overline{g}'(m, u) \geqq g(m', u) \geqq \inf_{m'' \in M'} g(m'', u))$.

However, since $(\forall u \in U)(\inf_{m \in M'} g(m'', u) \geqq \tau(u))$, we have $(\forall u \in U)(\overline{g}'(m, u) \geqq \tau(u))$; consequently $m$ is a solution of the problem. Q.E.D.

3. Proof of Proposition 3.5 ;

It is only required to show that $S(D)$ is a satisfactory subset with respect to $\tau_{S(D)}$, the fundamental aspiration level for $S(D)$, because of Proposition 3.4.

It is clear that

$$(\forall u \in U)(g(m, u) \geqq \tau_{S(D)}(u))$$

for any $m \in S(D)$, and so it is only required to show that

$$(\exists u \in U)(g(m, u) < \tau_{S(D)}(u))$$

for any $m \notin S(D)$. Suppose there exists $m \notin S(D)$ such that

$$(\forall u \in U)(g(m, u) \geqq \tau_{S(D)}(u)), \text{ then } S(D) \subset S(M - \{m\}, U, g')$$

because of A-3. However, both $M - \{m\} \ni m$ and $(\forall u \in U)(g(m, u) \geqq \tau_{S(D)}(u))$ imply that

$$(\forall u \in U)(\exists m' \in S(D) \subset S(M - \{m\}, U, g'))(g(m, u) \geqq g(m', u)),$$

which is Assumption A-4. Using A-4, we find $m \in S(M - \{m\}) \cup \{m\}$, $U, \overline{g}')$, that is, $m \in S(D)$ because of $(M - \{m\}) \cup \{m\} = M$. That is contradiction. Q.E.D.

4. Proof of Propositio- 3.9 ;

B-1, B-2 and B-3 are clearly valid. Let $\succcurlyeq_l$ be the linear order derived from B-1. First we will show that B-4 is true. Suppose $(\forall u \in U)(g(m, u) \geqq g(m', u))$ and $not (m \succcurlyeq_l m')$ for some $m$ and $m' \in M$. Then $not (m \succcurlyeq_l m')$ implies that $m' \succ_l m$ from B-1 so that $m' \in S(D)$ and $m \notin S(D)$.

from B-2.   Using B-3, we have $S(D) \subset S(M - \{m\}, U, g')$, which shows $m' \varepsilon$

$S(M - \{m\}, U, g')$ .   Since $(\forall u \varepsilon U)(g(m, u) \geq g(m', u))$ and $m' \varepsilon S(M - \{m\}, U, g')$, $m$

must be a satisfactory decision for $((M - \{m\}) \cup \{m\}, U, g')$   from A-4.

$(M - \{m\}) \cup \{m\}$       is actually $M$ , that is, $m \varepsilon S(D)$ .   This is

contradiction.   Consequently B-4 should hold.

Next we will show that B-5 is valid.   Since A-axiom system holds, Proposition 3.4 and 3.8 imply $S(D)$ is an satisfactory set with respect to some $\tau : U \to R.$ Let $m \varepsilon M$   be arbitrary.

(1)   Suppose $m \varepsilon S(D)$ .   Then for any $S \subset E(m)$   and for any $m' \varepsilon S$, $m'$ is also satisfactory.   The   $\wedge-$ closure property of $M$ implies that

$$g(\wedge S, u) = inf_{m \varepsilon S} g(m, u) \geq \tau(u) \quad \text{for each } u \varepsilon U.$$

This means that $\wedge S$   is satisfactory, that is, $\wedge S \gtrsim_l m.$

(2)   Suppose $m \notin S(D)$ .   Then for any $S \subset E(m)$   and for any $m' \varepsilon S$, $m'$ is also unsatisfactory.   Thus $g(\wedge S, u) = inf_{m \varepsilon S} g(m, u) < \tau(u)$ for

some $u \varepsilon U$ .   This means that $\wedge S$   is unsatisfactory, that is, $\wedge S \gtrsim_l m.$

Q.E.D.

## 5.   Proof of Proposition 3.10 ;

We will show that A-4 is obtained from B-axiom system.   Let us consider

$m \notin M$   such that $(\forall u \varepsilon U)(\exists m' \varepsilon S(D))(g(m, u) \geq g(m', u))$.

Then $(\forall u \varepsilon U)(g(m, u) \geq inf_{m' \varepsilon S(D)} g(m', u))$   holds.

However, all the alternatives of   $S(D)$   are equivalent, $inf_{m' \varepsilon S(D)} g(m', u)$

can be viewed as the performance by the alternative $\wedge S$   , that is, $inf_{m' \varepsilon S(D)}$

$g(m', u) = g(\wedge S(D), u)$ for each $u$ .   Thus we find $(\forall u \varepsilon U)(g(m, u) \geq g(\wedge S, u))$,

which implies $m \gtrsim_l \wedge S(D)$   by B-4.   Since $\wedge s(D)$ is satisfactory by B-5, we can conculde that $m$ is satisfactory.                           Q.E.D.

## 6.   Proof of Proposition 4.1 ;

Since tha proofs for both criteria are similar, we consider only the max-min criterion.

A-1 is true since $\gtrsim_m$   is clearly a linear order on $M$ .   Next, for any $m \notin$

$S(D)$, $S_m(D) = S_m(M - \{m\}, U, g')$   holds.       Hence A-3 is true.   Finally,

suppose $m \notin M$   and $(\forall u \varepsilon U)(\exists m' \notin S_m(D))(g(m, u) = g(m', u))$.

Let $r = inf_{u \varepsilon U} g(m', u)$. Then it is constant for any $m' \varepsilon S_m(D)$ .   Thus,

$$g\ (m,\ u)\ \geq g\ (m',\ u)\ \geq inf_{u\ \varepsilon\ U}\ g\ (m',u)\ =\ r\quad \text{for some }m'\ \varepsilon\ S_m\ (D).$$

Consequently, $g\ (m,\ u)\ \geq r$ holds ; which implies

$$m\ \varepsilon\ S_m\ (M\ \cup\ \{m\},\ U,\ \bar{g}\,').\qquad\qquad \text{Q.E.D.}$$

## 7. Proof of Proposition 4.6 ;

Since $(M,\ \geqslant)$ is a binary structure, $M'$ is the set of all maximal alternatives with respect to $\geqslant_m$ . Furthermore, since $M'$ is closed with respect to $\geqslant_m$ , there exists $m°\varepsilon\ M$ such that $m\ \geqslant_m\ m°$ for any $m\ \varepsilon\ M'$.

(1) For any $m\ \varepsilon\ M'\ (\forall u\ \varepsilon\ U)(g\ (m,\ u)\ \geq\ c\ \vcentcolon=\ inf_{u\ \varepsilon\ U}\ g\ (m,\ u))$ holds:

For, $inf_{u\ \varepsilon\ U}\ g\ (m,\ u)\ \geq\ inf_{u\ \varepsilon\ U}\ g\ (m°,\ u)\ =\ c$ because of $m\ \geq_m\ m°$.

(2) For any $m\ \&\ M'\ (\exists u\ \varepsilon\ U)(g\ (m,\ u)\ <\ c\ \vcentcolon=\ inf_{u\ \varepsilon\ U}\ g\ (m°,\ u))$ holds:

For, if there were $m\ \&\ M'$ such that $(\ \forall u\ \varepsilon\ U)(g\ (\ m\ ,\ u\ )\ \geq\ c)$ then $m\ \&\ M'$ would imply $m\ \lessdot_m\ m°$. Since $h$ is order homomorphism, we have $m\ \lessdot\ m°$ On the other hand, $(\ \forall u\ \varepsilon\ U\ )(g\ (\ m\ ,\ u\ )\ \geq\ c)$ means $m\ \geqslant_m\ m'$ , which implies $m\ \geqslant\ m°$ because $h$ is an order homomorphism. This is contradiction. Both (1) and (2) show $(\ M,\ \geqslant\ )$ is the satisfactory structure with a constant aspitation level.

Suppose $h$ were not an order homomorphism. Then there would be $m,\ m'\varepsilon\ M$ such that

$$m\ \geqslant_m\ m'\quad \text{and}\quad m'\ \geqslant\ m.$$

$m\ \geqslant_m\ m'$ implies that $inf_{u\varepsilon U}\ g\ (\ m\ ,\ u\ )\ \geq\ inf_{u\varepsilon U}\ g\ (\ m'\ ,\ u\ )$.

On the other hand, since $m'\!\!\downarrow m$ and the aspiration level is constant $c$,

$$(\forall u\ \varepsilon\ U\ )(g\ (\ m'\ ,\ u\ )\ \geq\ c\ )\quad \text{and}\quad (\ \exists\ u\ \varepsilon\ U\ )(g\ (\ m\ ,\ u\ )\ <\ c\ ).$$

Noting that $inf_{u\ \varepsilon\ U}g\ (\ m'\ ,\ u\ )\ \geq\ c$ and $inf_{u\ \varepsilon\ U}g\ (\ m\ ,\ u\ )\ <\ c$ , we

obtain $inf_{u\ \varepsilon\ U}\ g\ (\ m'\ ,\ u\ )\ >\ inf_{u\ \varepsilon\ U}\ g\ (\ m\ ,\ u\ )$.

That is contradiction. Q.E.D.

## 8. Proof of Proposition 4.7.

Since $(\ M,\ \geqslant\ )$ is a binary structure, $M'$ is the set of all maximal alternatives with respect to $\geqslant$ . Furthermore, since $M'$ is closed with respect to $\geqslant_r$ , there exists $m°\ \varepsilon\ M'$ such that $m\ \geqslant_r\ m°$ for any $m\ \varepsilon\ M'$ .

(1) For any $m\ \varepsilon\ M'(\forall u\ \varepsilon\ U)(g\ (\ m\ ,\ u\ )\ \geq\ sup_{m\ \varepsilon\ M}g\ (\ m\ ,\ u\ )\ -\ c\ )$ hols,

where $c = - inf_{u \in U} ( g ( m' , u ) - sup_{m \in M} g ( m , u ))$ :

For, since we have $m \succeq_r m°$,

$$inf_{u \in U} ( g ( m , u ) - sup_{m \in M} g ( m , u )) \geqq inf_{u \in U} ( g ( m° , u ) - sup_{m \in M} g (m , u ))$$

$$= - c.$$

Therefore, $( \forall u \in U )( g ( m , u ) \geqq sup_{m \in M} g ( m , u ) - c )$.

(2)  For any $m \nsucceq M'$ $( \exists u \in U)( g ( m , u ) < sup g ( m , u ) - c )$   holds,

where $c = - inf_{u \in U}( g ( m° , u ) - sup_{m \in M} g ( m , u )) \geqq 0$ :

For, if there exists $m \nsucceq M'$ such that

$( \forall u \in U )( g ( m , u ) \geqq sup_{m \in M} g ( m , u ) - c )$

Then $m \nsucceq M'$ implies $m \prec_r m°$.  Since $h$ is an order homomorphism, we
have $m \prec m°$.

On the other hand $( \forall u \in U )( g ( m , u ) \geqq sup_{m \in M} g ( m , u ) - c)$ means

$m \succeq_r m°$ , which implies $m \succeq m°$ because $h$ is an order homomorphism.

This is contradiction.

Both (1) and (2) shows that  $( M , \succeq )$  is the satisfactory structure with
a least opportunity loss aspiration level.

Suppose $h$ were not an order homomorphism.  Then there would be $m , m' \in M$
such that

$$m \succeq_r m' \quad and \quad m' \succ m .$$

$m \succeq_r m'$  implies that

$$inf_{u \in U}( g ( m , u ) - sup_{m \in M} g ( m , u)) \geqq inf_{u \in U}( g ( m' , u ) - sup_{m \in M} g ( m , u )).$$

On the other hand, since $m' \succ m$ and the aspiration level $\tau$ is $\tau ( u ) =$
$sup_{m \in M} g ( m , u ) - c$    for some  $c \geqq 0$,  we have

$( \forall u \in U )( g ( m' , u ) \geqq sup_{m \in M} g ( m , u ) - c )$

and $( \exists u \in U)( g ( m , u ) < sup_{m \in M} g ( m , u ) - c )$

These imply that

$inf_{u \in U} ( g ( m' , u ) - sup_{m \in M} g ( m , u )) \geqq - c$

and $inf_{u \in U} ( g ( m , u ) - sup_{m \in M} g ( m , u )) < - c$

so that $inf_{u \in U}( g ( m' , u) - sup_{m \in M} g ( m , u))$
     $> inf_{u \in U}( g (m, u) - sup_{m \in M} g (m, u) )$.

This is contratiction.

                                                              Q.E.D.

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