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COMBINED OVERHAUL AND REPLACEMENT POLICIES FOR DETERIORATING EQUIPMENT

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Abstract A combined overhaul/replacement policy is developed for a class of deteriorating equipment, where operating costs increase with age. It is assumed that the effect of an overhaul is to reduce these costs by a fixed amount, from the time of overhaul onwards. Two possible assumptions as to the dependence of overhauling effects on equipment age and two functional forms of this dependence are explored. Discounting considerations are omitted from the analysis and the possible effect of this omission on outcomes is investgated.

1. Introduction

Overhaul and replacement models, for a class of equipment where the main effect of deterioration is a gradual increase of operating costs, have been considered by numerous authors. In the classical "replacement only" models [1],[2],[3],[8], an optimal replacement period is derived so as to minimize the average unit cost of operating and replacement. The possible effect of overhauls on operating costs, and combined overhaul-replacement policies have been suggested [4], [6],[7],[9], introducing various simplifying assumptions as to the intervals between overhauls, their effect and the associated costs. A stochastic model for an overhaul-replacement policy has been demonstrated [5], assuming that the behaviour of deteriorating equipment is described by a set of transition probabilities from one state of serviceability to another.

In this study a set of overhaul-replacement models is developed, under different structures of cost and overhaul effectiveness.

The basic assumption about the equipment (following many authors, e.g. [1],[5],[9]) is that its deterioration process is readily described by a characteristic operating cost function, representing all expenses that vary with operation and/or age. Actually, this cost function combines two main components: (a) the age-dependent increasing cost of the various necessary inputs per unit of standard output (i.e. the diminishing productivity), and (b) the cost of disturbances caused to the whole system due to increasing failure rate and malefunction frequency of the said equipment.

Overhauls are carried out in order to improve equipment performance and thus reduce operating costs. It is assumed that the general characteristics of the operating cost function are retained, and the effect of an overhaul is a shift downwards ("improvement") in this function, after which it continues in parallel to its initial path. Replacement is considered to be a regeneration point of the whole life cycle, where the operating cost function starts again from its origin. A typical life cycle, where two overhauls are carried out prior to replacement is shown in Fig. 1.

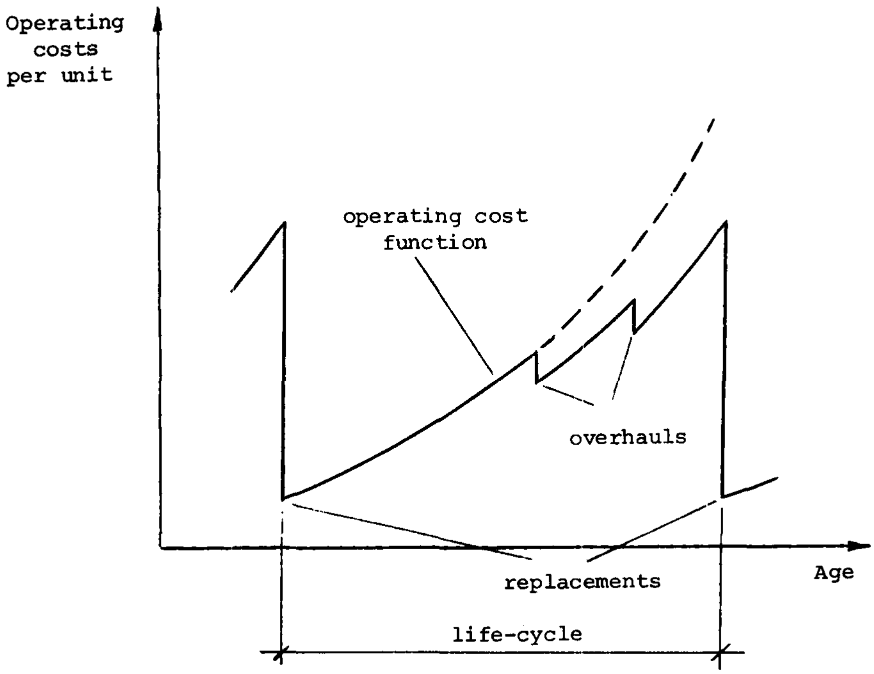

Fig. 1: Effect of replacement and overhauls on operating costs.

Experience shows that there is a connection between equipment age and the effect of an overhaul. Right after replacement, or former overhaul, the improvement brought about by an overhaul is relatively small. This improvement (reduction of the operating cost function) increases in some manner with time elapsed since the last renewal. The cost of an overhaul is taken to be constant, regardless of the age at which it is carried out. This is justified by the fact that an overhaul entails usually a fixed sequence of operations (such as: dismantling, component inspection and eventual replacement, re-assembly, test, etc.) with rather small variations in costs.

The objective here is to find a policy specifying the number and timing of overhauls, as well as the corresponding replacement period, so that the average cost per unit of time is minimized. In Section 2 several models are developed based on different assumptions and constraints, Section 3 brings the results of a numerical example, and in Section 4 possible effects of discounting considerations are discussed.

2. Derivation of Models

Consider a piece of equipment which deteriorates with time. Let $h(t)$ be the operating cost per unit time of this equipment and let S be the replacement cost. Assuming that the policy is to replace the equipment after a period T, the average cost q per unit time due to operation and replacement only is clearly:

(1) $$q = \frac{1}{T}\,[S + \int_0^T h(t)dt] \;.$$

Assume now that n overhauls are carried out between replacements, that t_i is the time of the i-th overhaul, and that the improvement brought about by an overhaul at time t_i follows an improvement function $g(t_i)$. This improvement, measured in cost per unit of time, manifests itself by a corresponding shift downwards of the operating cost function. Let G denote the total saving of operating costs due to overhauling, in a cycle, and let C be the (fixed) cost of an overhaul. The average cost per unit time of operating, overhauling and replacing equipment is then

(2) $$q = \frac{1}{T}\,[S + \int_0^T h(t)dt - G + nC] \;.$$

For the detailed definition of G , two assumptions have to be made regarding the time dependence of improvement:

(a) the point from which time in the improvement function is measured, the last replacement or the last overhaul;

(b) the form of the function $g(t)$.

Depending on these assumptions, and on the possibility that practically it might be preferable to carry out overhauls at equal intervals of time, - six possible models are analyzed. The models are numbered according to Table 1.

Table 1. Numbering of models.

		Effect dependent on time since last overhaul		Effect dependent on time since last replacement	
	Form of improvement function	Linear	S-form	Linear	S-form
Intervals between Overhauls	equal	No. 1	No. 2	No. 4	No. 5
	unrestricted		No. 3		No. 6

Model No. 1 covers two cases, unrestricted and equal intervals between overhauls, as it is proved that optimal results are obtained when the intervals are equal. It will also be shown that model 4 is identical to model 1.

Model No. 1. Here the improvement function is linear, its argument is the time elapsed since last overhaul, and no restriction is put on the intervals between overhauls. Consider a cycle of length T in which n overhauls are carried out. Let g_i be the improvement brought about by the i-th overhaul in the cycle, which is carried out at time t_i (since the beginning of the cycle), then

(3) $$g_i = g(t_i) = b(t_i - t_{i-1})$$

where b is a constant and $t_0 = 0$.

From the definition of the effect of overhauling it follows that the saving in operating costs due to an overhaul at t_i is $g_i(T - t_i)$, and the total saving per cycle due to n overhauls, carried out in this cycle, is

(4) $$G = \sum_{i=1}^{n} g_i(T - t_i) \quad .$$

Substituting for g_i , this becomes

(5) $$G = b \sum_{i=1}^{n} (t_i - t_{i-1})(T - t_i) .$$

For any cycle length T, overhauls should be scheduled so that the difference between gain and cost due to overhauling is maximized. Let $K = G - nC$ denote this difference. Differentiating K with respect to $t_1, t_2, \dots, t_n$, n equations are obtained from which the optimal overhaul times (for given T) can be calculated. In the case considered here

(6) $$K = b \sum_{i=1}^{n} (-t_i^2 + t_i T + t_i t_{i-1} - t_{i-1} T) - nC \quad .$$

For the i-th overhaul, $dK/dt_i = b(t_{i-1} - 2t_i + t_{i+1}) = 0$ and hence

(7) $$t_i = \frac{1}{2} (t_{i-1} + t_{i-1})$$

where $t_o = 0$ and $t_{n+1} = T$.

Thus, under the assumptions made, the maximal difference between gains and costs is attained when the n overhaul times divide the cycle T into $n+1$ equal periods. The time of the i-th overhaul is given by

(8) $$t_i = iT/n+1 \quad .$$

Inserting this result in (5), the total gain becomes

(9) $$G = \frac{1}{2} b \frac{n}{n+1} T^2 \ .$$

The average cost per unit time, (2), takes the form

(10) $$q = \frac{1}{T} [S + \int_0^T h(t)dt - \frac{1}{2} b \frac{n}{n+1} T^2 + nC] \ .$$

Differentiation with respect to T and n yields the following two expressions

(11) $$Th(T) - \frac{1}{2} b \frac{n}{n+1} T^2 - S - \int_0^T h(t)dt - nC = 0$$

(12) $$n = T(b/2C)^{\frac{1}{2}} - 1$$

from which the overall optimal policy (T^*, n^*) can be obtained using simple numerical techniques.

Model No. 2. A typical situation may exist where the improvement due to overhaul carried out shortly after a previous renewal is very small. Improvement becomes more substantial as the equipment grows older, until some point in time is reached after which further postponement of an overhaul does not add much to its effect. An S-form improvement function may suitably describe such a situation. Again, the independent variable in the improvement function is time since the last overhaul. Intervals between overhauls are taken to be equal.

A possible S-form improvement function for this case is:

$$g_i = g(t_i) = m\, exp[-\alpha e^{-\beta(t_i - t_{i-1})}] \tag{13}$$

Because of the imposed restriction, the time of the i-th overhaul is given by (8).

Substituting for g_i and t_i in (4), the total gain takes the form

$$G = \frac{1}{2}\, ntm\, exp[-\alpha e^{-\beta(T/n+1)}] \tag{14}$$

The objective function to be minimized (see (2)) becomes:

$$q = \frac{1}{T}\{S + \int_0^T h(t)dt - \frac{1}{2}\, nTm\, exp[-\alpha e^{-\beta(T/n+1)}] + nC\} \tag{15}$$

from which the following two expressions can be derived by differentiating with respect to T and n:

$$Th(T) - \frac{1}{2}\frac{n}{n+1}T^2 m\alpha\beta e^{-\beta(T/n+1)} exp[-\alpha e^{-\beta(T/n+1)}] - S - \int_0^T h(t)dt - nC = 0 \tag{16}$$

$$C - \frac{1}{2}\, Tm\, exp[-\alpha e^{-\beta(T/n+1)}]\{1 + \frac{n}{(n+1)^2}T\alpha\beta e^{-\beta(T/n+1)}\} = 0 \tag{17}$$

These can be solved numerically to obtain the optimal pair (T^*, n^*).

Model No. 3. Here, assumptions are the same as in model No. 2, except that the restriction of equal periods of time between overhauls is removed. Similarly to the procedure in the first model, the difference K between gain and cost of overhauls is brought to a minimum for any T.

Expression (6) takes here the form:

(18) $$K = m \sum_{i=1}^{n} \{(T-t_i)\exp[-\alpha e^{-\beta(t_i - t_{i-1})}]\} - nC$$

Differentiating with respect to t_i, one obtains:

(19) $$\{(T-t_i)\alpha\beta e^{-\beta(t_i - t_{i-1})} - 1\}\exp[-\alpha e^{-\beta(t_i - t_{i-1})}] -$$
$$- (T-t_{i+1})\alpha\beta e^{-\beta(t_{i+1}-t_i)} \exp[-\alpha e^{-\beta(t_{i+1}-t_i)}] = 0$$

For n overhauls, the corresponding set of overhaul times t_i can be found by solving the n equations obtained by inserting $i = 1,2,\ldots,n$ in (19) (with $t_o = 0$ and $t_{n+1} = T$).

Thus, for $n = 1$, T may be found from

(20) $$(T-t)\alpha\beta e^{-\beta t} - 1 = 0$$

for $n = 2$, t, and t_2 are calculated from

(21) $$\{(T-t_1)\alpha\beta e^{-\beta t_1} - 1\}\exp[-\alpha e^{-\beta t_1}] - (T-t_2)\alpha\beta e^{-\beta(t_2-t_1)}$$
$$\exp[-\alpha e^{-\beta(t_2-t_1)}] = 0$$

(22) $$(T-t_2)\alpha\beta e^{-\beta(t_2-t_1)} - 1 = 0$$

and so on.

Note that the objective function for this model is:

(23) $$q = \frac{1}{T} \{S + \int_o^T h(t)dt - m \sum_{i=1}^{n} (T-t_i)\exp[-\alpha e^{-\beta(t_i-t_{i-1})}] + nC\}$$

For each T and n, the optimal overhaul times can be calculated, and by their substitution in (23) the policy (T^*, n^*) for minimum cost per unit time may be found.

Model No. 4. In the following three models it is assumed that the improvement effect of overhauls is best described by taking the argument in the improvement function to be the time elapsed since the last replacement. The reduction in operating costs due to each overhaul is found by means of the improvement function, irrespective of former overhauls. However, this improvement affects costs only until the next renewal (overhaul or replacement). By this definition the saving in operating costs due to an overhaul at t_i is $g_i(t_{i+1} - t_i)$, and the total saving per cycle is:

$$G = \sum_{i=1}^{n} g_i(t_{i+1} - t_i) \quad (24)$$

If the improvement function is linear it is easily seen, from the geometric relations, that model No. 4 is identical to model No. 1. In a more formal way:

Expression (3) takes, here, the form:

$$g_i = bt_i \quad (25)$$

and the total gain due to n overhauls per cycle becomes

$$G = b \sum_{i=1}^{n} t_i(t_{i+1} - t_i) \ . \quad (26)$$

The difference between gain and cost is

$$K = b \sum_{i=1}^{n} (- t_i^2 + t_{i+1}t_i) - nC \ . \quad (27)$$

Differentiation with respect to t_i yields $t_i = \frac{1}{2}(t_{i-1} + t_{i+1})$, which is same as (7). Substituting $t_i = iT/(n+1)$ in (26) gives expression (9), and hence (10), (11) and (12) are applicable for model No. 4 as well.

Model No. 5. This model assumes an S-form improvement function, where time is measured from the last replacement, so that

$$g_i = m \ exp[-\alpha e^{-\beta t_i}] \quad (28)$$

For equal intervals between overhauls, the total gain due to n overhauls per replacement cycle is obtained by substituting (8) and (28) in (24):

$$G = m \sum_{i=1}^{n} (t_{i+1} - t_i) exp[-\alpha e^{-\beta t_i}] = m \frac{T}{n+1} \sum_{i=1}^{n} exp[-\alpha e^{-\beta i(T/n+1)}] \quad (29)$$

The objective function is

$$(30) \qquad q = \frac{1}{T}\{S + \int_0^T h(t)dt - m\,\frac{T}{n+1}\sum_{i=1}^{n} exp[-\alpha e^{-\beta i(T/n+1)}] + nC\}$$

from which the optimal policy (T^*,n^*) can be found.

Model No. 6. Lifting the restriction of equal periods between overhauls, optimal overhaul times are once more obtained by maximization of the difference between gain and cost due to overhauling:

$$(31) \qquad K = m\sum_{i=1}^{n}(t_{i+1} - t_i)\ exp[-\alpha e^{-\beta t_i}] - nC$$

The optimal set of overhaul times can be calculated from the n equations, inserting $i = 1,2,\ldots,n$ in

$$(32) \qquad \{(t_{i+1}-t_i)\alpha\beta e^{-\beta t_i} - 1\}\ exp[-\alpha e^{-\beta t_i}] + exp[-\alpha e^{-\beta t_{i-1}}] = 0$$

The objective function for this model is

$$(33) \qquad q = \frac{1}{T}\{S + \int_0^T h(t)dt - m\sum_{i=1}^{n}(t_{i+1}-t_i)exp[-\alpha e^{-\beta t_i}] + nC\}$$

where from (T^*,n^*) may be obtained.

3. Numerical Example

Consider a situation where the operating cost function of a piece of equipment can be approximated linearily by $h(t) = 4000 + 8000t$ money units (m.u.) per year. Replacement and overhaul costs are 70000 m.u. and 5000 m.u. respectively.

A set of empirical data about overhaul times and effects is available which can be analyzed to yield possible improvement functions. Let g_i be the improvement due to the i-th overhaul, at time t_i after replacement, then the following alternative forms are obtained:

(a) assuming a linear relationship (models No. 1 or No. 4),
$g_i = 4000\ (t_i - t_{i-1})$ m.u./year,

(b) assuming the overhauling effect is dependent on time since the last overhaul, and an S-form relationship (models No. 2 and No. 3),
$g_i = 12000e^{-3.29e^{-1.10(t_i-t_{i-1})}}$ m.u./year,

(c) assuming the effect is dependent on time since the last replacement, and an S-form relationship (models No. 5 and No. 6)

$$g_i = 20000(e^{-.58e^{-.89t_i}} - e^{-.58e^{-.89t_{i-1}}}) \text{ m.u./year.}$$

Numerical results of applying the six models presented in the previous sections are summarized in Table 2.

Table 2. Results of numerical example

	Improvement function	Linear	S-form: Effect dependent on time since last overhaul	S-form: Effect dependent on time since last replacement
Equal intervals	Model No.	1,4	2	5
	Number of overhauls	3	1	2
	Overhaul times t_1 =	1.45	2.50	1.83
	t_2 =	2.90		3.67
	t_3 =	4.35		
	Replacement at T =	5.8	5.0	5.5
	Average cost m.u./year	(33200)	(36100)	(33100)
Unrestricted intervals	Model No.	1,4	3	6
	Number of overhauls	(as above)	1	2
	Overhaul times t_1 =		2.125	2.25
	t_2 =			3.62
	Replacement at T =		5.0	5.5
	Average cost m.u./year		(36000)	(32800)

As could be foreseen, the assumption of an S-form improvement function leads to postponement of the first overhaul, reduction of the number of overhauls and shortening of replacement period - relative to the results when assuming a linear improvement function. Also, the average cost with unrestricted intervals is somewhat lower (though not significantly) then when equal intervals are imposed.

There is, of course, no point in comparing the resulting costs under different improvement function assumptions.

4. Discounting

Discounting considerations have been omitted from the previous analysis. The effect of introducing these depends on the discount rate and the values of the relevant parameters. Let q be the uniform (average) cost per unit time equivalent to the actual costs per cycle. Assuming replacement only, a linear operating cost function $h(t)=at$ and continuous discounting at a rate r, eq. (1) is rewritten to become:

(34) $$\int_0^T qe^{-rt}dt = Se^{-rT} + \int_0^T ate^{-rt}dt \ .$$

From this, the following expression is derived for the optimal replacement period(*):

(35) $$rT + e^{-rT} = 1 + \frac{S}{a}r^2 \ .$$

Considering model No. 1 (or No. 4), equally spaced overhauls and $h(t)=at$, expression (10) becomes:

(*) It should be noted that the above line of thinking yields the same results as traditional discounting considerations. Suppose one assumes that replacement intervals are equal and let $T_0 = 0$, $T_1, T_2, \ldots$ be times of successive replacements all of length T, then the present value of the total cost over an infinite horizon (y being the time measured from origin) is:

$$Q = \sum_{i=0}^{\infty} \left[\int_{T_i}^{T_{i+1}} a(y - T_i)e^{-ry}dy + Se^{-rT_{i+1}}\right]$$

Substituting $t = y - T_i$ one obtains

$$Q = \sum_{i=0}^{\infty} \left[e^{-rT_i}\int_0^T ate^{-rT}dt + Se^{-r(i+1)T}\right]$$

which can be reduced to

$$Q = \frac{\int_0^T ate^{-rt}dt + Se^{-rT}}{1 - e^{-rT}}$$

Minimizing Q in the above expression leads to results identical to those obtained from the minimization of q in (34).

(36)
$$\int_0^T q e^{-rt} dt = S e^{-rT} + \int_0^T a t e^{-rt} dt - \sum_{i=1}^{n} \int_{\frac{T}{n+1} i}^{T} b \frac{T}{n+1} e^{-rt} dt + \sum_{i=1}^{n} C e^{-ri(T/n+1)} .$$

As an illustration of the effect of discounting on results, Fig. 2 depicts the ratio T_r/T_o between the optimal replacement period taking into account a discount rate r and the replacement period when $r=0$, for the above case and the data of the numerical example. It is seen that for a realistic situation where, say, $r=10\%$ and $n=2$, the ratio T_r/T_o does not exceed 1.10. Failing to introduce discounting considerations results, in this case, in adopting a replacement period shorter by 10% than the optimal, which in turn corresponds to some 3% increase in the average cost per unit time. Considering the approximate nature of the pertinent data, this is a relatively small change which hardly justifies encumbering the various expressions.

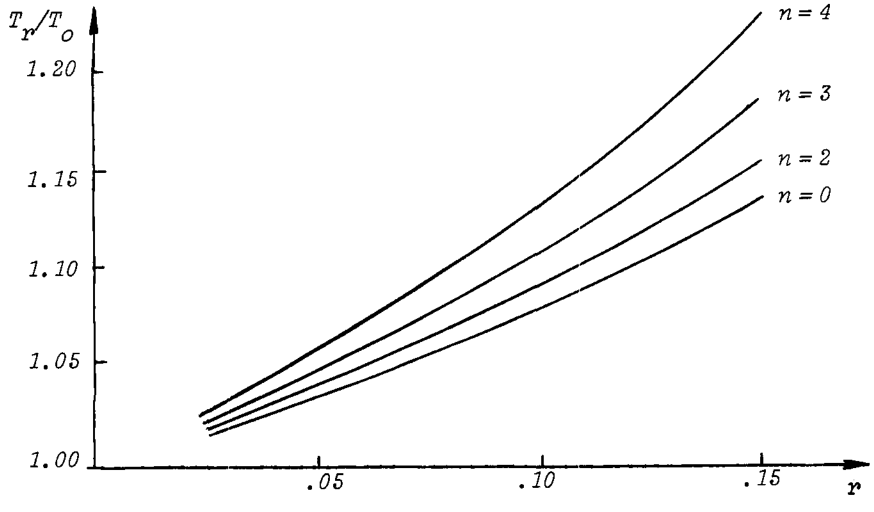

Fig. 2. Dependence of the optimal replacement period on discount rate.

Summary

An approach has been outlined which enables to define combined overhaul/replacement policies for minimum average cost per unit time, during the whole life cycle of equipment.

Some possible assumptions regarding the dependence of overhauling effects on time since last replacement or time since last overhaul are investigated. Indeed, various other assumptions of this kind (e.g. dependence of overhaul effects on both time since replacement and time since last overhaul, etc.) as well as different forms of the improvement function are plausible. Sound technical judgement and careful analysis of available data should lead to making the right assumptions and choosing the appropriate model.

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