Journal of the Operations Research
Society of Japan
Vol.21, No.2, June, 1978

© 1978 The Operations Research Society of Japan

MAXIMUM-FLOW PROBLEM IN DISCRETE CONTINUOUS COMPOUND SYSTEM AND ITS NUMERICAL APPROACH

Azuma Taguchi
University of Tokyo

(Received November 8, 1977)

Abstract We discuss some of the concepts related to the flow problems in a discrete-continuous compound system, and propose an approximation method as well as a numerical algorithm for the maximum-flow problem in a network which is assumed to be locally uniform and sufficiently dense. The algorithm is derived by means of the finite-element technique, and is tested by taking up as an example the maximum-flow problems on the road network of Tokyo Metropolitan Area.

1. Introduction

The theory of flows in networks is now well established from the standpoint of theory as well as of application, and many effective practical algorithms are being developed for dealing with problems of flows in networks. However, in general, applications of these algorithms to practical problems are limited due to the fact that they require prohibitively large computation time and memory for considerably large networks.

One of the ways to circumvent this difficulty would be to consider an approximate model. The model to be considered in this paper is based on the theory of flows in continua proposed by M. Iri in [5]. Continuum models of flow problems have been considered by Monge [7], Appell [1] and Kantorovitch [6], and recently by R. E. Gomory and T. C. Hu [3] for the isotropic case, and the general anisotropic case was formulated by M. Iri in invariant tensorial terms to include the continuous counterparts of all the concepts appearing in the established formalisms of the theory of flows in networks.

Each point of the continua concerned is endowed with characteristics of "capacity" and/or "distance", which are the continuous counterparts of branch characteristics in network flow problems. These characteristics are generally anisotropic, and give rise to a sort of saturation phenomena in the sense that flux (or velocity) of flows cannot increase beyond the prescribed limits. In other words, the problems to be considered are highly anisotropic and non-linear. Therefore, to treat the problems, we have to use a method fairly different from the one usually employed in fluid mechanics or elasticity-plasticity theory.

In the present paper, we shall consider the continuous approximation to the maximum flow problem in a network which is supposed to be locally uniform and sufficiently dense, and propose a numerical approach by means of the finite-element technique.

2. Basic concepts

An urban network may be subdivided into two kinds, one being a fine network composed from densely distributed narrow roads and the other a coarse network of arterial roads, expressways, etc. The network of the former kind can be regarded approximately as a continuum, but we had better treat the network of the latter kind as an ordinary network. Thus we are led to a compound model consisting partly of a continuum and partly of a discrete network.

We shall begin with the definitions of basic concepts of the discrete subsystem and the continuum subsystem of our compound model.

2.1. Discrete subsystem

The discrete subsystem is a network G which consists of the set of nodes N and the set of branches E. Each node is identified by a number i ($\in\{1,\ldots,m\}$), where m is the number of nodes in N. The branch starting from node i and ending at node j is denoted by a pair (i,j) ($\in\{1,\ldots,m\}\times\{1,\ldots,m\}$). (No parallel branches are assumed to exist.) The set of nodes through which flows pass between the discrete subsystem and the continuum will be denoted by $\tilde{N}$ ($\subset N$).

The set of successors of node i is denoted by $A^+(i)$, and the set of predecessors by $A^-(i)$:

$$A^+(i) = \{j \mid (i,j) \in E\} ,$$

$$A^-(i) = \{j \mid (j,i) \in E\} .$$

The flow in a branch (i,j) is denoted by f_{ij}. The flows are assumed to satisfy the continuity equation at each node of $N - \tilde{N}$:

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

(2.1) $$\sum_{j \varepsilon A^{+}(i)} f_{ij} - \sum_{j \varepsilon A^{-}(i)} f_{ji} = 0 \text{ for every } i \; \varepsilon \; N - \tilde{N}.$$

Flow f_{ij} is subject to the capacity constraint:

(2.2) $$f_{ij} \; \varepsilon \; [0, a_{ij}] \quad \text{for every } (i,j) \; \varepsilon \; F,$$

where a_{ij} is a positive number.

2.2. Continuum subsystem

The continuum subsystem is a connected subregion V of the n-dimensional Euclidean space with a sufficiently smooth $(n-1)$-dimensional boundary surface ∂V. We denote the coordinates of a point in V by x^{κ} $(\kappa=1,\ldots,n)$, the $(n-1)$-dimensional surface element oriented outward by dS_{κ}, and the n-dimensional volume element of V by dV. Also, we make a small simply connected subregion V_i of V with a smooth surface correspond to each node i of $\tilde{N}$ of the discrete subsystem.

A flow ξ^{κ} is a contravariant vector-density field in V which satisfies the continuity equation:

(2.3) $$\frac{\partial \xi^{\kappa}}{\partial x^{\kappa}} = 0 \quad \text{at every point } x \text{ in } V - \bigcap_{i \varepsilon \tilde{N}} V_i .$$

Here, as well as in the following, we adopt the summation convention. (Thus, for example $\sum_{\kappa=1}^{n}$ is omitted on the left-hand side of equation (2.3).)

The capacity convex of the continuum at position x is a closed convex set $C(x)$ (containing the origin) of the vector space of ξ's at x, and any feasible flow ξ should satisfy

(2.4) $$\xi \; \varepsilon \; C(x) \quad \text{at every point } x \text{ in } V.$$

The capacity convex is a natural extension of the capacity of a branch of a network, which is a closed interval as shown in (2.2). An example of a capacity convex in the 2-dimensional case is shown in Fig. 1.

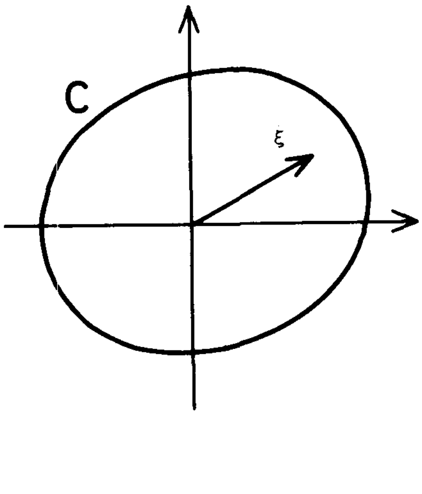


Fig. 1. Capacity convex

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

2.3. Connection of flows in G and flows in V

Let node i be an element of $\tilde{N}$ and V_i the subregion of V corresponding to node i. We shall connect the discrete subsystem G with the continuum subsystem V by making the amount

(2.5) $$\theta_i = -\sum_{j\in A^+(i)} f_{ij} + \sum_{j\in A^-(i)} f_{ji}$$

of flow going out of G at node i ($\in \tilde{N}$) balance the integral of the divergence of flow in V_i (Fig. 2). In order to do so, we put

(2.6) $$\frac{\partial \xi^\kappa}{\partial x^\kappa} = \theta_i \rho_i(x),$$

where $\rho_i(x)$ is a scalar-density field with a compact support in V_i such that

(2.7) $$\int_{V_i} \rho_i(x) dV = 1.$$

Then, by virtue of the Gauss-Stokes theorem, we have in fact

(2.8) $$\int_{\partial V_i} \xi^\kappa dS_\kappa = \theta_i \int_{V_i} \rho_i(x) dV = \theta_i .$$

2.4. Cost

We shall assume that the cost characteristic of V is represented by a function $\phi_c(x,\xi)$ of point x and flow ξ , which is a scalar-density field qua function of x and is convex qua function of ξ. Similarly, the cost characteristic of f_{ij} is represented by convex functions $\phi_{ij}(f_{ij})$ of flows f_{ij} in branches (i,j).

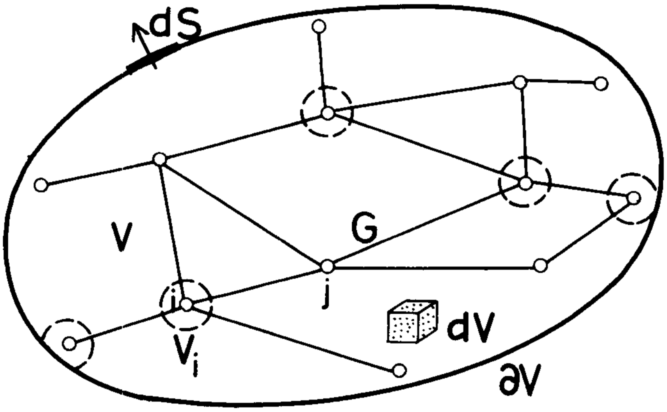


Fig. 2. Continuum V and network G

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

3. Maximum Flow Problem

We now define the maximum flow (with minimum cost) problem for a compound system with continuum V and network G.

We partition the boundary ∂V of V and the set of nodes N of G, respectively, into three parts, the entrance, the exit and the rest, and denote them as follows.

P : entrance $(\subset \partial V)$,

Q : exit $(\subset \partial V)$,

R : $\partial V - P \cup Q$,

where $P \cap Q = \phi$,

N_P : set of entrance nodes $(\subset N)$,

N_Q : set of exit nodes $(\subset N)$,

N_R : $N - N_P \cup N_Q$,

where $N_P \cap N_Q = \phi$.

Then, the maximum flow problem is defined as the problem of determining ξ^κ and f_{ij}'s which

(*P1*) minimizes

$$\Phi_1(\xi, f_{ij}) = \gamma\{ \int_V \phi_c(x,\xi)dV + \sum_{(i,j)\varepsilon E} \phi_{ij}(f_{ij}) \} + \int_P \xi^\kappa dS_\kappa - \sum_{i\varepsilon N_P} (\sum_{j\varepsilon A^+(i)} f_{ij} - \sum_{j\varepsilon A^-(i)} f_{ji}) - \sum_{i\varepsilon N_P\cap \tilde{N}} \theta_i \tag{3.1}$$

subject to the constraints:

$$\frac{\partial \xi^\kappa}{\partial x^\kappa} = 0 \qquad \text{in } V - \bigcup_{i\varepsilon \tilde{N}} V_i, \tag{3.2}$$

$$\frac{\partial \xi^\kappa}{\partial x^\kappa} = \theta_i \rho_i(x) \qquad \text{in } V_i,\ i \varepsilon N, \tag{3.3}$$

$$\xi^\kappa dS_\kappa = 0 \qquad \text{on } R, \tag{3.4}$$

$$\xi^\kappa \varepsilon C(x) \qquad \text{in } V, \tag{3.5}$$

$$\sum_{j\varepsilon A^+(i)} f_{ij} - \sum_{j\varepsilon A^-(i)} f_{ji} = 0 \qquad i \varepsilon N_R - \tilde{N}, \tag{3.6}$$

$$\sum_{j\varepsilon A^+(i)} f_{ij} - \sum_{j\varepsilon A^-(i)} f_{ji} = -\theta_i \qquad i \varepsilon N_R \cap \tilde{N}, \tag{3.7}$$

$$0 \le f_{ij} \le a_{ij} \qquad (i,j) \varepsilon E, \tag{3.8}$$

where γ is a nonegative constant.

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

Obviously if $\gamma = 0$, problem *P1* is reduced to the maximum flow problem in an ordinary sense. However, in that case, ξ^κ and f_{ij}'s are in general not determined uniquely. In order to guarantee the uniqueness of the solution, we may introduce a positive but sufficiently small γ, with strictly convex functions ϕ_c and ϕ_{ij}.

The problem dual to *P1* is defined as the problem of determining ζ, η_κ, λ_i's and μ_{ij}'s so as to

(*P1**) maximize

$$\Psi_1(\zeta,\eta,\lambda_i,\mu_{ij}) = -\int_V \psi_c(x,\eta)dV - \sum_{(i,j)\in E} \psi_{ij}(\mu_{ij}) \tag{3.9}$$

subject to the constraints:

$$\eta_\kappa = -\frac{\partial\zeta}{\partial x^\kappa} \quad \text{in } V, \tag{3.10}$$

$$\zeta = 1 \quad \text{on } P, \tag{3.11}$$

$$\zeta = 0 \quad \text{on } Q, \tag{3.12}$$

$$\mu_{ij} = \lambda_i - \lambda_j \quad (i,j) \in E, \tag{3.13}$$

$$\lambda_i = 1 \quad i \in N_P, \tag{3.14}$$

$$\lambda_i = 0 \quad i \in N_Q, \tag{3.15}$$

$$\int_{V_i} \zeta(x)\rho_i(x)dV = \lambda_i \quad i \in \tilde{N}, \tag{3.16}$$

where ζ and λ_i's play the roles of the Lagrangean multipliers for the primal constraints (3.2)∿(3.4) and (3.6)∿(3.7), respectively, ζ is a scalar field and η_κ is a covariant vector field. Futhermore $\psi_c(x,\eta)$ and $\psi_{ij}(\mu_{ij})$ are the functions conjugate to $\gamma\phi_c(x,\xi)$ and $\gamma\phi_{ij}(f_{ij})$, respectively:

$$\psi_c(x,\eta) = \max_{\xi\in C(x)} \{\xi^\kappa\eta_\kappa - \gamma\phi_c(x,\xi)\}, \tag{3.17}$$

$$\psi_{ij}(f_{ij}) = \max_{0\le f_{ij}\le a_{ij}} \{f_{ij}\mu_{ij} - \gamma\phi_{ij}(f_{ij})\}. \tag{3.18}$$

$\psi_c(x,\eta)$ is a scalar-density field qua function of x and is convex qua function of η, and $\psi_{ij}(\mu_{ij})$ is a convex function of μ_{ij}. ζ and λ_i's will be called "potential", and η and μ_{ij} will be called "tension". For a detailed discussion concerning the duality correspondence as well as the continuous counterpart of the "maximum-flow minimum-cut theorem", see Iri[5].

To be rigorous, we have at this stage to go into theoretical argument on the conditions under which problems *P1* and *P1** have solutions, and on the dual correspondence between the quantities and relations in *P1* and those in *P1**. However, it would require lengthy mathematical discussions, so that we

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

shall proceed directly to a numerical method of finding an approximate solutions of the problem in the following and postpone the mathematical discussions elsewhere. We shall assume, for the time being, that our problems *P1* and *P1** have solutions and that the solutions of *P1* and *P1** satisfy the following relations of complementarity:

$$\xi^{\kappa} = \frac{\partial\psi_c(x,\eta)}{\partial\eta_\kappa}, \qquad f_{ij} = \frac{\partial\psi_{ij}(\mu_{ij})}{\partial\mu_{ij}},$$

where $\partial\psi_c/\partial\eta_\kappa$ and $\partial\psi_{ij}/\partial\mu_{ij}$ are to be read as "subgradients" when they are not uniquely determined as "derivatives".

4. Numerical Calculation

From the point of view of numerical calculations, problem *P1** has the advantage over *P1* since the former is essentially a scalar-variable problem with constraints easy to treat, whereas the latter is a vector-variable problem with constraints expressed by differential equations. Hence, we shall try to derive a method for finding numerical solutions to the problem based on problem *P1** . We shall make use of the finite-element technique to numerically approximate the continuous part of the problem.

In the following we shall take up 2-dimensional continua, but our method is easy to extend to higher dimensions.

4.1. Formulation of the approximate discretized problem *P2**

The region V is divided into subregions A_e which are triangular elements (Fig. 3). We denote the set of subscripts e of A_e by L, the vertices of triangular elements by J's, and the set of vertices by I. In addition, we use the following notations:

I_P : the set of vertices on entrance P,
I_Q : the set of vertices on exit Q,
$L(J)$: the set of subscripts of the elements with vertex J.

We adopt a piecewise linear function $\zeta^{(a)}(x)$ as an approximate representation of $\zeta(x)$, which is defined as, within each element,

$$\zeta^{(a)}(x) = \sum_J F_J(x)\zeta_J , \tag{4.1}$$

where $F_J(x)$ is a linear local interpolation function associated with vertex J (i.e. $F_J(x)$ is equal to the barycentric coordinate (corresponding to vertex J) of point x with the three vertices of the elements as the base points), ζ_J is a value of potential at vertex J and the summation is taken over the

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

three vertices contained in each element. We adopt a piecewise constant function $\xi^{(a)\kappa}(x)$ as an approximate representation of $\xi^{\kappa}(x)$, which is constant within each element. Futhermore we assume that the capacity convex is uniform within each element and denote it by C_e.

In order to facilitate the calculation, we define the division of V_i and the function $\rho_i(x)$ associated with V_i, as follows.

We put a point J_i anywhere in V_i (if $\partial V \cap \partial V_i \neq \phi$, then we may put J_i

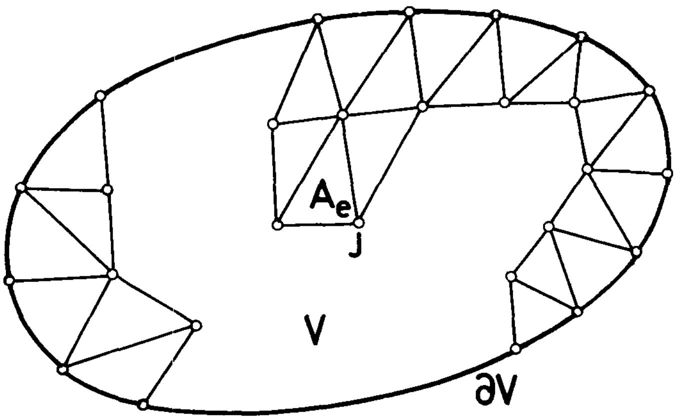
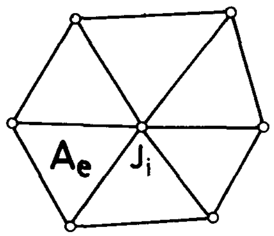
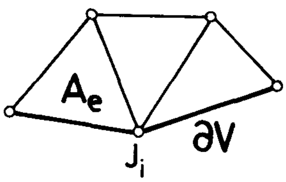


Fig. 3. Division of V into elements



(a) J_i is in V.



(b) J_i is on ∂V.

Fig. 4. Division of V_i

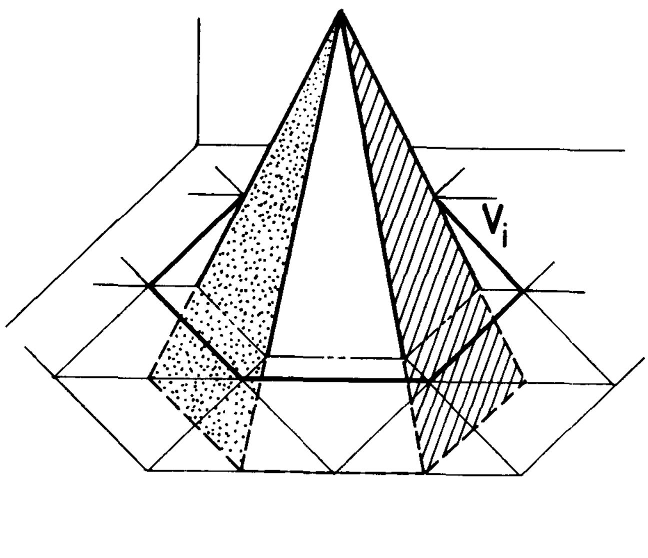


Fig. 5. Function $\rho_i(x)$ defined on V_i of a regular division

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

on $\partial V \cap \partial V_i$), divide V_i into triangular elements with J_i as an vertex (as shown in Fig. 4), and define $\rho_i(x)$ by

(4.2) $$\rho_i(x) = \frac{3}{\Delta_i}(4F_{J_i}(x) - 1)$$

within each element A_e (Fig. 5), where

Δ_i = the sum of the areas of the elements in V_i.

It is easy to see that the $\rho_i(x)$ thus defined satisfies

(4.3) $$\int_{V_i} \zeta^{(a)}(x)\rho_i(x)dV = \zeta_{J_i}$$

as well as (2.8).

Substitution of (4.1) into (3.17) followed by the integration over an element A_e yields the expression

(4.4) $$\psi_e(\eta^{(a)}) = \int_{A_e} \psi_c(x,\eta^{(a)})dV = \int_{A_e} \max_{\xi^{(a)}\varepsilon C_e} \{\xi^{(a)\kappa}\eta_\kappa^{(a)} - \phi_c(\xi^{(a)})\}dV$$
$$= \max_{\xi^{(a)}\varepsilon C_e} \{\xi^{(a)\kappa}\eta_\kappa^{(a)} - \phi_c(\xi^{(a)})\}\Delta_e,$$

where

$$\eta_\kappa^{(a)} = -\frac{\partial\zeta^{(a)}}{\partial x^\kappa} = -\sum_J \zeta_J\frac{\partial F_J}{\partial x^\kappa},$$

and

Δ_e = the area of element A_e.

In terms of the quantities introduced so far, the discretized form of problem *P1** (which we shall call problem *P2**) is defined as the problem of determining $\zeta^{(a)}$, $\eta_\kappa^{(a)}$, λ_i's and μ_{ij}'s so as to

(*P2**) maximize

(4.5) $$\Psi_2(\zeta^{(a)},\eta^{(a)},\lambda_i,\mu_{ij}) = -\sum_{e\varepsilon L} \psi_e(\eta^{(a)}) - \sum_{(i,j)\varepsilon E} \psi_{ij}(\mu_{ij})$$

subject to the constraints:

(4.6) $$\eta_\kappa^{(a)} = -\frac{\partial\zeta^{(a)}}{\partial x^\kappa} \quad \text{in } V,$$

(4.7) $$\zeta_J = 1 \quad J \in I_P,$$

(4.8) $$\zeta_J = 0 \quad J \in I_Q,$$

(4.9) $$\mu_{ij} = \lambda_i - \lambda_j \quad (i,j) \in E,$$

(4.10) $$\lambda_i = 1 \quad i \in N_P,$$

(4.11) $$\lambda_i = 0 \quad i \in N_Q,$$

(4.12) $$\zeta_{J_i} = \lambda_i \quad i \in \tilde{N},$$

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

Problem $P2^*$ is a convex optimization problem with variables of a finite number. Therefore, if cost characteristics ψ_e and ψ_{ij} are specified, its numerical solution will be obtained by means of any of ordinary techniques in convex programming. In numerical examples discussed in Section 5, Powell's method based on conjugate directions [9] was used to solve $P2^*$.

4.2. A procedure for determining the capacity convex C_e from the road map

First, we shall consider the concept of superposition of component continua.

We may approximate a network composed of densely distributed parallel branches with equal capacity by a continuum with the capacity convex

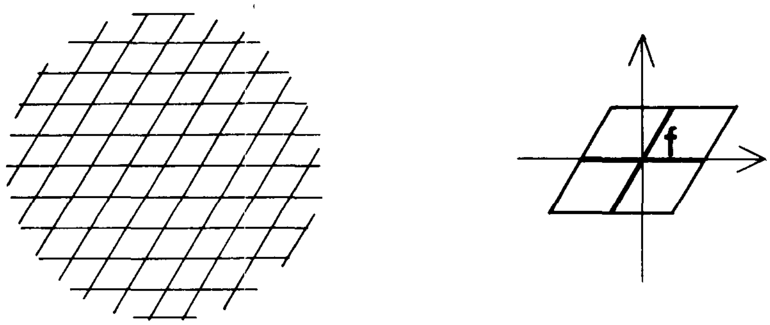
$$C = \{\xi^{\kappa} \mid \xi^{\kappa} = kf^{\kappa},\ \ 0 \leq k \leq 1\}$$

where f^{κ} is a vector which is parallel to the branches and whose length is equal to the density multiplied by the capacity. Futhermore, the network composed of more than one set of densely distributed parallel branches may be appropriately approximated by a continuum with the capacity convex C which is the superposition of the component continua with the capacity convex C_i corresponding to each set of parallel branches, i.e. we may put

$$C = \sum_i C_i$$

where " $\sum$ " means the sum of the subsets of a vector space. The intuitive meaning of this rule is illustrated in Fig. 6.

If A_e is an element of triangulation of the area covering a given network G_o and if the network is sufficiently uniform in A_e, we may be allowed to



(a) Network (b) Capacity convex of the approximate continuum

Fig. 6. Superposition of continua

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

consider only the branches intersecting the three edges of A_e. We shall denote by E_+ the set of those branches intersecting an edge of A_e oriented from the inside of A_e to the outside, and by E_- those oriented from the outside to the inside. For each branch b of E_+, we make correspond a network G_b composed of densely distributed parallel branches whose density multiplied by capacity is equal to

$$f_b^\kappa = \frac{a_b}{l_b} e_b^\kappa ,$$

where e_b^κ is the unit vector parallel to b, a_b the capacity of b, and l_b the distance between the two lines which are parallel to b and circumscribed about A_e (Fig. 7).

We denote by C_b the capacity convex of the continuum corresponding to G_b, and define the continuum with capacity convex

$$C_+ = \sum_{b \in E_+} C_b . \tag{4.13}$$

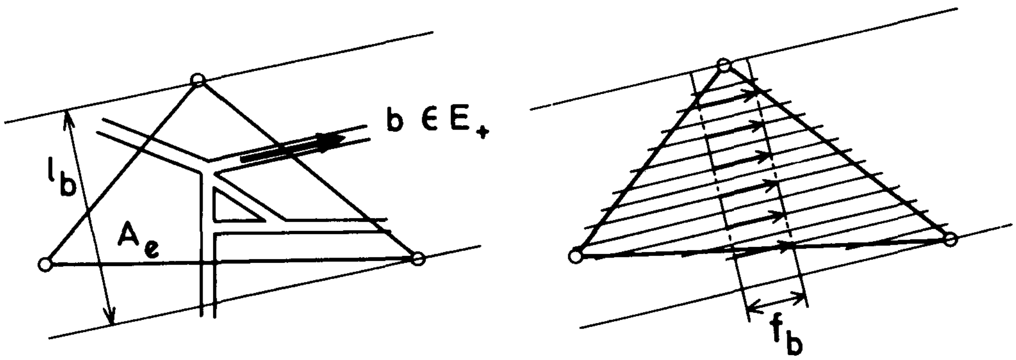
Similarly, for the branches of E_-, we have another continuum with capacity convex

$$C_- = \sum_{b \in E_-} C_b . \tag{4.14}$$

It is obvious that both C_+ and C_- overestimate the capacity of the network (or its approximate continuum), so that

$$C_e = C_+ \cap C_- \tag{4.15}$$

will be a better approximation of the capacity convex of the continuum



(a) Branch b intersecting an edge of A_e (b) Network G_b

Fig. 7. Road network G_0 and element A_e

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

approximating G_o (Fig. 8). We shall adopt C_e determined in this manner as the capacity convex of the element A_e of the approximate continuum.

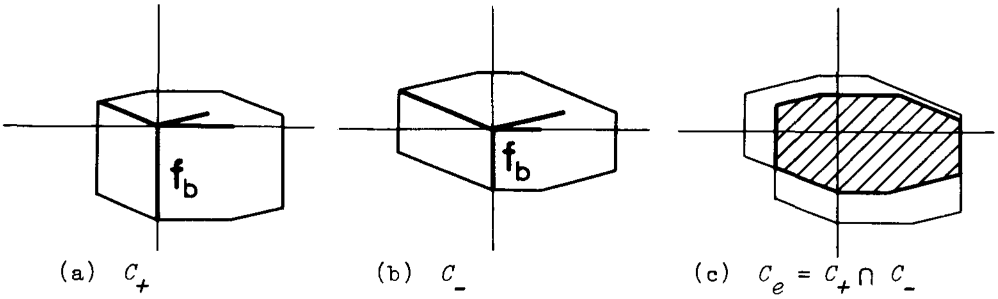


Fig. 8. Capacity convex $C_e = C_+ \cap C_-$

4.3. Relation between the original problem *P1* and the dual *P2* of *P2**

The approximate problem *P2** was not derived directly from the original problem *P1* but from *P1**, the dual of *P1*. In this section, we shall consider the relation of the problem *P2*, which is the dual of *P2**, to the original *P1* , as well as to the network G_o. For simplicity, we shall confine our consideration to the continuum subsystem V and neglect the term ϕ_c.

We shall use the following notation (as is illustrated in Fig. 9).

l_J : the polygonal curve (in the set of elements $\{A_e \mid e \in L(J)\}$) which connects the middle points of the edges incident to vertex J.

l_P : the polygonal curve (in the set of elements $\{A_e \mid e \in L(J), J \in I_P\}$) which connects the middle points of the edges incident to vertices of I_P but not on P.

For a path (sequence of edges) S which separates P and Q, we denote by I_S the set of vertices of the edges.

l_S : the polygonal curve (in the set of elements $\{A_e \mid e \in L(J)$, A_e lies between S and Q, $J \in I_S\}$) which connects the middle points of the edges (between S and Q) incident to vertices of I_S but not belonging to S.

According to this notation, the problem, *P2*, dual to *P2** may be described as the problem of

(*P2*) minimizing

(4.16) $$\Phi_2(\xi^{(a)},\omega) = -\sum_{J \in I_P} \omega_J$$

subject to the constraints:

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

(4.17) $\omega_J \quad = 0 \qquad J \in I - (I_P \cup I_Q)$,

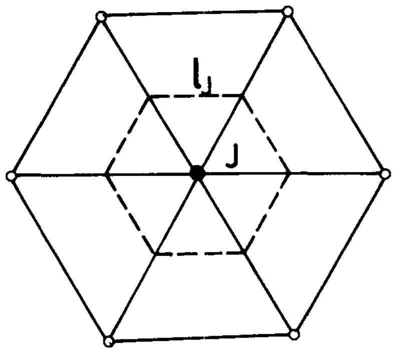
(4.18) $\xi^{(a)\kappa} \in C_e \qquad e \in L,$

where

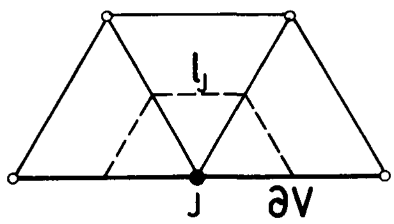
(4.19) $$\omega_J \quad = - \sum_{e \in L(J)} \Delta_e \frac{\partial F_J}{\partial x^\kappa} \xi^{(a)\kappa}.$$

As from the definition of $F_J(x)$, $-\Delta_e \; \partial F_J/\partial x^\kappa$ in (4.19) is a vector which is parallel to the outward normal of the edge of A_e opposite to J, and which is one half as long as that edge. Since $\xi^{(a)}$ is constant within each element, ω_J expresses the amount of flux crossing l_J from the side of J to the opposite edges. Therefore, the condition that ω_J vanishes may be regarded as an approximate form of the continuity equations (3.2) and (3.4).

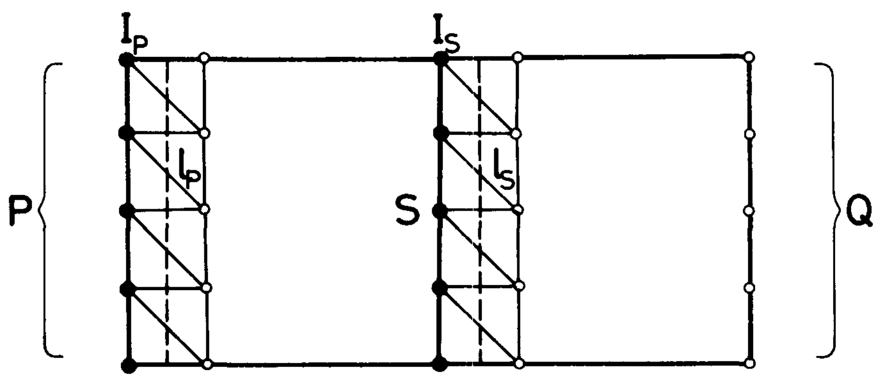
From the characterization of ω_J mentioned above, it follows that $\sum_{J \in I_P} \omega_J$



(a) l_J when J is in V



(b) l_J when J is on ∂V



(c) l_P and l_S

Fig. 9. Definition of the curves l_J, l_P and l_S

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

is the amount of flux crossing l_P from the side of P to that of Q, and that it corresponds to the term $-\int_P \xi^\kappa dS_\kappa$ in (3.1). Since (4.17) holds at each J between P and Q for an arbitrary given S, the amount of flux crossing l_S from P to Q is equal to $\sum_{J\in I_P} \omega_J$. This implies the relation

$$-\min \Phi_2 \leq \Omega(l_S) \ ,$$

where $\Omega(l_S)$ is the maximum amount of flux which can cross l_S under the capacity constraints of the elements containing l_S.

The l_S's correspond to cuts in capacitated networks. From the construction of capacity convexes in Section 4.2, it will be seen that the "capacity" of a "cut" l_S is approximately equal to the average value of the capacities of those cuts which consists of the branches of the original network distributed in the elements containing l_S. Therefore, to obtain a good numerical solution in accuracy, we must not divide the region V in such a way

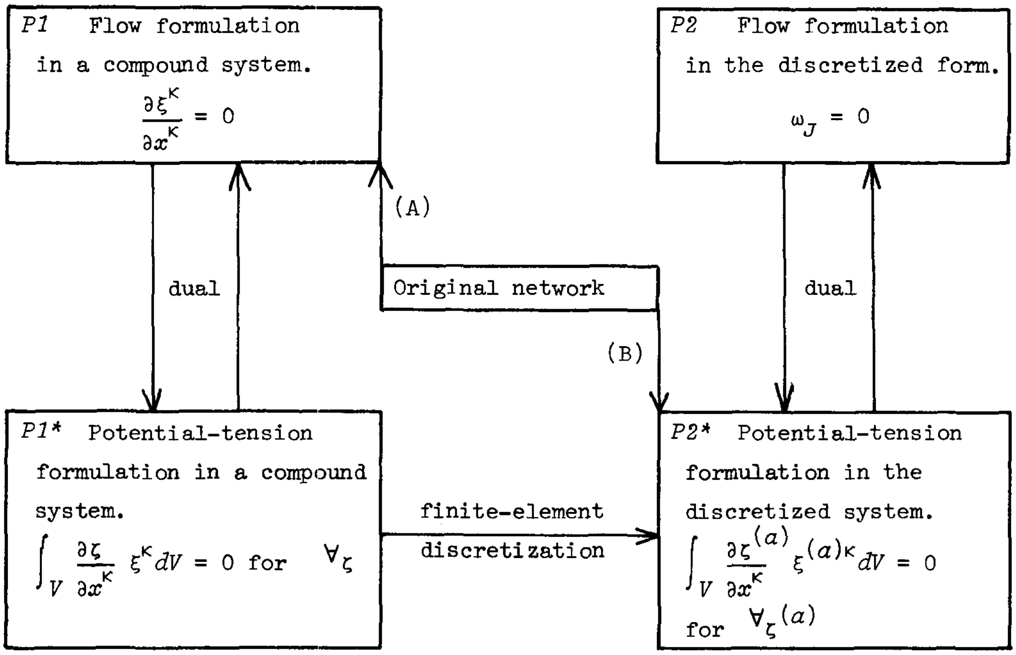


Fig. 10. Relations among the problems *P1*, *P1**, *P2* and *P2**.

The expressions represent the nondivergence conditions of flow in continua.

(A) Modelling in terms of a discrete-continuous compound system.

(B) Calculation of capacity convexes of finite elements from the capacities of the branches in the original network.

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

that the variation of the structure of the original network within an element may be very large in an element and very small in another, i.e. the division into elements should be uniform with respect to the variation of structure of the network.

The relations among the problems derived in Section 3 and Section 4 are summarized in Fig. 10.

5. Numerical Examples

We took up as an example the maximum-flow problems on the road network of Tokyo Metropolitan Area, shown in Fig. 12, regarding the expressways as the discrete subsystem and the remaining roads as the continuum subsystem. The numerical solutions to the finite-element approximation of the compound system were compared with the direct solutions of the maximum-flow problems in the original network.

For simplicity, we assumed that the maximum traffic volume passing through a road is proportional to the width of the road. The ratio of the traffic volume to the width of the road was assumed to be 2 cars/minute/meter for expressways and 1 car/minute/meter for the rest.

The numerical calculations were carried out on the HITAC 8800/8700 of the Computer Center of the University of Tokyo.

5.1. Cost characteristic

We used two types of functions as the cost characteristic $\phi_c(\xi)$ and $\phi_{ij}(f_{ij})$ with respect to flow.

Type 1 : An isotropic quadratic function of flow normalized with reference to the capacity, defined as

$$\phi_c(\xi) = \varepsilon_1 \frac{|\xi^\kappa|^2}{r_e^2}, \tag{5.1}$$

$$\phi_{ij}(f_{ij}) = (\varepsilon_1 \frac{f_{ij}^2}{a_{ij}^2})(l_{ij} \frac{a_{ij}}{k_{ij}}), \tag{5.2}$$

where

$|\xi^\kappa|$: the Euclidean norm of flow ξ^κ,

r_e : the radius of the circle with the center at the origin with which C_e is enclosed within bounds,

l_{ij} : the length of branch (i,j),

a_{ij} : the capacity of branch (i,j),

k_{ij} : the maximum traffic volume per unit width of the road,

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

ε_1 : a positive number (5.0 in the following calculation).

Type 2 : An isotropic function approximately linear in the Euclidean norm of flow but slightly modified to become strictly convex, defined as

$$(5.3) \qquad \phi_c(\xi) = \varepsilon_2\sqrt{|\xi^K|^2 + \delta^2 r_e{}^2}\ ,$$

$$(5.4) \qquad \phi_{ij}(f_{ij}) = \varepsilon_2\sqrt{f_{ij}{}^2 + \delta^2 a_{ij}{}^2}\ l_{ij}\ ,$$

where r_e, a_{ij} and l_{ij} are the same as those defined in the above, and

δ : a small positive number (0.15 in the following calculation),

ε_2 : a positive number (0.6 in the following calculation).

The factor $l_{ij}a_{ij}/k_{ij}$ in (5.2) and that l_{ij} in (5.4) are introduced to make the physical dimension of $\phi_{ij}(f_{ij})$ coincide with that of $\int_V \phi_c(\xi)dV$.

The factors ε_1 and ε_2 are taken to be so small that ϕ_c and ϕ_{ij} may not affect the maximality of flux feasible from P to Q.

These cost functions may be physically interpreted as follows. In type 1 functions, the factor $|\xi^K|/r_e$ in (5.1) (or f_{ij}/a_{ij} in (5.2)) represents

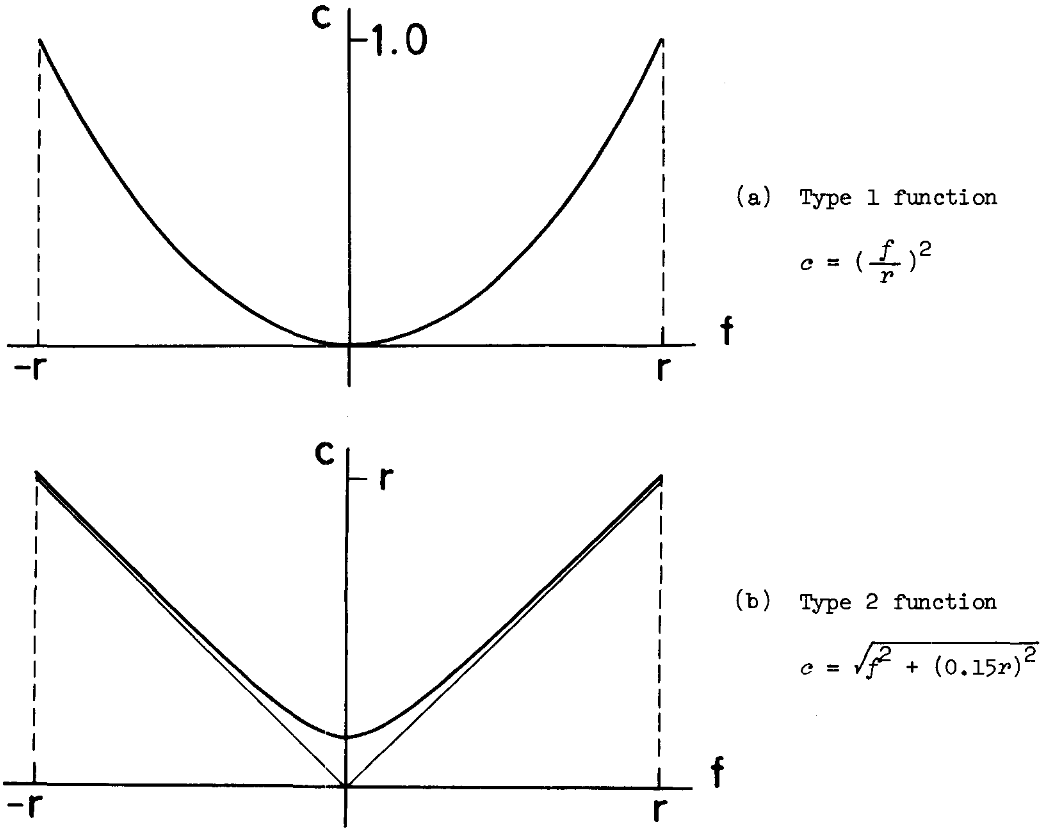


Fig. 11. Example of cost functions in the one-dimensional case

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

the congestion rate at a point in an element (or in a branch). Therefore the marginal cost along a route is proportional to the integral (or sum) of the congestion rates along the route. Type 2 functions are approximately linear in the Euclidean norm of flow, so that the marginal cost along a route is nearly proportional to the length of the route.

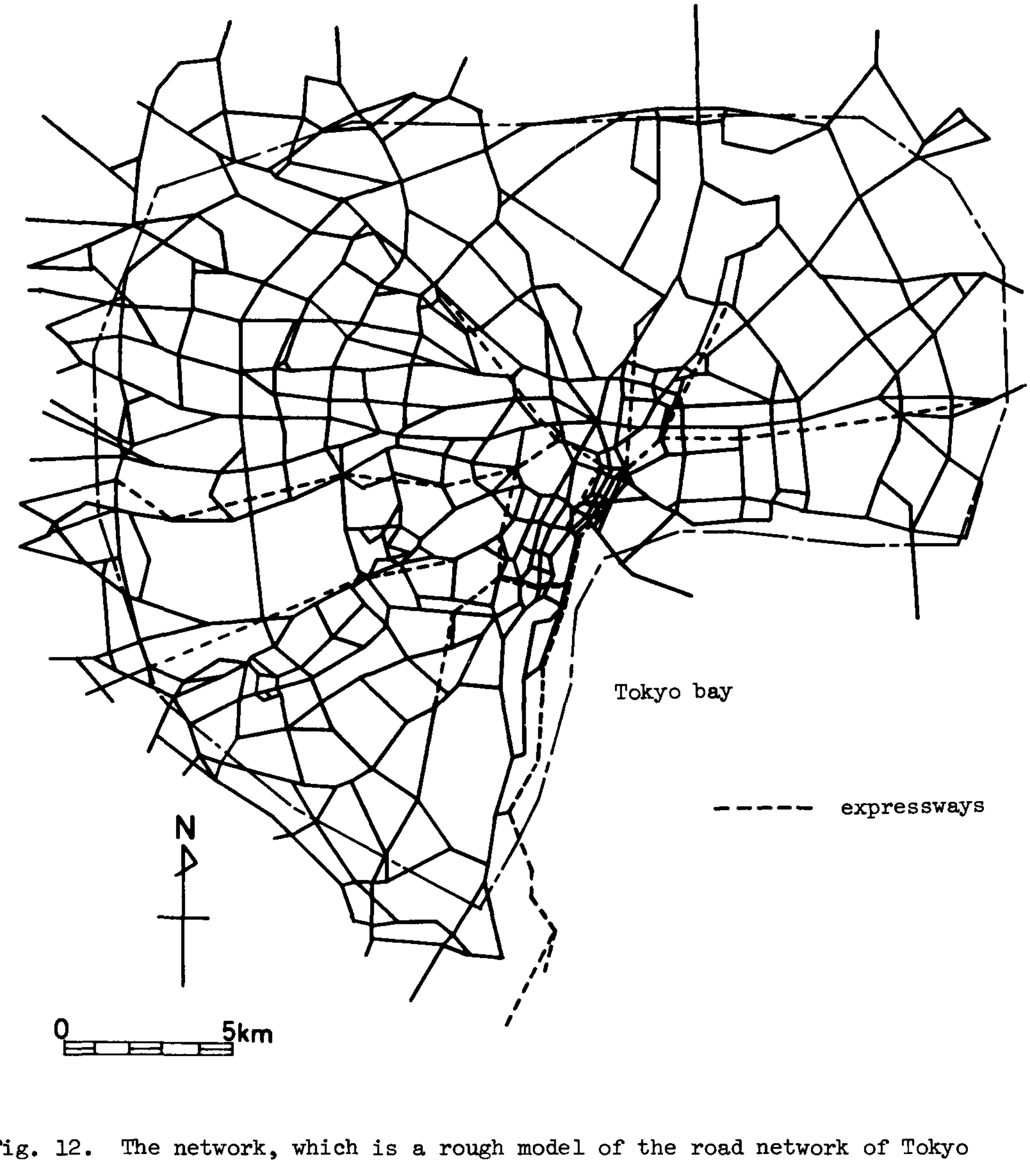


Fig. 12. The network, which is a rough model of the road network of Tokyo Metropolitan Area, consisting of 520 intersections (or nodes) and 850 links (or branches). The area circumscribed by the chain line is divided into triangular elements such as shown in Figs. 15 and 16

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

5.2. Computational results

We have considered the following two cases as to the location of entrance and exit.

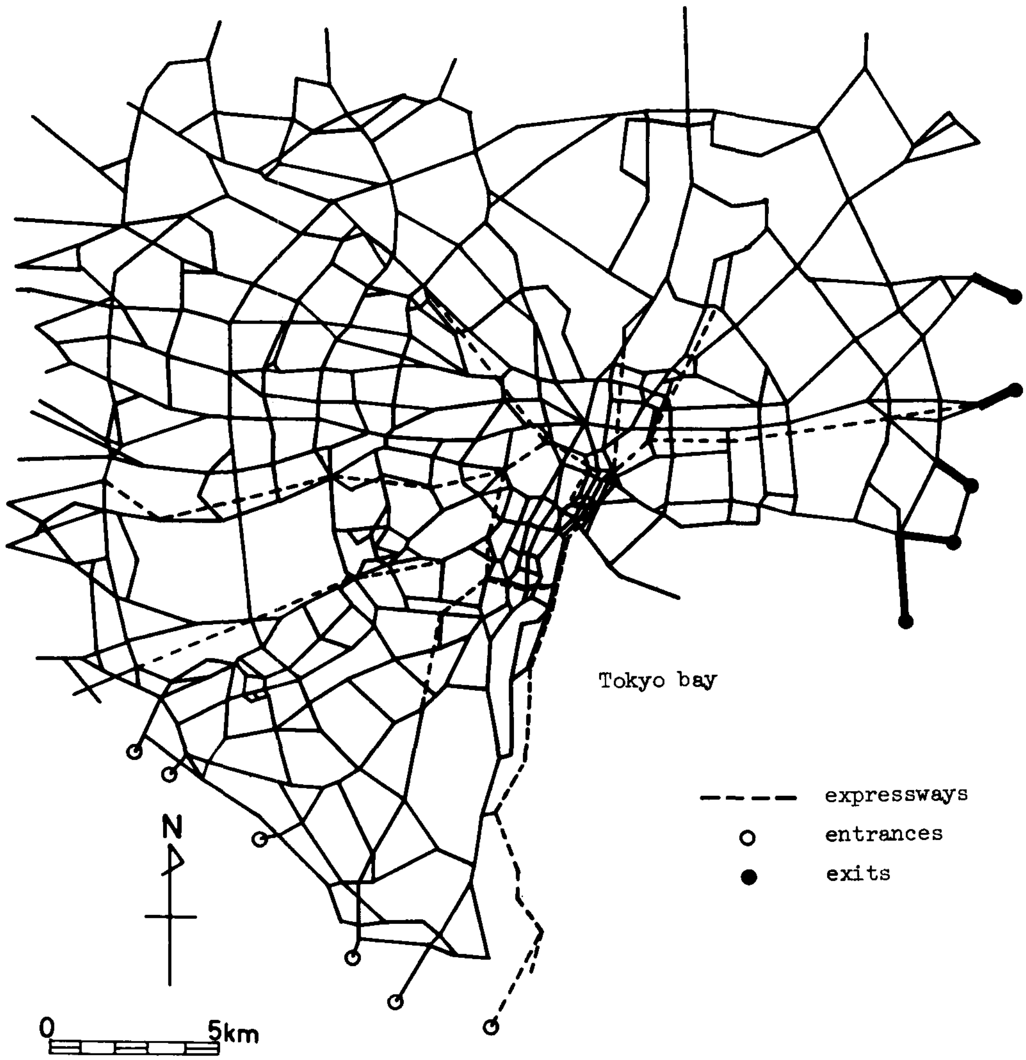


Fig. 13. Entrance nodes, exit nodes and the minimum cut for case-1 entrance-exit in the original network (computation No. 1). The minimum cutset consists of the branches denoted by bold lines.

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

Case 1 : Particles enter the area from the south-west suburbs and leave it to the east suburbs (Fig. 13).

Case 2 : Particles enter the area from all the directions except those of the bay and gather to the center of the city (Fig. 14).

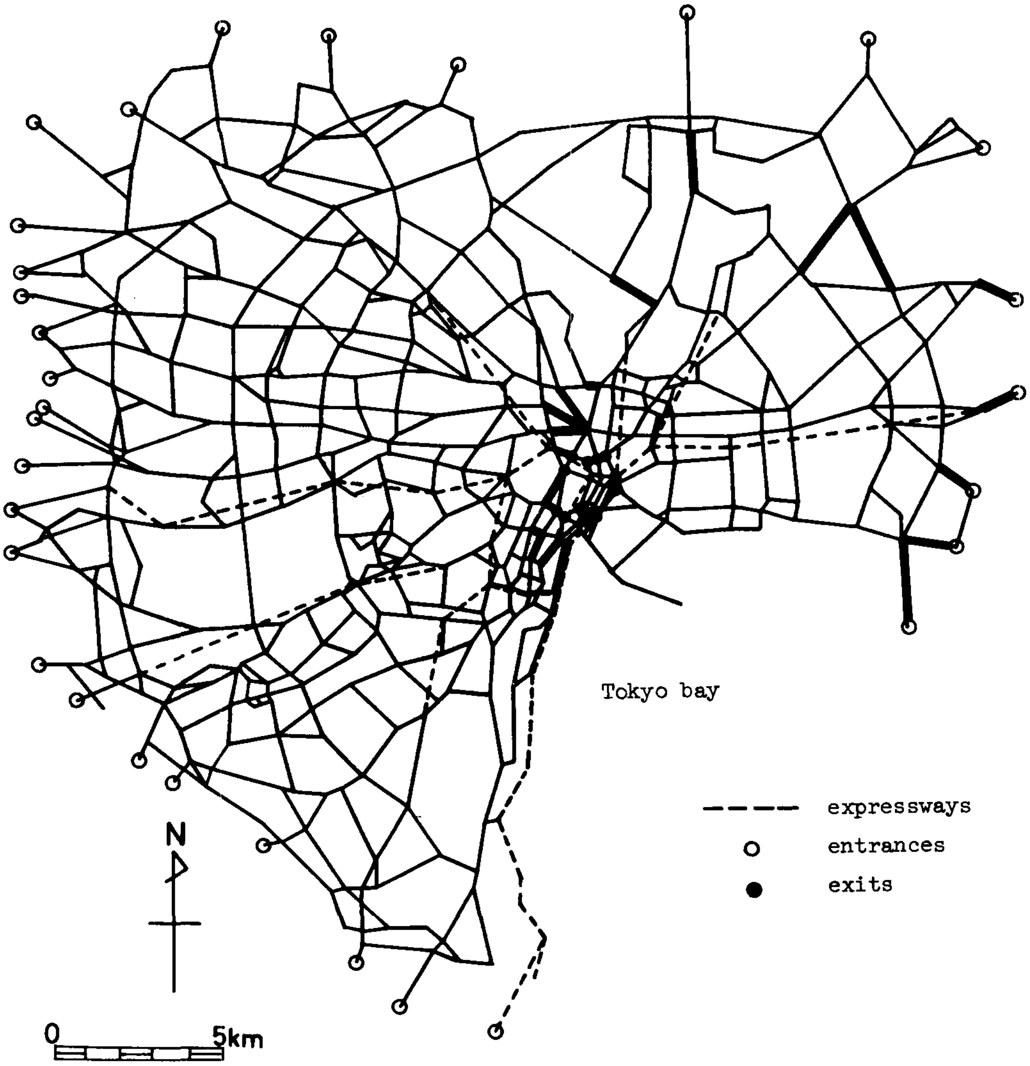


Fig. 14. Entrance nodes, exit nodes and the minimum cut for case-2 entrance-exit in the original network (computation No. 2). The minimum cutset consists of the branches denoted by bold lines.

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

For those two cases, the maximum amounts of flux in the original network were calculated by means of the ordinary network flow algorithm [4],[11]. The computation times are shown in Table 1, and the minimum cuts are illustrated in Figs. 13 and 14.

Table 1. Maximum flow in the original network

No.	entrance and exit (see Figs. 13 and 14)	maximum flow (cars/minute)	computation time* (cpu sec.)
1	case-1	40.	1.
2	case-2	261.	11.

* On the HITAC 8700/8800 with OS/7 at the Computer Center of Tokyo University.

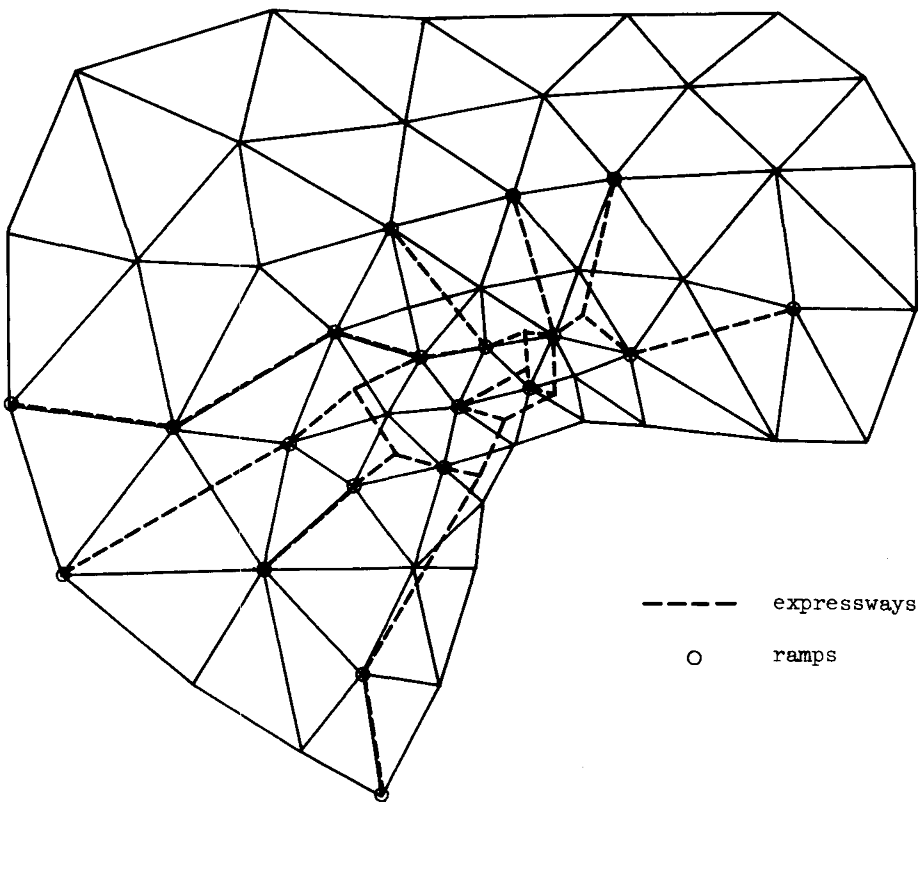


Fig. 15. Discretization with coarse mesh

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

The cases we have treated by means of the finite-element approximation are shown in Table 2. The meshes used are illustrated in Figs. 15 and 16

Table 2. Examples of computation by means of the approximate model

No.	entrance and exit	cost	mesh	Fig.
3	case-1	type-1	coarse	17
4	case-1	type-2	coarse	18
5	case-2	type-1	coarse	19
6	case-2	type-2	coarse	20
7	case-2	type-2	fine	21

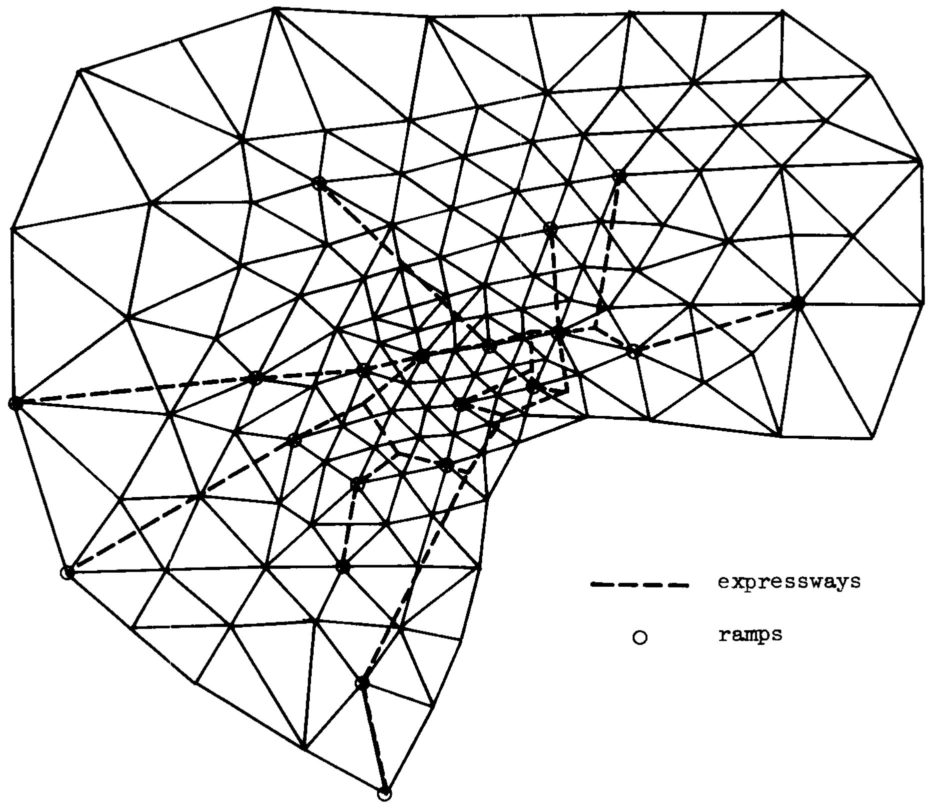


Fig. 16. Discretization with fine mesh

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

together with the discrete subsystem, where the nodes of $\tilde{N}$ defined in Section 2.1 are designated as "ramps". Table 3 shows the maximum amount of flux in the respective cases.

The flow configurations obtained by the finite-element approximation are shown in Figs. 17∿18, where the intensity of flow field in each element is represented by the density of arrows there and the equi-potential lines are drawn at intervals of 0.05. The "minimum cuts" are visualized as the areas where the equi-potential lines are drawn densely.

Comparision of the solutions of the original network with those of the approximate model with respect to the location of the minimum cuts — Fig. 13

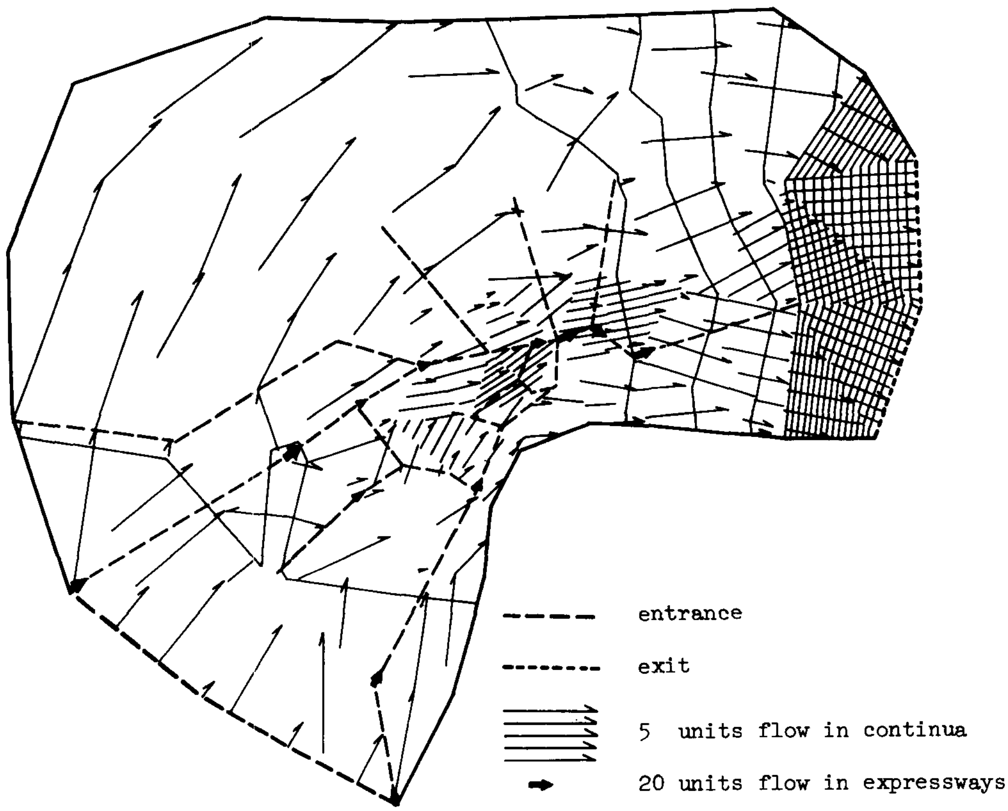


Fig. 17. Result of computation No. 3 — case-1 entrance-exit, coarse mesh and type-1 cost

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

with Fig. 17 (or Fig. 18) for Case 1 and Fig. 14 with Fig. 19 (or Fig. 20) for Case 2 — will allow us to say that they show fairly good coincidence with each other in the large.

The effect of the cost functions may be found in the flow configurations in Fig. 17 and Fig. 18, and the difference between Fig. 19 and Fig. 20 is rather ambiguous.

Table 3 shows that the maximum amount of flux calculated with the coarse mesh is about 20% larger than the value in the original network for Case 2. Primarily, the procedure of the calculation of capacity convexes and the coarseness of the division into elements would be responsible for this.

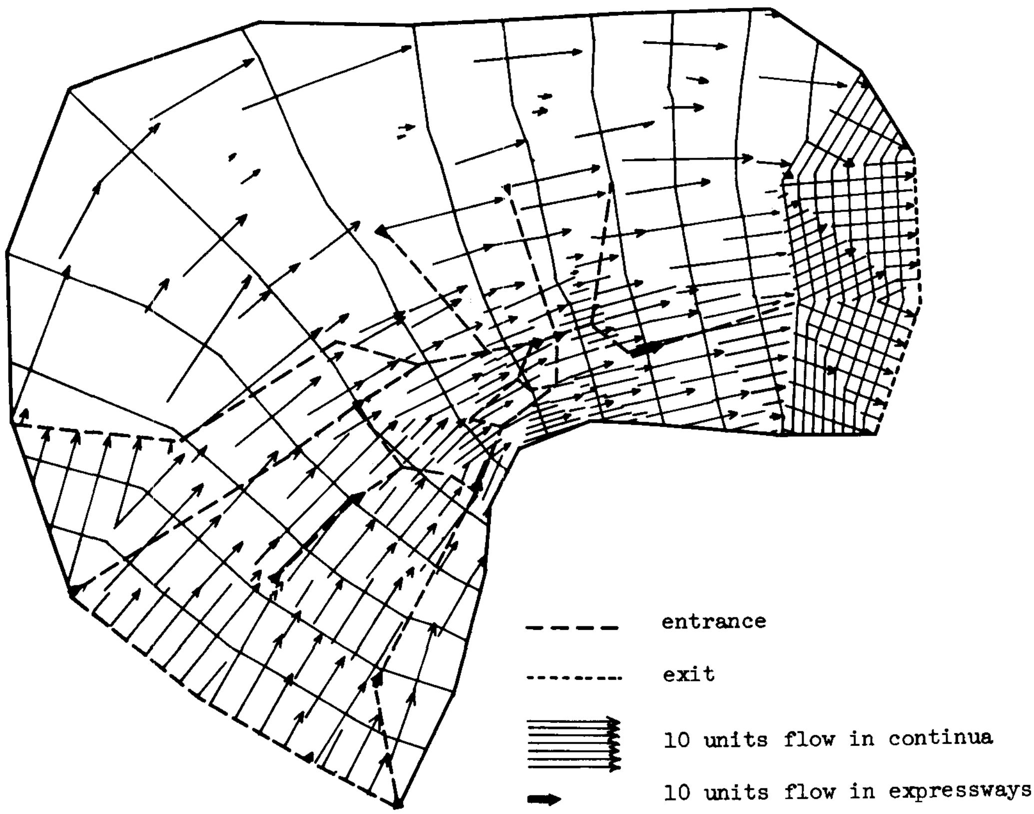


Fig. 18. Results of computation No. 4 — case-1 entrance exit, coarse mesh and type-2 cost

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

In fact, the solution for Case 2 with the fine mesh of Fig. 16 yields a better result with the maximum amount of flux different relatively by 13% from the value in the original network.

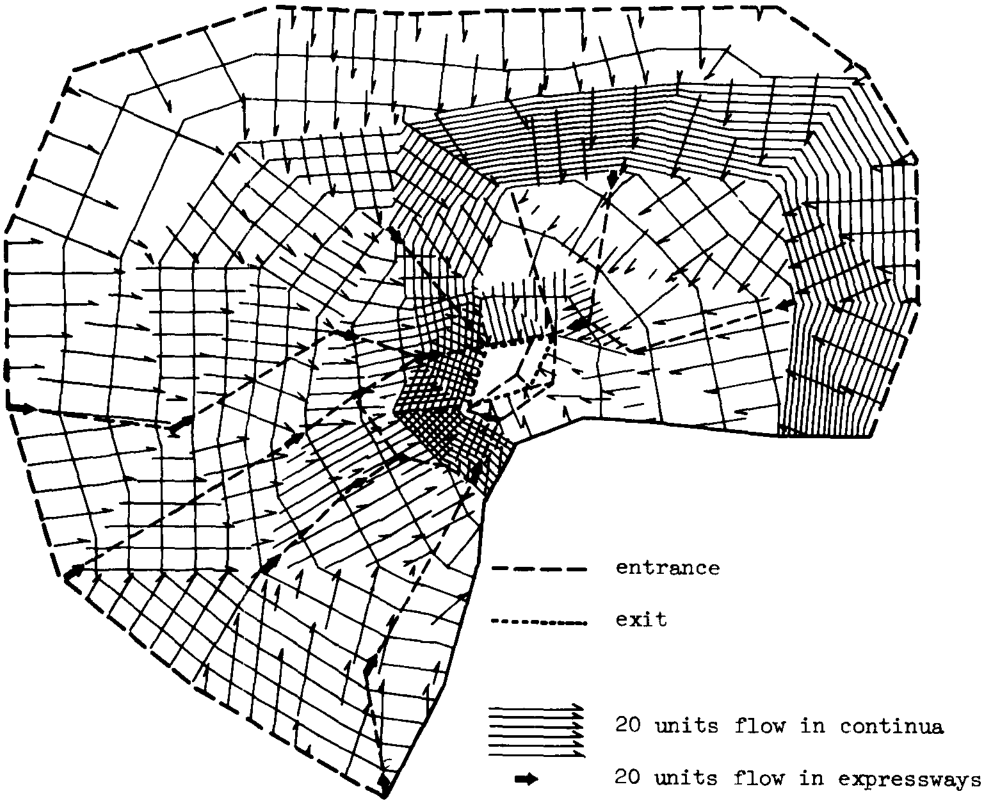


Fig. 19. Result of computation No. 5 — case-2 entrance exit, coarse mesh and type-1 cost

Table 3. Maximum flow in the approximate model

No.	maximum flow (cars/minute)	relative error to the network solution	computation time* (cpu sec.)
3	40.9	0.02	18.
4	40.9	0.02	19.
5	311.0	0.19	16.
6	316.7	0.21	19.
7	295.6	0.13	57.

* On the HITAC 8700/8800 with OS/7 at the Computer Center of Tokyo University.

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

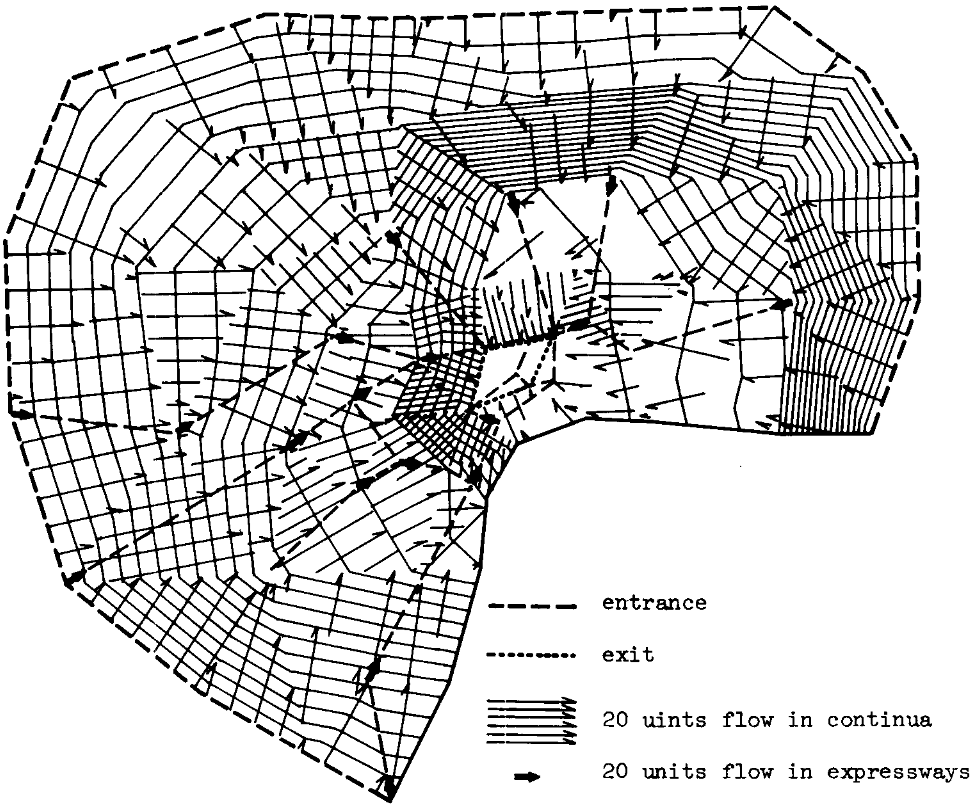


Fig. 20. Result of computation No. 6 — case-2 entrance-exit, coarse mesh and type-2 cost

6. Conclusion

In this paper some of the concepts related to the flow problems in a discrete-continuous compound system have been discussed, and an approximation method as well as a numerical algorithm has been proposed and tested for the maximum-flow problem. The effectiveness of the method has been demonstrated.

It might be emphasized that our approximation model has a number of advantages over the network model. For example, however fine or dense the network may become, the computational complexity of the continuum approximation does not grow very much, i.e. the accuracy of the numerical solution will

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

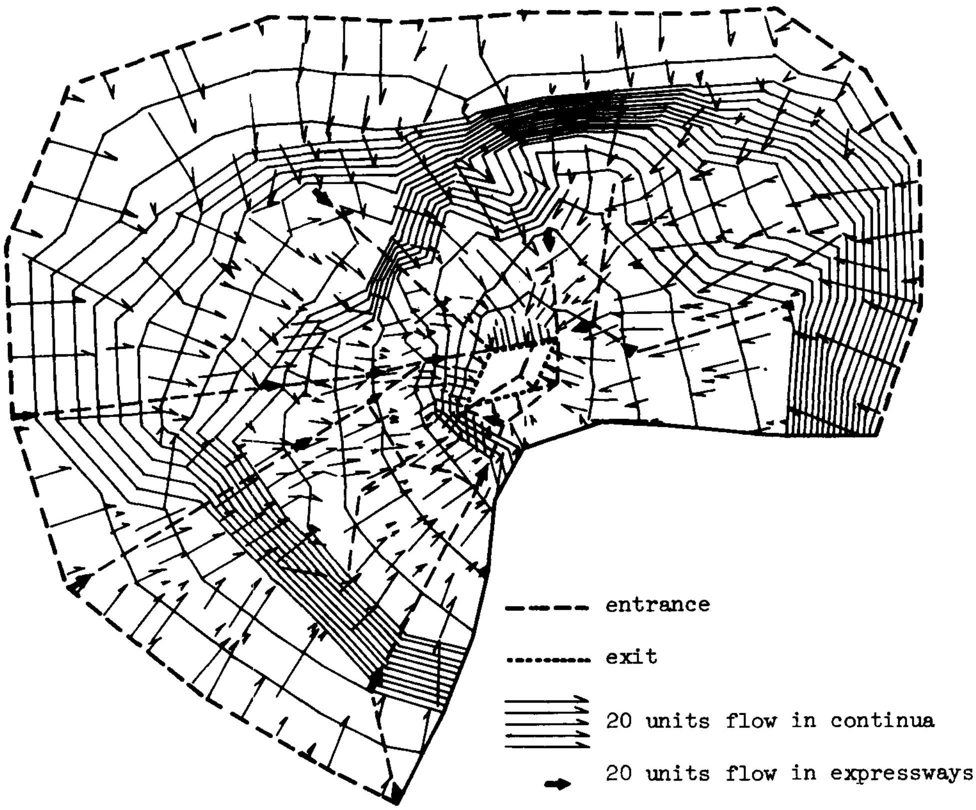


Fig. 21. Result of computation No. 7 — case-2 entrance-exit, fine mesh and type-2 cost

be improved even with the same mesh if the network becomes finer, and the computation time and memory strage necessary will not increase. On the other hand, if we would solve the network problem directly by means of network-flow algorithms, the computation time as well as memory strage necessary would increase at least proportional to the number of nodes and branches in the network. Futhermore, from the standpoints of data gathering, the continuum model is easier to build since it requires only the data concerning the directions and capacities of the branches crossing the edges of triangular elements.

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

Other linear network flow problems, such as the shortest path problem and the minimum cost transportation problem, can be formulated by means of the continuum model in a similar manner (see Iri [5]). Also, for these problems, numerical algorithms were derived and tested in the same way as we proposed in this paper, and our continuum model proved to be effective as well. The results on those problems will be published before long.

One might naturally hope that the model would be applicable as well to the multicommodity flow problem, but in this case we encounter difficulties of treating the problem of two- (or higher-)dimensional flows that do not occur with the problem on a network. Let a nonnegative number $\underset{d}{f_{ij}}$ be a flow for commodity d $(=1,\ldots,t)$ in branch (i,j), and a_{ij} be a capacity of branch (i,j). Different commodities share in a branch the same "resource", namely the branch capacity; that is, the flow $\underset{d}{f_{ij}}$'s are subject to the capacity constraints

$$\sum_{d=1}^{t} \underset{d}{f_{ij}} \leq a_{ij} \qquad \text{for every branch } (i,j).$$

The capacity constraints become complicated in the continuum model for two- (or higher-)dimensional flows as follows. Let $\underset{d}{\xi^\kappa}$ $(\kappa=1,\ldots,n)$ be a flow (vector-density) field of commodity d $(=1,\ldots,t)$ and $C(x)$ be a capacity convex set at point x. Since $\underset{d}{\xi^\kappa}$'s denote the amount of different materials for different d's, it is not sufficient to satisfy the capacity constraint that the vector sum of all $\underset{d}{\xi^\kappa}$'s is contained in $C(x)$, but the set of every combination of $\underset{d}{\xi^\kappa}$'s

$$K(x) = \{\xi^\kappa \mid \xi^\kappa = \sum_{d=1}^{t} s_d \underset{d}{\xi^\kappa} \ , \ s_d = 0 \text{ or } 1\}$$

must be contained in $C(x)$. For two-dimensional flows, the convex hull of $K(x)$ is a polygon with as many as $2t$ (in general) edges so that the capacity constraints become the more complicated the lager the number of commodities is. Thus, it is not easy to reduce the complexity of computation of the multicommodity flow problem by means of the continuum-model approach.

Acknowledgements

The author would like to express his sincere thanks to Professor Masao Iri for his continuing guidance and encouragement.

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

References

[1] Appell, P.: *Le problème géométrique des déblais et remblais.* Mémorial des Sciences Mathématiques, Fasc. XXVII, Académie des Sciences de Paris, Gauthier-Villars, Paris, 1928.

[2] Ekeland, I. and Temam, R.: *Convex Analysis and Variational Problems.* North-Holland, Amsterdam, 1976.

[3] Hu, T. C.: *Integer Programming and Network Flows.* Addison-Wesley, Reading, 1969.

[4] Iri, M.: *Network Flow, Transportation and scheduling --- Theory and Algorithms.* Academic Press, New York, 1969.

[5] Iri, M.: Theory of Flows in Continua as Approximation to Flows in Networks. *Proceedings of the IXth International Symposium on Mathematical Programming,* held at Budapest, 1976, North-Holland, (to appear).

[6] Kantorovitch, L.: On the Translocation of Masses. *Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS,* Vol. XXXVII, No. 7-8 (1942), pp. 199 ∿ 201 (Mathematics).

[7] Monge, G.: Déblai et remblai. Mémoires de l'Académie des Sciences, 1781.

[8] Oden, J. T. and Reddy, J. N.: *An Introduction to the Mathematical Theory of Finite Elements.* Wiley-Interscience, New York, 1976.

[9] Powell, M. J. D.: An Efficient Method for Finding the Minimum of a Function of Several Variables without Calculating Derivatives. *The Computer Journal,* Vol. 7, 1964, pp. 155 ∿ 162.

[10] Rockafellar, R. T.: *Convex Analysis.* Princeton University Press, Princeton, 1970.

[11] Tomizawa, N.: A Computational Experiment on a Practical Large-scale Problem (in Japanese). *Fundamental Studies of the Techniques for Processing Network Problems in Operations Research by Computers*, Iri, M. et al. (eds.), Technical Report Series T-73-2, Operations Research Society of Japan, 1973, pp. 115 ∿ 133.

[12] Yamamoto, Y.: A Theory of Saturation. *Proceedings of the IXth Japan National Congress for Applied Mechanics,* 1961.

Azuma TAGUCHI

Department of Mathematical Engineering
and Instrumentation Physics
Faculty of Engineering
University of Tokyo, Hongo
Bunkyo-ku, Tokyo 113 Japan

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.