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A SUMMARY OF OPTIMUM PREVENTIVE MAINTENANCE POLICIES MAXIMIZING INTERVAL RELIABILITY

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Abstract This paper considers the interval reliability of a repairable system for the time interval [T, T+x]. Optimum preventive maintenance policies maximizing (i) the limiting interval reliability when T is constant, and (ii) the interval reliability when T is exponential, and minimizing (iii) the expected cost per unit of time, are derived. It is shown that under suitable conditions there exist optimum policies which are given by unique solutions of equations. Further this paper treats discrete case.

1. Introduction

Preventive maintenance (pm) of a repairable system is of great interest in reliability theory. Morse [4] derived the optimum pm policy maximizing the steady-state availability. Barlow and Hunter [1] pointed out that this problem is reduced to an age replacement problem. Nakagawa [5] summarized many of results in pm problems of repairable systems. In this paper, we discuss optimum pm policies maximizing the interval reliability.

Barlow and Proschan [2, p. 8] defined that "Interval reliability R(x,T)" is the probability that at a specified time T, the system is operating and will continue to operate for an interval of time x. The interval reliability is simply called "reliability" when T = 0, and further, become "pointwise availability at time T" as $x \to 0$. Thus, the interval reliability is one of the most important quantities from the viewpoints of reliability and availability, and is useful in practical situations. For example, consider the model of a standby generator. Suppose that T is the time until the electric power stops and x is the required time until the electric power recoveries again. Then, the interval reliability represents the probability that a standby generator will be able to operate during the interruption of the electric power.

Consider the pm policy for a repairable system. We adopt the interval reliability as an appropriate objective function because the interval reliability holds two quantities of reliability and availability. Then, we give optimum pm policies maximizing the limiting interval reliability when T is constant and maximizing the interval reliability when T has an exponential distribution. It is shown that the failure rate defined in (4.1) of [2] plays an important role in obtaining the optimum pm policy. Further, by introducing a repair cost, a pm cost, and a cost of system failure during [T, T+x], we derive the optimum pm policy minimizing the expected cost per unit of time in the steady-state.

Finally, we consider a discrete system in which the time to system failure is measured by the number of cycles. Such examples are switching devices, railroad tracks, ball bearings and typewriters. We show how to convert the results in continuous model to ones in discrete model.

As concluding remarks, we mention briefly the interval availability which is another quantity for the time interval [T, T+x], and give an example.

2. Optimum Preventive Maintenance Policy

Consider a system which is repaired upon failure and then returned to operation. We assume that the failure time has a distribution $F(t)$ with mean $1/\lambda$ and the repair time has a distribution $G(t)$ with mean $1/\mu$. Further consider the pm of the system in which it is repaired at failure or is maintained preventively at t_0, whichever occurs first (for more detailed assumptions of the pm policy, refers to [1]). However, the pm of the operating system is not made during the interval [T, T+x] even if the time for pm comes. Assume that the distribution of time to pm completion is the same as the repair time distribution $G(t)$. We derive an optimum pm time t_0^* which maximizes the limiting interval reliability.

We set a pm time t_0 to the operating system and obtain the interval reliability $R(x,T;t_0)$. For simplicity of equations, we denote that $A(t)$ is a distribution of a degenerate random variable placing unit mass at t_0, i.e.,

$$A(t) \equiv \begin{cases} 0 & \text{for } t < t_0, \\ 1 & \text{for } t \geq t_0. \end{cases}$$

Let $M(t)$ be the expected number of occurences of system recoveries during

(0, t] if the system begins to operate at time 0. Then, by the method similar to [2, p. 82], the interval reliability is easily given by

$$(1)\qquad R(x,T;t_0) = \bar{F}(T+x)\bar{A}(T) + \int_0^T \bar{F}(T+x-u)\bar{A}(T-u)dM(u),$$

where $\bar{F}(t) \equiv 1 - F(t)$, $\bar{A}(t) \equiv 1 - A(t)$, and $M(t)$ is given by the following renewal-type equation:

$$M(t) = [\int_0^t \bar{A}(u)dF(u) + \int_0^t \bar{F}(u)dA(u)]*G(t)*[1 + M(t)],$$

where the asterisk represents the Stieltjes convolution. Forming the Laplace transform of (1), we have that

$$(2)\qquad R^*(x,s;t_0) \equiv \int_0^\infty e^{-sT} R(x,T;t_0)dT$$

$$= \frac{e^{sx}\int_x^{t_0+x} e^{-st}\bar{F}(t)dt}{1 - G^*(s) + sG^*(s)\int_0^{t_0} e^{-st}\bar{F}(t)dt},$$

where $G^*(s)$ denotes the Laplace-Stieltjes transform of $G(t)$. Thus, the limiting interval reliability is

$$(3)\qquad R(x;t_0) \equiv \lim_{s\to 0} sR^*(x,s;t_0)$$

$$= \frac{\int_x^{t_0+x}\bar{F}(t)dt}{\int_0^{t_0}\bar{F}(t)dt + 1/\mu}.$$

We seek an optimum pm time t_0^* maximizing $R(x;t_0)$ in (3) for $x > 0$. Let

$$H(t) \equiv \frac{F(t+x) - F(t)}{\bar{F}(t)} \qquad \text{for } x > 0,\ t \geq 0, \text{ and } F(t) < 1.$$

Then, $H(t)$ is called the failure rate and has the same properties of $r(t)$ ($\equiv f(t)/\bar{F}(t)$, where $f(t)$ is a density of $F(t)$) (see [2, p. 23]). Further, let

$$\theta_1 \equiv \frac{\int_0^x \overline{F}(t)dt + 1/\mu}{1/\lambda + 1/\mu} .$$

Then, we have the following theorem:

Theorem 1. Suppose that the failure rate H(t) is continuous and monotonically increasing.

(i) If $H(\infty) > \theta_1$, then there exists a finite and unique t_0^* $(0 < t_0^* < \infty)$ which satisfies

$$(4) \qquad H(t_0)[\int_0^{t_0} \overline{F}(t)dt + 1/\mu] - \int_0^{t_0} [\overline{F}(t) - \overline{F}(t+x)]dt = 1/\mu .$$

(ii) If $H(\infty) \leq \theta_1$, then the optimum pm time is $t_0^* = \infty$, i.e., no pm is made.

Proof: Differentiating $R(x;t_0)$ in (3) with respect to t_0 and putting it zero, we have (4). Let

$$q_1(t_0) \equiv H(t_0)[\int_0^{t_0} \overline{F}(t)dt + 1/\mu] - \int_0^{t_0} [\overline{F}(t) - \overline{F}(t+x)]dt.$$

Then, $q_1(t_0)$ is monotonically increasing since

$$q_1'(t_0) = H'(t_0)[\int_0^{t_0} \overline{F}(t)dt + 1/\mu] > 0 \qquad \text{for } t_0 > 0,$$

and

$$q_1(0) = F(x)/\mu,$$

$$q_1(\infty) = H(\infty)(1/\lambda + 1/\mu) - \int_0^x \overline{F}(t)dt.$$

If $H(\infty) > \theta_1$, then $q_1(\infty) > 1/\mu > q_1(0)$. Thus, from the monotonicity and the continuity of $q_1(t_0)$, there exists a unique and finite t_0^* satisfying (4), which maximizes $R(x;t_0)$. Otherwise, if $H(\infty) \leq \theta_1$, then $q_1(\infty) \leq 1/\mu$. Thus, the optimum pm policy is $t_0^* = \infty$.

It is clear that if the failure rate H(t) is non-increasing, then the optimum pm time is $t_0^* = \infty$. In case (i) of the above theorem, the limiting interval reliability is

$$(5) \qquad R(x;t_0^*) = 1 - H(t_0^*).$$

Further, we have the following upper limit of the optimum pm time t_0^*.

Theorem 2. Suppose that the failure rate $H(t)$ is continuous, monotonically increasing, and $H(\infty) > \theta_1$. Then, there exists a finite and unique $\bar{t}_0$ which satisfies $H(\bar{t}_0) = \theta_1$ and $t_0^* < \bar{t}_0$.

Proof: From the assumption that $H(t_0)$ is monotonically increasing, we easily have that

$$(6)\qquad H(t_0) < \frac{\int_{t_0}^{\infty} \bar{F}(t)dt - \int_{t_0+x}^{\infty} \bar{F}(t)dt}{\int_{t_0}^{\infty} \bar{F}(t)dt} \qquad \text{for } 0 \leq t_0 < \infty.$$

Thus, we have the inequality

$$q_1(t_0) > H(t_0)(1/\lambda + 1/\mu) - \int_0^x \bar{F}(t)dt \qquad \text{for } 0 \leq t_0 < \infty.$$

Therefore, if there exists a $\bar{t}_0$ satisfying

$$H(\bar{t}_0)(1/\lambda + 1/\mu) - \int_0^x \bar{F}(t)dt = 1/\mu,$$

i.e.,

$$H(\bar{t}_0) = \theta_1,$$

then $t_0^* < \bar{t}_0$. It is easily seen that the root of $H(\bar{t}_0) = \theta_1$ is finite and unique since $H(t_0)$ is monotonically increasing, $H(0) < \theta_1$, and $H(\infty) > \theta_1$.

If the time for pm has $G_2(t)$ with mean $1/\mu_2$ and the time for repair has $G_1(t)$ with mean $1/\mu_1$, both of which may be different from each other, then the limiting interval reliability is easily given by

$$(7)\qquad R(x;t_0) = \frac{\int_x^{t_0+x} \bar{F}(t)dt}{\int_0^{t_0} \bar{F}(t)dt + F(t_0)/\mu_1 + \bar{F}(t_0)/\mu_2}.$$

Next, consider the pm policy when T is a random variable with an exponential distribution with mean $1/\alpha$. Then, the interval reliability is, from [2],

$$(8)\qquad R(x,\alpha;t_0) \equiv \int_0^{\infty} \alpha e^{-\alpha T} R(x,T;t_0)dT$$

$$= \frac{\alpha e^{\alpha x} \int_x^{t_0+x} e^{-\alpha t}\, \overline{F}(t)dt}{1 - G^*(\alpha) + \alpha G^*(\alpha)\int_0^{t_0} e^{-\alpha t}\, \overline{F}(t)dt} .$$

Let

$$\theta_2 \equiv \frac{1 - F^*(\alpha)G^*(\alpha) - \alpha G^*(\alpha)e^{\alpha x}\int_x^{\infty} e^{-\alpha t}\, \overline{F}(t)dt}{1 - F^*(\alpha)G^*(\alpha)} .$$

Then, we have the following theorems similar to Theorems 1 and 2.

Theorem 3. Suppose that the failure rate $H(t)$ is continuous and monotonically increasing.

(i) If $H(\infty) > \theta_2$, then there exists a finite and unique t_0^* $(0 < t_0^* < \infty)$ which satisfies

(9) $$H(t_0)[\alpha G^*(\alpha)\int_0^{t_0} e^{-\alpha t}\, \overline{F}(t)dt + 1 - G^*(\alpha)] - \alpha G^*(\alpha)\int_0^{t_0} e^{-\alpha t}[\overline{F}(t) - \overline{F}(t+x)]dt = 1 - G^*(\alpha).$$

(ii) If $H(\infty) \leq \theta_2$, then the optimum pm time is $t_0^* = \infty$, i.e., no pm is made.

Theorem 4. Suppose that the failure rate $H(t)$ is continuous, monotonically increasing, and $H(\infty) > \theta_2$. Then, there exists a finite and unique $\overline{t}_0$ which satisfies $H(t_0) = \theta_2$ and $t_0^* < \overline{t}_0$.

Finally, we introduce the following costs: If c_1 is the cost incurred for each repaired unit, c_2 for each preventively maintained unit, and c_3 for the system failure during the interval $[T, T+x]$, then the expected cost per unit of time in the steady-state is

(10) $$C(t_0) \equiv c_1M_1 + c_2M_2 + c_3[1 - R(x;t_0)],$$

where M_j $(j = 1, 2)$ represents the expected number of repaired units (preventively maintained units) per unit of time, respectively, in the steady-state. Then, we easily have that

(11) $$C(t_0) = \frac{c_1F(t_0) + c_2\overline{F}(t_0) - c_3\int_x^{t_0+x} \overline{F}(t)dt}{\int_0^{t_0} \overline{F}(t)dt + 1/\mu} + c_3.$$

Assume that $C(\infty) < C(0)$, i.e.,

$$c_2(1/\lambda + 1/\mu) > (c_1 - c_3\int_x^\infty \overline{F}(t)dt)(1/\mu) \qquad \text{for } x > 0.$$

For, the expected cost of the system in which no pm is made would be less than that of the system in which pm is always made. This assumption is plausible in actual situations. Let

$$\theta_3 \equiv 1/(1/\lambda + 1/\mu).$$

Then, we have the following optimum pm policy minimizing the expected cost in (11).

Theorem 5. Suppose that $c_1 > c_2$, $c_2/\theta_3 > (c_1 - c_3\int_x^\infty \overline{F}(t)dt)(1/\mu)$, and the failure rate $H(t)$ is continuous and monotonically increasing.

(i) If $[(c_1 - c_2)r(\infty) + c_3H(\infty)] > c_3\theta_1 + c_1\theta_3$, then there exists a finite and unique t_0^* $(0 < t_0^* < \infty)$ which satisfies

$$(12)\qquad [(c_1 - c_2)r(t_0) + c_3H(t_0)][\int_0^{t_0} \overline{F}(t)dt + 1/\mu] - (c_1 - c_2)F(t_0) - c_3\int_0^{t_0} [\overline{F}(t) - \overline{F}(t+x)]dt = c_2 + c_3/\mu.$$

(ii) If $[(c_1 - c_2)r(\infty) + c_3H(\infty)] \leq c_3\theta_1 + c_1\theta_3$, then the optimum pm time is $t_0^* = \infty$, i.e., no pm is made.

Proof: Differentiating $C(t_0)$ with respect to t_0 and putting it zero, we have (12). Let

$$q_2(t_0) \equiv [(c_1 - c_2)r(t_0) + c_3H(t_0)][\int_0^{t_0} \overline{F}(t)dt + 1/\mu] - (c_1 - c_2)F(t_0) - c_3\int_0^{t_0} [\overline{F}(t) - \overline{F}(t+x)]dt.$$

Then, $q_2(t_0)$ is monotonically increasing if $H(t_0)$ is monotonically increasing, since $r(t_0)$ and $H(t_0)$ have the same property. Further,

$$q_2(0) = [(c_1 - c_2)r(0) + c_3F(x)](1/\mu),$$

$$q_2(\infty) = [(c_1 - c_2)r(\infty) + c_3H(\infty)](1/\lambda + 1/\mu) - c_1 + c_2 - c_3\int_0^x \overline{F}(t)dt,$$

and note that $q_2(0) < c_2 + c_3/\mu$ from the assumption of $c_2/\theta_3 > (c_1 - c_3$

$\cdot\int_x^\infty \bar{F}(t)dt/\mu)$. Thus, in a similar way of proving Theorem 1, we can obtain Theorem 5.

Further, we can obtain the upper limit of the optimum pm time t_0^*, although we omit here.

3. Optimum PM Policy for The System Which Operates at Discrete Times

Consider a repairable system which operates at discrete times (see [6]). The time to system failure is discrete, e.g., a few cycles to failure, and the repair is done by the number of cycles. A typical example of such a model is airplane tires. In other cases, the life times are sometimes not recorded at the exact instant of any failure, but done per day, per month, per year, and so on. In this section, we convert from the continuous model to a discrete model.

We assume that the probability of system failure at cycle j (j = 1, 2, ...) is f(j) and its mean cycle is $1/\lambda \equiv \sum_{j=1}^{\infty} jf(j)$, and the probability that repair of the system is completed at cycle j (j = 1, 2, ...) is g(j) and its mean cycle is $1/\mu \equiv \sum_{j=1}^{\infty} jg(j)$.

We suppose that the system should operate at cycles N, N+1, ..., N+n-1 (n, N = 1, 2, ...), and obtain such a probability. We define that the interval reliability R(n,N) for the discrete model is the probability that the system will be operating during [N, N+n-1], if the system begins to operate at time 0. Note that R(1,N) is the probability that the system is operating at cycle N and R(n,1) is the probability that the system has not failed during cycle [1,n].

Further, consider the pm of the above discrete model. That is, if the system operates for n_0 cycles without failure, we stop its operation for pm. We assume that the probability that pm of the system is completed at cycle j is also the same as the probability g(j) of the repair completion. Then, the limiting interval reliability is

$$R(n;n_0) = \frac{\sum_{k=1}^{n_0} \sum_{j=k+n}^{\infty} f(j)}{\sum_{k=1}^{n_0} \sum_{j=k}^{\infty} f(j) + 1/\mu}, \tag{13}$$

which corresponds to (3) in the continuous model.

Let

$$H(k) \equiv \frac{\sum_{j=k}^{\infty} f(j) - \sum_{j=k+n}^{\infty} f(j)}{\sum_{j=k}^{\infty} f(j)} \qquad (k = 1, 2, \ldots),$$

$$\theta_4 \equiv \frac{\sum_{k=1}^{n} \sum_{j=k}^{\infty} f(j) + 1/\mu}{1/\lambda + 1/\mu} .$$

Then, Theorems 1 and 2 are rewritten as the following theorems:

Theorem 6. Suppose that the failure rate $H(k)$ is monotonically increasing.

(i) If $H(\infty) > \theta_4$, then there exists a finite and unique n_0^* $(1 \leq n_0^* < \infty)$ which satisfies

(14) $$L(n_0^*) \geq 1/\mu \text{ and } L(n_0^* - 1) < 1/\mu,$$

where

(15) $$L(k) \equiv \begin{cases} H(k+1)\left[\sum_{i=1}^{k} \sum_{j=i}^{\infty} f(j) + 1/\mu\right] - \sum_{i=1}^{k} \sum_{j=i}^{i+n-1} f(j) & (k = 1,2,\ldots), \\ 0 & (k = 0). \end{cases}$$

(ii) If $H(\infty) \leq \theta_4$, then the optimum pm time is $n_0^* = \infty$, i.e., no pm is made.

Theorem 7. Suppose that the failure rate $H(k)$ is monotonically increasing and $H(\infty) > \theta_4$. Then, there exists a finite and minimum $\bar{n}_0$ which satisfies $H(\bar{n}_0+1) \geq \theta_4$ and $n_0^* \leq \bar{n}_0$.

The above theorems are easily proved since $L(k)$ in (15) is monotonically increasing if $H(k)$ is monotonically increasing. Further, note that $H(k)$ is monotonically increasing if and only if the failure rate $r(k)$ $(\equiv f(k)/\sum_{j=k}^{\infty} f(j))$ is monotonically increasing, which is defined as the failure rate of a discrete distribution in [2].

6. Concluding Remarks

We have discussed the optimum pm policies which maximize the limiting interval reliability and the interval reliability when T is exponentially distributed, and which minimize the expected cost under suitable conditions. It has been shown in the theorems that the optimum pm time t_0^* can be given by unique and finite solution of the equations. It is of great interest that the failure rate H(t) plays a similar role as that of the failure rate r(t) in the age replacement problem. We have also discussed the optimum policy for discrete case.

Another quantity of some interest for the time interval [T, T+x] is the interval availability $R_A(T, T+x)$, which is defined as the expected fraction of the interval [T, T+x] that the system is operating ([2]);

$$R_A(T,T+x) = (1/x)\int_T^{T+x} P[X(t) = 1]dt,$$

where P[X(t) = 1] is the probability that the system is operating at time t. Thus, from the definition of the interval reliability, we have that

$$R_A(T,T+x) = (1/x)\int_T^{T+x} R(0,t)dt. \tag{16}$$

Thus, the interval availability is easily deduced from the interval reliability.

Finally, we give examples of optimum pm policies maximizing the limiting interval reliability. First, suppose that the failure time distribution is a gamma distribution with order 2, i.e., $dF(t) = \beta^2 te^{-\beta t}\, dt$. Then,

$$H(t) = 1 - e^{-\beta x} - \frac{\beta x}{1 + \beta t} e^{-\beta x},$$

$$\theta_1 = \frac{(2/\beta)(1 - e^{-\beta x}) - xe^{-\beta x} + 1/\mu}{2/\beta + 1/\mu}.$$

The failure rate H(t) is monotonically increasing with H(0) = F(x) and $H(\infty) = 1 - e^{-\beta x}$. From (i) of Theorem 1, if $x \leq 1/\mu$ then we should make no pm. Otherwise, the optimum time t_0^* is

$$\beta t_0^* (x - 1/\mu) - x(1 - e^{-\beta t_0^*}) = (1 + \beta x)/\mu,$$

and

$$R(x;t_0^*) = \frac{1 + \beta(t_0^* + x)}{1 + \beta t_0^*} e^{-\beta x}$$

From Theorem 2, we have that

$$\beta t_0^* < \frac{x + (\beta x + 1)/\mu}{x - 1/\mu} .$$

Next, suppose that the failure time distribution is a negative binomial with a shape parameter 2, i.e., $f(j) = jp^2q^{j-1}$ $(j = 1, 2, \ldots)$, where $q \equiv 1 - p$. Then, we have that

$$H(k) = 1 - q^n - \frac{npq^n}{kp + q} ,$$

$$\theta_4 = \frac{(1 - q^n)/\lambda - nq^n + 1/\mu}{1/\lambda + 1/\mu} .$$

The failure rate $H(k)$ is monotonically increasing with $H(\infty) = 1 - q^n$. From Theorem 6, if $n \leq 1/\mu$, we should make no pm. If $n > 1/\mu$, we should make pm at n_0^* cycle which is the minimum such that

$$\frac{n[n_0p - q(1 - q^{n_0})]}{p(n_0 + n) + 1} > \frac{1}{\mu} .$$

Further, from the inequality $n_0^* \leq \bar{n}_0$,

$$n_0^* \leq \frac{n}{n - 1/\mu} \left(\frac{1 + q}{p} + \frac{1}{\mu}\right) - \frac{1}{p} .$$

For example, if $p = 0.2$ and $1/\mu = 1$, then we have $n_0^* = 15, 10, 8, 7$ for $n = 2, 3, 4, 5$, respectively. In particular, when $n = 5$, $\bar{n}_0 = 8$, $R(5,7) = 0.473$, and $R(5,\infty) = 0.458$.

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