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ALGORITHMS FOR SOLVING THE INDEPENDENT-FLOW PROBLEMS

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Abstract Given a capacitated network with the entrance-vertex set V_1 and the exit-vertex set V_2 on which polymatroids are defined, an independent flow is a flow in the network such that a vector corresponding to the supplies in V_1 and a vector corresponding to the demands in V_2 are, respectively, independent vectors of the polymatroids on V_1 and V_2. The independent-flow problems considered in the present paper are the following two: (1) to find a maximum independent flow; and (2) to find an optimal independent flow, i.e., a maximum independent flow of the minimum cost when a cost is given to each arc. We present several theorems which algorithmically characterize optimal independent flows and we propose algorithms for solving the independent-flow problems based on the theorems.

1. Introduction

Given a capacitated network with the entrance-vertex set V_1 and the exit-vertex set V_2 on which polymatroids are, respectively, defined, an independent flow is a flow in the network such that a vector corresponding to the supplies in V_1 and a vector corresponding to the demands in V_2 are, respectively, independent vectors of the polymatroids on V_1 and V_2. We shall consider the independent-flow problems: (1) to find a maximum independent flow; and (2) to find an optimal independent flow, i.e., a maximum independent flow of the minimum cost when a cost is given to each arc.

We first examine some fundamental properties, of a polymatroid, concerning the transformation of independent vectors of a polymatroid. Next, we define an auxiliary network associated with an independent flow and then, by using auxiliary networks, we provide several theorems which algorithmically characterize optimal independent flows (cf. a non-algorithmic characterization of maximum independent flows made by C. J. H. McDiarmid [10]). Based on these theorems, we propose algorithms for finding a maximum independent flow and an optimal independent flow.

The independent-flow problems include as special cases those recently treated in [4] ∿ [7], [9] and [11]. As the ordinary network-flow algorithms have played a significant role in solving many combinatorial problems (cf. [3] and [8]), the present paper will contribute toward solving combinatorial problems related to matroids and/or polymatroids from the point of view of flows in networks.

2. Definition of Polymatroid

Let E be a nonempty finite set and $(H,\leq)$ be a totally ordered additive group with a total order relation $\leq$. We can take as $(H,\leq)$, for example, the additive group of real numbers, rational numbers or integers with the ordinary order relation. We occasionally write $a \geq b$ if and only if $b \leq a$ $(a,b \in H)$. Also, we define a relation $<$ as

$$a < b \quad \text{if and only if} \quad a \leq b \quad \text{and} \quad a \neq b$$

and, similarly, a relation $>$.

Define H_+ as the set of all nonnegative elements of H, i.e.,

(2.1) $$H_+ = \{e \mid e \in H,\ 0 \leq e\}.$$

We denote by H^E (resp. H_+^E) the set of all functions from E into H (resp. H_+). Elements of H^E are expressed by x, y, z etc., and for each $\mathrm{x}(\in H^E)$ and each e $(\in E)$ we denote by $x(e)$ the image of e with respect to x. We shall regard functions in H^E (resp. H_+^E) as H-(resp. H_+-)valued vectors with coordinates indexed by E.

For x $(\in H^E)$ and A $(\subseteq E)$, let us define

(2.2) $$x(A) = \sum_{e \in A} x(e)$$

and

(2.3) $$|\mathrm{x}| = x(E).$$

A function f from 2^E, the set of all subsets of E, into H_+ is called a *β-function* [2] from 2^E into H_+ if it satisfies

(2.4) $$f(\emptyset) = 0,$$

(2.5) $$f(A) \leq f(B) \qquad (A \subseteq B \subseteq E),$$

(2.6) $$f(A \cup B) + f(A \cap B) \leq f(A) + f(B) \qquad (A,\ B \subseteq E).$$

The (2.6) means that f is a *submodular function*.

We define a *polymatroid* P as a pair (E,ρ), where ρ is a β-function from 2^E into H_+, and we also denote the polymatroid by $\mathrm{P}(E,\rho)$. E is called the *ground set* and ρ the *ground-set rank function* of the polymatroid

$\mathrm{P}(E,\rho)$. A vector x in H_+^E is called an *independent vector* of $\mathrm{P}(E,\rho)$ if x satisfies

(2.7) $\quad x(A) \leqq \rho(A) \qquad (A \subseteq E)$.

(For polymatroids, also see [2], [10], [12] and [13].)

3. Fundamental Properties of a Polymatroid

Let $\hat{P}$ be the set of all independent vectors of a polymatroid $\mathrm{P}(E,\rho)$. Define the *saturation function* sat, from $\hat{P}$ into 2^E, with respect to $\mathrm{P}(E,\rho)$ as follows.

"For each x ($\varepsilon\ \hat{P}$) and u ($\varepsilon\ E$), $u\ \varepsilon$ sat(x) if and only if for any d (> 0) the vector y ($\varepsilon\ H_+^E$) defined by

(3.1) $$y(u) = x(u) + d,$$ $$y(v) = x(v) \qquad (v\ \varepsilon\ E - \{u\})$$

does not belong to $\hat{P}$, i.e., y is not an independent vector."

Informally, sat(x) is the "saturated" subset of ground set E with respect to x in $\hat{P}$. Note that the saturation function is a generalization of the closure function of a matroid. We can easily see from (2.7) that, for x ($\varepsilon\ \hat{P}$) and u ($\varepsilon\ E$), u belongs to sat(x) if and only if

(3.2) $\quad x(A) = \rho(A)$

for some A ($\subseteq E$) such that $u\ \varepsilon\ A$.

For x ($\varepsilon\ \hat{P}$) and u ($\varepsilon\ E$ − sat(x)), denote by $c^+(x,u)$ the maximum value of d (> 0) for which the vector y defined by (3.1) belongs to $\hat{P}$. $c^+(x,u)$ is given in terms of the ground-set rank function ρ by

(3.3) $\quad c^+(x,u) = \min\{\rho(A) - x(A) \mid A \subseteq E, u \varepsilon A\} \ (> 0)$.

For x ($\varepsilon\ \hat{P}$) and u ($\varepsilon\ E$) such that $u\ \varepsilon$ sat(x), let us define dep(x,u) ($\subseteq E$) as follows.

" $v\ \varepsilon$ dep(x,u) if and only if

(i) $v = u$

or

(ii) $v\ \varepsilon$ sat(x) − $\{u\}$ and there exists d (> 0) such that the vector y defined by

(3.4) $$y(u) = x(u) + d, \qquad y(v) = x(v) - d,$$ $$y(w) = x(w) \qquad (w\ \varepsilon\ E - \{u,v\})$$

belongs to $\hat{P}$."

Define dep(x,u) = $\emptyset$ if $u \notin$ sat(x). The function dep from $\hat{P} \times E$ into 2^E is called the *dependence function* with respect to $\mathrm{P}(E,\rho)$. When $v\ \varepsilon$ dep(x,u) − $\{u\}$, we denote by $\tilde{c}(x,u,v)$ the maximum value of d (> 0) for which the vector y defined by (3.4) belongs to $\hat{P}$. We can easily show that $\tilde{c}(x,u,v)$ is given in terms of ρ by

(3.5) $\quad \tilde{c}(x,u,v) = \min\{\rho(A) - x(A) \mid A \subseteq E, u \in A, v \notin A\} \ (\leq x(v)).$

Now, we show several lemmas which are fundamental in the theory of polymatroid and will be used in the subsequent sections.

Lemma 1. If, for x $(\in \hat{P})$ and A, B $(\subseteq E)$,

(3.6) $\quad x(A) = \rho(A), \qquad x(B) = \rho(B),$

then we also have

(3.7) $\quad x(A \cup B) = \rho(A \cup B), \qquad x(A \cap B) = \rho(A \cap B).$

Proof: The lemma follows from the submodularity of ρ, i.e.,

(3.8) $\quad \rho(A) - x(A) + \rho(B) - x(B)$

$\geq \rho(A \cup B) - x(A \cup B) + \rho(A \cap B) - x(A \cap B) \geq 0.$

Q.E.D.

Lemma 2. Suppose $x \in \hat{P}$ and define

(3.9) $\quad \mathcal{A}_1 = \{A \mid A \subseteq E, x(A) = \rho(A)\}.$

Then, we have

(3.10) $\quad \mathrm{sat}(x) \in \mathcal{A}_1, \qquad \mathrm{sat}(x) \supseteq A \qquad (A \in \mathcal{A}_1).$

Proof: The lemma follows from the definition of the saturation function and Lemma 1 (also see (3.2)). Q.E.D.

Lemma 3. Suppose $x \in \hat{P}$ and $u \in \mathrm{sat}(x)$. Define

(3.11) $\quad \mathcal{A}_2 = \{A \mid A \subseteq E, u \in A, x(A) = \rho(A)\}.$

Then, we have

(3.12) $\quad \mathrm{dep}(x,u) \in \mathcal{A}_2, \qquad \mathrm{dep}(x,u) \subseteq A \qquad (A \in \mathcal{A}_2).$

Proof: The lemma follows from the definition of the dependence function and Lemma 1. Q.E.D.

It should be noted that from Lemmas 2 and 3 there holds

$$\mathrm{dep}(x,u) \subseteq \mathrm{sat}(x).$$

It should be also noted that $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) in Lemma 2 (resp. Lemma 3) forms a distributive lattice with respect to set inclusion and that $\mathrm{sat}(x)$ (resp. $\mathrm{dep}(x,u)$) is the maximum element (resp. the minimum element) of $\mathcal{A}_1$ (resp. $\mathcal{A}_2$).

Lemma 4. Suppose $x \in \hat{P}$, $u \in \mathrm{sat}(x)$ and $v \in \mathrm{dep}(x,u) - \{u\}$. For an arbitrary d satisfying

(3.13) $\quad 0 < d \leq \tilde{c}(x,u,v),$

let y be a vector (in R_+^E) defined by

$$(3.14)\quad \begin{aligned} &y(u) = x(u) + d, \qquad y(v) = x(v) - d, \\ &y(w) = x(w) \quad (w \in E - \{u,v\}). \end{aligned}$$

Then, we have

$$(3.15)\quad y \in \hat{P},$$

$$(3.16)\quad \text{sat}(y) = \text{sat}(x).$$

Proof: The fact that $y \in \hat{P}$ follows from (3.13), (3.14) and the definition of $\tilde{c}(x,u,v)$.

Moreover, since for $B_0 \equiv \text{sat}(x)$

$$y(B_0) = x(B_0) = \rho(B_0)$$

and since for any $A\ (\supseteq \text{sat}(x))$

$$y(A) = x(A),$$

(3.16) follows from Lemma 2. Q.E.D.

Lemma 5. Under the assumption of Lemma 4, there holds

$$(3.17)\quad c^+(y,w) = c^+(x,w) \qquad (w \in E - \text{sat}(x)).$$

Proof: By Lemma 2 and the definition of x and y, we have for $B_0 \equiv \text{sat}(x)$

$$(3.18)\quad y(B_0) = x(B_0) = \rho(B_0),$$

$$(3.19)\quad y(A) = x(A) \qquad (A \supseteq B_0).$$

Moreover, for any independent vector z, if $z(B) = \rho(B)$ for some $B\ (\subseteq E)$, then

$$(3.20)\quad \rho(A) - z(A) \geqq \rho(A \cup B) - z(A \cup B) \qquad (A \subseteq E).$$

(See (3.8).) Therefore, from (3.18) $\sim$ (3.20) and (3.3) we get (3.17).

Q.E.D.

Lemma 6. Suppose $x \in \hat{P}$ and let u_1, u_2 and v_2 be three distinct elements of E such that

$$(3.21)\quad \begin{aligned} &u_i \in \text{sat}(x) \quad (i = 1, 2), \\ &v_2 \in \text{dep}(x,u_2), \qquad v_2 \notin \text{dep}(x,u_1). \end{aligned}$$

For an arbitrary d satisfying

$$(3.22)\quad 0 < d \leqq \tilde{c}(x,u_2,v_2),$$

let y be a vector (in $\hat{P}$) defined by

$$(3.23)\quad \begin{aligned} &y(u_2) = x(u_2) + d, \qquad y(v_2) = x(v_2) - d, \\ &y(w) = x(w) \quad (w \in E - \{u_2,v_2\}). \end{aligned}$$

Then, we have

$$(3.24)\quad u_1 \in \text{sat}(y),$$

$$(3.25)\quad \text{dep}(y,u_1) = \text{dep}(x,u_1).$$

Proof: The relation (3.24) follows from (3.21) and Lemma 4.

Furthermore, by the assumption and Lemmas 1 and 3,

(3.26) $u_2, v_2 \notin \mathrm{dep}(x,u_1)$.

Set

(3.27) $C_0 = \mathrm{dep}(x,u_1)$.

From (3.26), (3.27) and Lemma 3, we have

(3.28) $C_0 \in \mathcal{A}_2^y \equiv \{A \mid A \subseteq E, u_1 \in A, y(A)=\rho(A)\}$,

(3.29) $y(A) = x(A) \quad (A \subseteq C_0)$.

From (3.28) and (3.29), the set C_0 must be the minimum element of $\mathcal{A}_2^y$. We thus have (3.25). Q.E.D.

Lemma 7. Suppose $x \in \hat{P}$ and let u_1, u_2, v_1 and v_2 be four distinct elements of E satisfying (3.21) and (3.30):

(3.30) $v_1 \in \mathrm{dep}(x,u_1)$.

Also, let y be a vector given by (3.23) for d satisfying (3.22). Then, we have

(3.31) $\tilde{c}(y,u_1,v_1) = \tilde{c}(x,u_1,v_1)$.

Proof: For any independent vector z, if $z(B) = \rho(B)$ for some B ($\subseteq E$), then there holds

(3.32) $\rho(A) - z(A) \geqq \rho(A \cap B) - z(A \cap B) \quad (A \subseteq E)$.

(See (3.8).) Set $C_0 = \mathrm{dep}(x,u_1)$. Then, from (3.26) and Lemma 3,

(3.33) $y(C_0) = x(C_0) = \rho(C_0)$.

Furthermore, from (3.26) we have

(3.34) $y(A) = x(A) \quad (A \subseteq C_0)$.

Lemma 7 follows from (3.32) ∿ (3.34) and (3.5). Q.E.D.

Lemma 8. Suppose $x \in \hat{P}$ and let u_i, v_i $(i = 1, 2, \cdots, q)$ be $2q$ distinct elements of E such that

$$u_i \in \mathrm{sat}(x), \quad v_i \in \mathrm{dep}(x,u_i) \quad (i = 1, 2, \cdots, q),$$

$$v_j \notin \mathrm{dep}(x,u_i) \quad (1 \leqq i < j \leqq q).$$

For arbitrary d_i $(i = 1, 2, \cdots, q)$ satisfying

(3.36) $0 < d_i \leqq \tilde{c}(x,u_i,v_i) \quad (i = 1, 2, \cdots, q)$,

let y be a vector (in R_+^E) defined by

$$\text{(3.37)} \quad \left.\begin{array}{l} y(u_i) = x(u_i) + d_i \\ y(v_i) = x(v_i) - d_i \end{array}\right\} \quad i = 1, 2, \cdots, q,$$
$$y(w) = x(w) \quad (w \in E - \{u_1,u_2,\cdots,u_q,v_1,v_2,\cdots,v_q\}).$$

Then, we have

(3.38) $y \in \hat{P}$,

(3.39) $\mathrm{sat}(y) = \mathrm{sat}(x)$,

(3.40) $c^+(y,w) = c^+(x,w) \quad (w \in E - \mathrm{sat}(x))$.

Proof: Lemma 8 can be easily shown by induction on q using Lemmas 4 ∿ 7. Q.E.D.

Lemma 8 will play an important role in developing algorithms for solving the independent-flow problems. It should be noted that (3.38) ∿ (3.40) hold if the assumption of Lemma 8 is valid by appropriately numbering u_i's and v_i's.

Lemma 9. Suppose $x, y \in \hat{P}$ and $\mathrm{sat}(x) = E$. Consider a bipartite graph $G^*(X,Y;C^*)$ with end-vertex sets

(3.41) $X = E, \quad Y = E' \equiv \{e' | e \in E\}$

and an arc set

(3.42) $C^* = \{(u,v') | u,v \in E,\ u \in \mathrm{dep}(x,v)\} \subseteq E \times E'$.

Here, the prime denotes a copy. The capacities of the arcs in C^* are assumed to be infinity. Then, there exists a flow g $(\in H_+^{C^*})$ in G^* such that

(3.43) $$g(\delta^+ u) \leqq x(u) \quad (u \in X),$$
$$g(\delta^- v') = y(v) \quad (v' \in Y),$$

where $\delta^+ u = \{(u,v') | v' \in Y, (u,v') \in C^*\}$ and $\delta^- v' = \{(u,v') | u \in X, (u,v') \in C^*\}$.

Proof: For any subset B of E, let A be given by

(3.44) $A = \{u | u \in X, v' \in B', (u,v') \in C^*\} \quad (\subseteq X)$.

It follows from (3.42), (3.44) and Lemmas 1 and 3 that

(3.45) $x(A) = \rho(A)$,

(3.46) $B \subseteq A$.

Therefore, we see from (3.45) and (3.46) that

(3.47) $y(B) \leqq y(A) \leqq \rho(A) = x(A)$.

Consequently, from (3.44) and (3.47), the existence of a flow g in G^* satisfying (3.43) is shown by the supply-demand theorem for bipartite networks [1, p.84]. Q.E.D.

4. The Independent-Flow Problems

Now, we shall formulate the independent-flow problems. Let $G(V,A^*;V_1,V_2)$ be a finite directed graph with a vertex set V, an arc set A^* and two distinguished vertex subsets V_1 and V_2 of V, where incidence functions ∂^+ and ∂^- from A^* into V are defined. If $\partial^+ a = u$ and $\partial^- a = v$ for a $(\in A^*)$ and u, v $(\in V)$, then u and v are, respectively, called the *initial vertex* and the *terminal vertex* of a. For simplifying the presentation, we assume that $V_1 \cap V_2 = \emptyset$ and that for each u, v (ε V) there is at most one arc with its initial vertex u and the terminal vertex v.

Suppose that for each i (= 1, 2) a polymatroid $\mathrm{P}_i(V_i, \rho_i)$ is defined on the vertex set V_i and that to each arc a (ε A^*) a *capacity* $c(a)$ (ε $H_+ - \{0\}$) and a *cost* $\gamma(a)$ (ε H) are given. Let N be a network represented by an ordered quintuple:

(4.1) $\quad \mathrm{N} = (G(V,A^*;V_1,V_2),\mathrm{P}_1,\mathrm{P}_2,\mathrm{c},\gamma),$

where c (resp. γ) is the vector in $H_+^{A^*}$ (resp. H^{A^*}) with its components $c(a)$ ($a\varepsilon A^*$) (resp. $\gamma(a)$ ($a\varepsilon A^*$)).

We call an ordered triple $(\mathrm{s},\mathrm{f},\mathrm{t})$ an *independent flow* in N if

(4.2) $\quad \mathrm{s}$ is an independent vector of $\mathrm{P}_1(V_1,\rho_1)$,

(4.3) $\quad \mathrm{t}$ is an independent vector of $\mathrm{P}_2(V_2,\rho_2)$,

(4.4) $\quad 0 \leqq f(a) \leqq c(a) \qquad (a \;\varepsilon\; A^*),$

(4.5) $\quad s(v) + f(\delta^- v) = f(\delta^+ v) \qquad (v \;\varepsilon\; V_1),$

(4.6) $\quad f(\delta^- v) = f(\delta^+ v) + t(v) \qquad (v \;\varepsilon\; V_2),$

(4.7) $\quad f(\delta^- v) = f(\delta^+ v) \qquad (v \;\varepsilon\; V - V_1 \cup V_2),$

where, for v (ε V), δ^+ and δ^- are defined by

(4.8)
$$\delta^+ v = \{a \mid \partial^+ a = v, a\varepsilon A^*\},$$
$$\delta^- v = \{a \mid \partial^- a = v, a\varepsilon A^*\}.$$

$|\mathrm{s}|$ (= $|\mathrm{t}|$) is the *flow value* of $(\mathrm{s},\mathrm{f},\mathrm{t})$. A *maximum independent flow* is an independent flow of the maximum flow-value. The *cost* $C(\mathrm{f})$ *of an independent flow* $(\mathrm{s},\mathrm{f},\mathrm{t})$ is defined by

(4.9) $\quad C(\mathrm{f}) = \sum_{a\varepsilon A^*} f(a)\gamma(a).$

Here, we assume that H is a totally-ordered commutative ring so that the multiplication in (4.9) is defined. A maximum independent flow of the minimum cost is called an *optimal independent flow*.

We shall consider the following two problems (called the *independent-flow problems*):

(I) to find a maximum independent flow in N (regardless of the cost);

(II) to find an optimal independent flow in N.

We assume that, considering $\gamma(a)$ ($a\varepsilon A^*$) as the length of a, there is no directed cycle of negative length in $G(V,A^*;V_1,V_2)$. Also, we shall occasionally express an independent flow $(\mathrm{s},\mathrm{f},\mathrm{t})$ in N by f alone.

5. Auxiliary Network Associated with an Independent Flow

Given an independent flow (s,f,t) in $N = (G(V,A^*;V_1,V_2),P_1,P_2,c,\gamma)$ we define the *auxiliary network* $\overline{N}_f = (\overline{G}_f(\overline{V},\overline{A}),\overline{c},\overline{\gamma})$ *associated with* (s,f,t) as follows. The $\overline{G}_f(\overline{V},\overline{A})$ is a directed graph, called the *auxiliary graph associated with* (s,f,t), with a vertex set $\overline{V}$ and an arc set $\overline{A}$, where

(5.1) $\quad \overline{V} = V \cup \{s^*,t^*\}$

and $\overline{A}$ is composed of the following six disjoint parts:

(5.2) $\quad A_0 = \{a \mid a \in A^*, f(a) < c(a)\}$,

(5.3) $\quad B^* = \{a^* \mid a \in A^*, 0 < f(a), \partial^+ a^* = \partial^- a, \partial^- a^* = \partial^+ a\}$,

(5.4) $\quad A_1 = \{(u,v) \mid v \in \mathrm{sat}_1(s), u \in \mathrm{dep}_1(s,v) - \{v\}\}$,

(5.5) $\quad A_2 = \{(u,v) \mid u \in \mathrm{sat}_2(t), v \in \mathrm{dep}_2(t,u) - \{u\}\}$,

(5.6) $\quad S_1 = \{(s^*,v) \mid v \in V_1 - \mathrm{sat}_1(s)\} \cup \{(v,s^*) \mid s(v) > 0\}$,

(5.7) $\quad S_2 = \{(v,t^*) \mid v \in V_2 - \mathrm{sat}_2(t)\} \cup \{(t^*,v) \mid t(v) > 0\}$.

Here, for each i (= 1, 2), sat_i and dep_i are, respectively, the saturation function and the dependence function defined with respect to the polymatroid $P_i(V_i,\rho_i)$. Furthermore, $\overline{c}$ is a vector $(\in R_+^{\overline{A}})$ given by

$$
\begin{aligned}
(5.8) \quad \overline{c}(a) &= c(a) - f(a) && \text{if } a \in A_0, \\
&= f(\hat{a}) && \text{if } a \in B^* \text{ and } \hat{a} \text{ is the arc in } A^* \text{ such that } \partial^+ a = \partial^- \hat{a} \text{ and } \partial^- a = \partial^+ \hat{a}, \\
&= \tilde{c}_1(s,v,u) && \text{if } a = (u,v) \in A_1, \\
&= \tilde{c}_2(t,u,v) && \text{if } a = (u,v) \in A_2, \\
&= c_1^+(s,v) && \text{if } a = (s^*,v) \in S_1, \\
&= s(v) && \text{if } a = (v,s^*) \in S_1 \\
&= c_2^+(t,v) && \text{if } a = (v,t^*) \in S_2, \\
&= t(v) && \text{if } a = (t^*,v) \in S_2.
\end{aligned}
$$

Here, for each i (= 1, 2), $\tilde{c}_i$ and c_i^+ are, respectively, $\tilde{c}$ and c^+ defined with respect to the polymatroid $P_i(V_i,\rho_i)$. Also, $\overline{\gamma}$ is a vector (in $R^{\overline{A}}$) given by

$$
\begin{aligned}
(5.9) \quad \overline{\gamma}(a) &= \gamma(a) && \text{if } a \in A_0, \\
&= -\gamma(\hat{a}) && \text{if } a \in B^* \text{ and } \hat{a} \text{ is the arc in } A^* \text{ such that } \partial^+ a = \partial^- \hat{a} \text{ and } \partial^- a = \partial^+ \hat{a}, \\
&= 0 && \text{otherwise.}
\end{aligned}
$$

For each arc a $(\in \overline{A})$, $\overline{c}(a)$ is considered as the capacity of a and $\overline{\gamma}(a)$ as the length of a in $\overline{N}_f$.

6. Algorithmic Characterization of Optimal Independent Flows

We shall provide theorems which algorithmically characterize optimal independent flows and give a basis for obtaining algorithms for finding an optimal independent flow and a maximum independent flow. We shall only outline the proofs of the theorems since the theorems can be proven in a manner similar to those in [4] and [5].

Theorem 1. Let (s,f,t) be an independent flow in N. The independent flow (s,f,t) is of the minimum cost among all independent flows, in N, having flow values equal to $|s|$ $(= |t|)$ if and only if there exists no negative directed cycle in the auxiliary graph $\overline{G}_f$ associated with (s,f,t). (Here, a negative directed cycle is a directed cycle of negative length.)

Proof: *The "only if" part*: Suppose there exists at least one negative directed cycle in $\overline{G}_f$. Then, change the flow f along a negative directed cycle of the fewest arcs in $\overline{G}_f$. The resultant flow is an independent flow due to Lemma 8 (the end-vertices of the arcs, in A_1 (or A_2) of $\overline{G}_f$, lying on the negative directed cycle satisfy the assumption of Lemma 8 because the negative directed cycle is composed of the fewest arcs) and has a less cost than that of f. This contadicts the assumption.

The "if" part: Let $\overline{f}$ be an arbitrary independent flow, in N, having the flow value equal to $|s|$. Construct a circulation in $\overline{N}_f$ which corresponds to the difference $\overline{f} - f$ as follows. Let g be a vector (in $R_+^{\overline{A}}$) given by

$$\begin{aligned} g(a) &= \overline{f}(a) - f(a) && \text{if } a \in A_0 \text{ and } \overline{f}(a) > f(a), \\ &= f(\hat{a}) - \overline{f}(\hat{a}) && \text{if } a \in B^* \text{ and } \hat{a} \text{ is an arc in } A^* \text{ such} \\ & && \quad \text{that } \partial^+ a = \partial^- \hat{a},\ \partial^- a = \partial^+ \hat{a} \text{ and } f(\hat{a}) > \overline{f}(\hat{a}), \\ &= 0 && \text{otherwise.} \end{aligned}$$

Then, by appropriately assigning a flow to each arc of $S_1 \cup A_1 \cup S_2 \cup A_2$ and by adding it to g, we get a circulation in $\overline{N}_f$. (Here, we neglect the capacities in $\overline{N}_f$. Assigning a flow to each arc of A_1 and A_2 in an appropriate manner is slightly involved but the existence of such a way of assignment can be shown by employing Lemma 9.) Since the difference $\overline{f} - f$ is thus realized as a circulation in $\overline{N}_f$ and since there exists no negative directed cycle in $\overline{G}_f$, the cost of $\overline{f}$ must be greater than or equal to that of f. Q.E.D.

Theorem 2. Suppose that (s,f,t) is an independent flow, in N, of the minimum cost among all independent flows having flow values equal to $|s|$ and that there exists at least one directed path from s^* to t^* in the auxiliary graph $\overline{G}_{\mathrm{f}}$ associated with (s,f,t). Let P^* be a shortest directed path from s^* to t^* in $\overline{G}_{\mathrm{f}}$. (When there exists more than one such path, let us take as P^* such a shortest path of the fewest arcs.) Moreover, define d by

(6.1) $\quad d = \min\{\overline{c}(a) \mid a$ is an arc lying on $P^*\} > 0$

and a vector $\tilde{\mathrm{f}}$ (in R^{A^*}) by

(6.2) $\quad \tilde{\mathrm{f}}(a) = d$ if a is an arc in A_0 lying on P^*,
$\quad = -d$ if a is an arc in A^* such that $\hat{a}$ is an arc, in B^*, lying on P^* and that $\partial^+ a = \partial^- \hat{a}$, $\partial^- a = \partial^+ \hat{a}$,
$\quad = 0$ otherwise.

Then, $\mathrm{f}^* \equiv \mathrm{f} + \tilde{\mathrm{f}}$ is an independent flow in N having the flow value equal to $|\mathrm{s}| + d$ $(> |\mathrm{s}|)$.

Proof: Because of the assumption and Theorem 1, a shortest directed path P^*, from s^* to t^* in $\overline{G}_{\mathrm{f}}$, of the fewest arcs exists. That the flow f^* is an independent flow in N can be shown by the use of Lemma 8. Q.E.D.

Here, it may be noted that, for f and f* in Theorem 2,

(6.3) $\quad C(\mathrm{f}^*) - C(\mathrm{f}) = d\cdot$(the length of P^* in $\overline{G}_{\mathrm{f}}$).

Corollary. Suppose that (s,f,t) is an independent flow in N and that there exists at least one directed path from s^* to t^* in $\overline{G}_{\mathrm{f}}$. Let P^* be a directed path, from s^* to t^* in $\overline{G}_{\mathrm{f}}$, of the fewest arcs and define a vector $\tilde{\mathrm{f}}$ (in R^{A^*}) by (6.1) and (6.2). Then, $\mathrm{f} + \tilde{\mathrm{f}}$ is an independent flow having the flow value equal to $|\mathrm{s}| + d$ $(> |\mathrm{s}|)$ in N.

Proof: Suppose the cost vector γ of N is a zero vector. The corollary then follows from Theorem 2. Q.E.D.

Theorem 3. f^* $(\equiv \mathrm{f} + \tilde{\mathrm{f}})$ of Theorem 2 is an independent flow of the minimum cost among all independent flows, in N, having flow values equal to $|\mathrm{s}| + d$.

Proof: Let $(\overline{\mathrm{s}}, \overline{\mathrm{f}}, \overline{\mathrm{t}})$ be an arbitrary independent flow, in N, having the flow value equal to $|\mathrm{s}| + d$. Similarly as in the proof of the "if" part of Theorem 1, we can construct a flow, from s^* to t^* in $\overline{\mathrm{N}}_{\mathrm{f}}$, which corresponds to the difference $\overline{\mathrm{f}} - \mathrm{f}$ and has the flow value equal to d. Decompose the constructed flow into flows f_i $(i \in I)$ along directed paths P_i from s^* to t^* (whose flow values sum up to d) and circulations g_j $(j \in J)$ along directed cycles Q_j. Note that the lengths of paths P_i $(i \in I)$ are less than or equal to that of P^* and that by the assumption the lengths of cycles Q_j $(j \in J)$ are nonnegative. Therefore, the cost of $\overline{\mathrm{f}}$ must be greater than or equal to that of f^* because of (6.3). Q.E.D.

Theorem 4. Suppose that (s,f,t) is an independent flow in N. If there is no directed path from s^* to t^* in the auxiliary graph $\overline{G}_f$ associated with (s,f,t), then (s,f,t) is a maximum independent flow in N.

Proof: If there exists an independent flow $(\overline{s},\overline{f},\overline{t})$, in N, having the flow value greater than $|s|$, then as in the proof of Theorem 3 we can construct a flow, from s^* to t^* in $\overline{N}_f$, having the flow value equal to $|\overline{s}| - |s|$ (> 0). Consequently, there must exist a directed path from s^* to t^* in $\overline{G}_f$, which contradicts the assumption. Q.E.D.

For every independent flow $(\overline{s},\overline{f},\overline{t})$ in N and every vertex subset U $(\subseteq V)$ there holds

(6.4) $$|\overline{s}| \leqq \rho_1(V_1-U) + c(A^*\cap(U\times(V-U))) + \rho_2(V_2\cap U).$$

On the other hand, when the assumption of Theorem 4 holds, let U^* be the set of vertices (in V) which are reachable from s^* along a directed path in $\overline{G}_f$. Then, by the definition of the auxiliary network we can easily show that (s,f,t) of Theorem 4 satisfies

(6.5) $$|s| = \rho_1(V_1-U^*) + c(A^*\cap(U^*\times(V-U^*))) + \rho_2(V_2\cap U^*).$$

Consequently, from (6.4) and (6.5) we have

(6.6) $$\max\{|s| \mid (s,f,t) \text{ is an independent flow in } N\} = \min\{\rho_1(V_1-U)+c(A^*\cap(U\times(V-U)))+\rho_2(V_2\cap U) \mid U\subseteq V\}.$$

7. Algorithms

Based on the theorems and the corollary in the preceeding section, we can propose algorithms for finding an optimal independent flow (and a maximum independent flow) in N. The theorems and the corollary may not, however, be sufficient to guarantee the finite termination of the algorithms described below. We thus assume that $\rho_i(A)$ $(A\subseteq V_i,\ i = 1, 2)$ and $c(a)$ $(a \in A^*)$ are integral multiples of a common element of H. We also assume that auxiliary networks can be efficiently constructed.

(I) *Primal-dual algorithm for finding an optimal independent flow*

1° Set $f = 0$.

2° Construct the auxiliary network $\overline{N}_f = (\overline{G}_f(\overline{V},\overline{A}),\overline{c},\overline{\gamma})$ associated with the independent flow f. If there is no directed path from s^* to t^* in $\overline{G}_f$, then the algorithm terminates and f is a solution; or else go to 3°.

3° Find a shortest directed path P^*, from s^* to t^*, of the fewest arcs in $\overline{G}_f$. Let $\tilde{f}$ be a vector (in H^{A^*}) given by (6.1) and (6.2).

Then set

$$f \leftarrow f + \overset{\gamma}{f}$$

and go back to 2°.

(I') *Algorithm for finding a maximum independent flow*

Algorithm (I) described above is employed as an algorithm for finding a maximum independent flow in N by assuming that the cost vector γ is a zero vector. In this case, we should take as P^* in 3° a path, from s^* to t^*, of the fewest arcs in $\overline{G}_f$. Also, the initial independent flow f in 1° can be arbitrarily chosen instead of f = 0 as far as $f(a)$ $(a \in A^*)$, $\rho_i(A)$ $(A \subseteq V_i,\ i = 1, 2)$ and $c(a)$ $(a \in A^*)$ are commensurable.

(II) *Primal algorithm for finding an optimal independent flow*

1° Find a maximum independent flow f in N (using Algorithm (I') described above).

2° Construct the auxiliary network $\overline{N}_f$ associated with the independent flow f. If there is no negative directed cycle in $\overline{G}_f$, then the algorithm terminates and f is a solution; or else go to 3°.

3° Find a negative directed cycle Q^* of the fewest arcs in $\overline{G}_f$. Then, change the flow f along the cycle Q^* by as great a flow value as possible (similarly as described in Step 3° of Algorithm (I)). Set the resultant flow as f again and go back to 2°.

The validity of Algorithms (I), (II) and (I') is clear because of the theorems, the corollary and their proofs in the preceeding section. The algorithms terminate in a finite number of steps due to the integrality of the data $\rho_i(A)$ $(A \subseteq V_i,\ i = 1, 2)$ and $c(a)$ $(a \in A^*)$. Since in the course of carrying out the algorithms we get only integral independent flows, we have the following.

Theorem 5. If $\rho_i(A)$ $(A \subseteq V_i,\ i = 1, 2)$ and $c(a)$ $(a \in A^*)$ are integral multiples of a common element (say, e_0) of H, then there is an optimal (or maximum) independent flow f^o in N with $f^o(a)$ $(a \in A^*)$ being integral multiples of e_0. In fact, such an integral optimal (or maximum) independent flow can be found by using Algorithm (I) or (II) (or (I')).

It should be noted that, if the integrality of the data is not assumed, Algorithms (I), (II) and (I') may not terminate in a finite number of steps but that, when we truncate the calculation, the flow then obtained by Algorithm (I) is an independent flow of the minimum cost among all independent flows of the same flow-value, while the flow then obtained by Algorithm (II) is a maximum independent flow having an improved cost less than the initial one (though for Algorithm (II) a maximum independent flow may not be easily found).

The author has not been able to succeed in determining whether Algorithms (I), (II) and (I') terminate in a finite number of steps for network N with general polymatroids and capacities.

Finally, it should be noted that with a slight modification the argument through the present paper is also valid for nonnegative submodular functions instead of β-functions.

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