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A GENERALIZED CHANCE CONSTRAINT PROGRAMMING PROBLEM

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ABSTRACT

This paper considers a generalized chance constraint programming problem having a controllable probability level α with which the chance constraint should be satisfied. Several properties of this problem are derived and, based on these properties, an algorithm is also proposed.

1. Introduction

Many types of chance constrained programming problem have been considered [1-7, 9, 10] since Charnes and Cooper [1] introduced chance constraints into mathematical programming problems. Especially, S. Kataoka [6] proposed an important problem called P model which dealt with randomness of coefficients in the objective function and gave an algorithm for an optimal solution giving the highest objective value to a chance constraint which should be satisfied with a fixed probability level α.

Though the larger α is favorable since constraints are to be satisfied with high probability, it may make the objective value smaller. Hence, we consider α to be a decision variable and optimize a linear function of original variable x and this decision variable α. That is, this paper generalizes Kataoka's idea to the case with controllable probability level α. Besides this generalization a different algorithm from his technique is proposed. It is based on a parametric approach.

In Section 2, problem P and its deterministic equivalent problem P' are formulated. Section 3 treats subproblem P^q and auxiliary problems P_R^q used to solve P^q. Several useful properties of P^q and P_R^q are also derived. Based on these properties, Section 4 proposes a solution algorithm for P^q. In Section 5, some theorems useful to reduce the computational pains are derived. In Section 6, an algorithm for deterministic equivalent problem P' is given. To illustrate our method, an example is also given in Section 7. Finally, Section 8 summarizes our results and suggests further developements.

2. Problem Formulation

We consider the following generalized chance constraint programming problem P.

$$\underline{\mathrm{P}}: \quad \text{Minimize} \quad f - \lambda\alpha$$

$$\text{subjecto to Prob}\,\{c'x \leqq f\} \geqq \alpha$$

$$(2.1) \qquad x \in S$$

$$1 \geqq \alpha > \frac{1}{2}$$

where $S = \{x |\ Ax \geqq b,\ x \geqq 0\}$, c, x are n- vectors; b is an m- vector; A is an $m \times n$ matrix; λ is a positive scalar†; f, α are scalars. c is a random variable vector with multivariate normal distribution function $N(\bar{c}, V)$, where $\bar{c}$ is a mean vector and V is a variance covariance matrix. V is assumed to be positive definite. We assume that $\min\{\vec{c}\ x |\ x \in S\}$ exists, i.e., finite.

Let $F(q)$ denote the distribution function of the standard normal distribution, $N(0,1)$. Then problem P can be transformed into the following deterministic problem P'††. (For detail, see Appendix 1.)

$$\underline{\mathrm{P'}}: \quad \text{minimize} \quad g(x,q) \triangleq \vec{c}\,x + q\sqrt{x'Vx} - \lambda F(q)$$

$$(2.2) \qquad \text{subject to} \quad x \in S$$

$$q > 0$$

where $q = F^{-1}(\alpha)$.

Problem P' has a nonlinear objective function of x and q. In the next Section 3, several useful properties to solve P' are derived in order to overcome the difficulty arising from this nonlinearity.

3. Subproblems P^q

For each $q > 0$, the following subproblem P^q of P' is defined.

$$(3.1)\quad \underline{P^q}\ :\ \text{Minimize } \bar{c}'x + q\sqrt{x'Vx} - \lambda F(q) \quad \text{subject to } x \in S$$

Let $x(q)$ and $g(q)$ denote an optimal solution of P^q and the optimal value, respectively. Then the following properties hold. (See Appendix 2 for proofs.)

Property (i) $x(q)$ is unique.

(ii) $\bar{c}'x(q) + q\sqrt{x(q)'Vx(q)}$ is a monotone increasing function of q.

(iii) $\sqrt{x(q)'Vx(q)}$ is a nonincreasing function of q.

(iv) $\bar{c}'x(q)$ is a nondecreasing function of $q > 0$.

In order to solve P^q, the following auxiliary problem P^q_R of P^q is considered for each $R > 0$.

$$(3.2)\quad \underline{P^q_R}\ :\ \text{Minimize } \frac{R}{q}\bar{c}'x + \frac{1}{2}x'Vx \quad \text{subject to } x \in S$$

Since P^q_R is a convex quadratic programming problem, the optimal solution of P^q_R , denoted by $x^q(R)$, may be found by a known method. Especially, Wolfe's long form [11] may be suitable because it solves parametric quadratic programming problem P^q_R for all $R > 0$.

By the convex programming theory, $x(q)$ is the x-part of the solution of the following Kuhn-Tucker condition:

† λ is a given constant for taking the effects of α into the objective function.

†† Though $x \neq 0$ is needed in the course of transformation from P to P', P' can include $x=0$ and P' substituted $x=0$ corresponds to P substituted $x=0$.

$$v + A'u - \frac{qVx}{\sqrt{x'Vx}} = \bar{c}$$

$$Ax - e = b$$

$$u'e = 0. \quad v'x = 0.$$

$$u, v, e, x \geqq 0$$

where v: n × 1 vector, u: m × 1 vector (Lagrange multiplier), e: m × 1 vector. While $x^q(R)$ is the x-part of the solution of the following Kuhn-Tucker condition :

$$v + A'u - Vx = \bar{c}\,\frac{R}{q}$$

$$Ax - e = b$$

$$u'e = 0. \quad v'x = 0$$

$$u, v, e, x \geqq 0$$

Therefore it is clear that if $x^q(R)$ satisfies

$$\sqrt{x^q(R)'\,Vx^q(R)} = R \quad ,$$

then it is also an optimal solution of p^q. Giving the following definition

$$K^q(R) \triangleq \sqrt{x^q(R)'Vx^q(R)} - R \quad , \tag{3.3}$$

then the above condition becomes

$$K^q(R) = 0 \quad ,$$

that is, $x^q(R)$ giving $K^q(R) = 0$ may be sought. Above Kuhn-Tucker condition is a linear complementary equations with parametrized right hand side with respect to R/q. A solution of this equation is determined by a certain basis B, and $x^q(R)$ (the x-part of the solution) is therefore linearly dependent on R on the closed interval on which the same basis B maintains the nonnegativity of the solution. In other words, $x^q(R)$ may be represented on the interval as follows:

$$x^q(R) = \frac{R}{q}\,r_B + t_B \qquad (L_B \leqq \frac{R}{q} \leqq U_B) \tag{3.4}$$

where r_B, t_B are constant n × 1 vectors determined by the basis B and L_B, U_B are the lower and upper bound specifying the interval, respectively. If $\hat{R}$ with $k^q(\hat{R}) = 0$ is found, $x(q)$ can be obtained by

$$x(q) = r_B\left(\frac{\hat{R}}{q}\right) + t_B$$

(Hereafter $\hat{R}$ for q is denoted by $R(q)$).

The condition (3.3) is equivalent to the existence of a root of the following quadratic equation Q in the interval $[\,qL_B, qU_B\,]$.

$$\underline{Q}\ :\qquad (q^2 - r_B{}'Vr_B)R^2 - 2r_B{}'Vt_BqR - q^2t_B{}'Vt_B = 0$$

Roots β_1, β_2 are given as follows:

$$\beta_1 = q\cdot\frac{r_B{}'Vt_B - \sqrt{D}}{q^2 - r_B{}'Vr_B} \qquad \beta_2 = q\cdot\frac{r_B{}'Vt_B + \sqrt{D}}{q^2 - r_B{}'Vr_B}$$

where $D = (r_B{}'Vt_B)^2 + t_B{}'Vt_B(q^2 - r_B{}'Vr_B)$.

Note that $q^2 = r_B{}'Vr_B$ cannot happen for $q > 0$ as shown in Appendix 3.

Even if neither β_1 nor β_2 belong to $[qL_B, qU_B]$, some informations can be deduced as shown in the next Theorem 1. If either β_1 or β_2 and not both†, belongs to the interval, we substitute β_1/q or β_2/q into the inequalities

$$L_B \leqq \frac{\beta_1}{q} \leqq U_B\ , \qquad L_B \leqq \frac{\beta_2}{q} \leqq U_B$$

respectively, and solve the inequality with respect to q (with fixed r_B, t_B) and determine the set of q (denoted by $I(B)$), in which same basis B is still optimal basis.

Theorem 1. $K^q(R)$ has a unique zero point $R(q)$ in $R > 0$. Moreover,

(a) $K^q(R) > 0 \iff 0 < R < R(q)$

(b) $K^q(R) < 0 \iff R > R(q)$

Proof : $K^q(R)$ is clearly a continuous function of R.††

By property (i) and the fact that $x^q(R)$ with $K^q(R)=0$ becomes $x(q)$, $K^q(R)$ must have unique zero point $R(q)$. Therefore $R(q)$ separates interval $R > 0$ into two intervals, so-called "positive interval" ($K^q(R) > 0$) and "negative interval" ($K^q(R) < 0$). For sufficient large $R=\bar{R}$, $x^q(R)$ is equal to $x \in S$ giving $\min \bar{c}'x$. By the assumption of finiteness of this x, $\sqrt{x^q(R)'Vx^q(R)}$ becomes a finite fixed value for $R \geqq \bar{R}$ Therefore $K^q(R) < 0$ for $R > R(q)$ is derived and $K^q(R) > 0$ for $R < R(q)$ is also derived. □

† It is easy to show that $\beta_1 < 0$ in case of $q^2 - r_B{}'Vr_B > 0$, β_1, $\beta_2 < 0$ in case of $q^2 - r_B{}'Vr_B < 0$ and $r_B{}'Vt_B > 0$ even if β_1, β_2 are real roots.

†† Since $K^q(R) = \sqrt{x^q(R)'Vt_B(R)} - R$, continuity of $K^q(R)$ is implied by the continuity of $x^q(R)$. Continuity of $x^q(R)$ is well known according to the theory of the parametric quadratic programming.

For the optimal solution $x^q(R)$, the following properties hold. (Proofs are quite similar for properties (iii)(iv) and omitted.)

Property (v) : $x^q(R)'Vx^q(R)$ is a nondecreasing function of R.

(vi) : $\bar{c}'x^q(R)$ is a nonincreasing function of R.

4. Algorithm 1 for Subproblem P^q

In this section, an algorithm for P^q(Algorithm 1) is proposed. In the algorithm, R_ℓ (R_u) is used to denote current lower bound (upper bound) of $R(q)$, respectively. First R_ℓ is set to 0 and R_u to a sufficiently large number M. Algorithm 1 starts with choosing an arbitrary positive number R_1. For each R, algorithm 1 calculates B, r_B, t_B, L_B and U_B. If neither β_1 nor β_2 belongs to $[qL_B, qU_B]$, then either R_ℓ or R_u is updated by using Theorem 1.

$(R_u - R_\ell)$ is at least halved after updating except the first iteration. Next, R is set to $(R_\ell + R_u)/2$ and same procedure is repeated. (Refer to Figure 1_a - Figure 1_e in the proof of Theorem 2.)

[Algorithm 1]

Step 0 : Set $R \leftarrow R_1$ (R_1 is an aribitrary positive number†), $R_u \leftarrow M$(M is a sufficiently large number††) and $R_\ell \leftarrow 0$. Go to Step 1.

Step 1 : Solve P^q_R and find B, r_B, t_B, L_B and U_B. Go to Step 2.

Step 2 : If $k^q(qL_B) < 0$, set $R_u \leftarrow qL_B$ and $R \leftarrow (R_u+R_\ell)/2$, and return to Step 1; if $K^q(qL_B)=0$, set $\beta \leftarrow qL_B$ and go to Step 4; if $K^q(qL_B) > 0$, go to Step 3.

Step 3: If $K^q(qU_B) < 0$, solve Q-equation and set $\beta \leftarrow \beta_2$ or β_1 (according to $L_B \leqq \beta_2 \leqq U_B$ or $L_B \leqq \beta_1 \leqq U_B$) and go to Step 4; if $K^q(qU_B)=0$, set $\beta \leftarrow qU_B$ and go to Step 4; if $K^q(qU_B) > 0$, set $R_\ell \leftarrow qU_B$ and $R \leftarrow (R_u+R_\ell)/2$ and return to Step 1.

Step 4: Set $x(q) \leftarrow \frac{\beta}{q} r_B + t_B$ and terminate.

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† If an optimal solution of certain subproblem $P^{\hat{q}}$ for $\hat{q} > q$ (or $\hat{q} < q$) is known, then $R_1 \geqq \sqrt{x(\hat{q})'Vx(\hat{q})}$ ($R_1 \leqq \sqrt{x(\hat{q})'Vx(\hat{q})}$) should be taken as an R_1.

†† M can be set to $\sqrt{x'Vx}$ using $x \in S$ minimizing $\bar{c}'x$.

Remark : (1) Several methods to choose the next R are possible, and efficiency of Algorithm 1 seems to greatly depend on the choice method.

(2) If $K^q(qL_B) < 0$, $K^q(qU_B) < 0$ necessarily holds by Theorem 1. Thus the test for $K^q(qU_B)$ is not needed. In case $K^q(qU_B) > 0$, $K^q(qL_B) > 0$ holds and the test for $K^q(qL_B)$ is also omitted.

(3) $[qL_B, qU_B] \subseteq [R_l, R_u]$ holds except the first L_B, U_B.

Theorem 2. Algorithm 1 terminates after finite iterations and it finds an optimal solution $x(q)$ of P^q upon termination.

Proof : **(Finiteness)** After each calculation of Step 1, five cases (a), (b), (c), (d), (e) (as illustrated in Figure 1a, 1b, 1c, 1d, and 1e below) are possible. (Note that the case of both $K^q(qL_B) < 0$ and $K^q(qU_B) > 0$ never occurs as pointed out in the above Remark.)

In case (d), (e), it is clear that

$$\beta = qL_B(qU_B)$$

and

$$x(q) = L_B r_B + t_B \qquad (x(q) = U_B r_B + t_B)$$

holds respectively. In case (c), either β_1 or β_2 (but not both) must belong to $[qL_B, qU_B]$ according to the continuity and "the mean value theorem" with respect to $K^q(R)$. In case (c), (d), (e), algorithm 1 jumps to Step 4 and terminates. In case (a), (b), neither β_1 nor β_2 belongs to the interval $[qL_B, qU_B]$ by Theorem 1. First, note that

$$qL_B \leqq (R_\ell + R_u)/2 \leqq qU_B \tag{4.1}$$

holds as is easily known from the updating procedure of R in Step 2 or Step 3.

<u>Case a</u> : R_u is set to qL_B as $K^q(qL_B) < 0$.

<u>Case b</u> : R_ℓ is set to qU_B as $K^q(qU_B) > 0$.

In any cases, it follows from (4.1) that the difference $R_u - R_\ell$ is at least halved except the first execution of Step 2 and Step 3. Therefore, after finite iterations, case(c), case(d) or case(e) occurs since $R(q)$ belongs to a certain interval $[qL_B, qU_B]$ with $qU_B - qL_B > 0$.†

(Validity) Termination condition itself proves validity of Algorithm 1. □

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† Even in the degenerate case, that is, $L_B = U_B$, another base $\bar{B}$ exists such that $L_{\bar{B}} = U_B$ (or $U_{\bar{B}} = L_B$) and $U_{\bar{B}} - L_{\bar{B}} > 0$ according to the theory of the parametric quadratic programming. Therefore, without any loss of generality, $qU_B - qL_B > 0$ can be assumed.

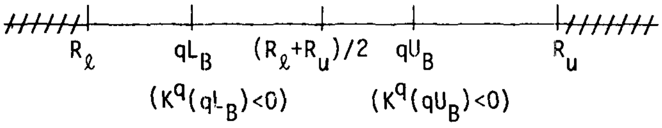

Figure 1.a. Case (a)

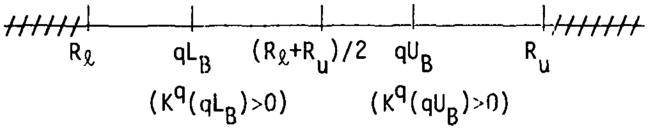

Figure 1.b. Case (b)

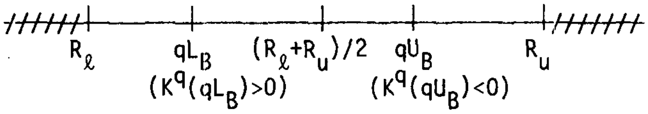

Figure 1.c. Case (c)

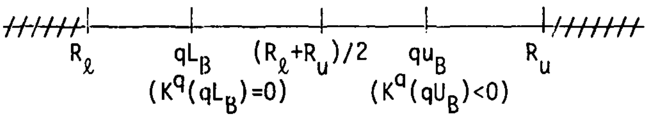

Figure 1.d. Case (d)

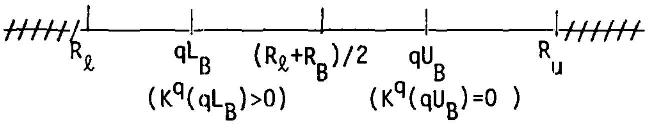

Figure 1.e. Case (e)

5. Properties of P^q

In this section, minimization of $g(x,q)$ defined as (2.2) of Section 2 is discussed. $g(x,q)$ is a convex function with respect to $q \geqq 0$ for fixed x since

$$\frac{\partial^2 g(x,q)}{\partial q^2} = \frac{q}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} q^2\right) \geqq 0$$

for $q > 0$. For each $x \in S$, $h(x)$ is defined as follows:

$$h(x) \triangleq \inf\{g(x,q) \mid q > 0\} \quad . \tag{5.1}$$

Then the optimal solution $q(x)$ giving $h(x)$ is given as follows:

$$q(x) = \begin{cases} \sqrt{\log\left(\dfrac{\lambda^2}{2\pi x' Vx}\right)} & \left(x' Vx < \dfrac{\lambda^2}{2\pi}\right) \\ 0 & \left(x' Vx > \dfrac{2}{2\pi}\right) \end{cases} \tag{5.2}$$

by searching the zero point of

$$\frac{\partial g(x,q)}{\partial q} = \sqrt{x'Vx} - \lambda f(q)$$

because of convexity of $g(x,q)$ with respect to q, where $f(q)$ denotes the probability density function of the standard normal distribution $N(0.1)$.

Theorem 3. The optimal solution (x^*, q^*) of P', if exists, satisfies

$$\lambda f(q^*) = \sqrt{x(q^*)' Vx(q^*)} \quad \text{or} \quad q^* = \log\left(\frac{\lambda^2}{2\pi x(q^*)' Vx(q^*)}\right)$$

proof : Proof is easy and so it is omitted.

Since

$$q^* = \log\left(\frac{\lambda^2}{2\pi x(q^*)' Vx(q^*)}\right) \leqq \log\left(\frac{\lambda^2}{2\pi R^2_{\min}}\right)$$

holds from this necessary condition of (x^*, q^*), if q^* exists, an upper bound of q^* (denoted by q_u) can be given by

$$q_u = \begin{cases} \log\left(\dfrac{\lambda^2}{2\pi R^2_{\min}}\right) & (2\pi R^2_{\min} < \lambda^2) \\ 0 & (\text{otherwise}) \end{cases} \tag{5.3}$$

where $R^2_{\min} \triangleq \min\{x' Vx \mid x \in S\}$.

Here, we define a transformation T from $\Gamma = \{q > 0\}$ to $\Gamma \cup \{0\}$. T plays a principal role in our algorithm given in the next section.

T:

$$T(q) = \begin{cases} \log\left(\dfrac{\lambda^2}{2\pi x(q)' Vx(q)}\right) & \left(x(q)' Vx(q) < \dfrac{\lambda^2}{2\pi}\right) \\ 0 & (\text{otherwise}) \end{cases}$$

Note that $T(q)$ is nondecreasing function of q because of property (iii) and Theorem 3 states a necessary condition of q^* is $q^* = T(q^*)$, that is, q^* is a fixed point of T. Unfortunately, this condition is not necessarily a sufficient condition.

Theorem 4. for $q_1 > 0$ and $q_2 = T(q_1)$,

$q^* \notin [q_2, q_1]$ in case $q_1 > q_2$

$q^* \notin [q_1, q_2]$ in case $q_1 < q_2$ holds.

proof: If $q_1 > q_2$, for any $\hat{q} \in [q_2, q_1]$,

$$T(\hat{q}) - \hat{q} < T(\hat{q}) - q_2 \leqq T(q_1) - q_2 = 0.$$

holds (since $T(q)$ is a nondecreasing function of q).

Therefore $\hat{q}$ does not satisfy the necessary condition of q^*, In case $q_1 < q_2$, the proof may be similarly done. []

Now, next property (vii) shows that $I(B)$ is a continuous interval if not empty.

Property (vii) : $I(B) = \phi$ or $I(B)$ consists of a continuous interval.

proof : Solving either inequality

$$L_B \leqq \frac{r_B' Vt_B - \sqrt{D}}{q^2 - r_B' Vr_B} \leqq U_B$$

(5.4) or

$$L_B \leqq \frac{r_B' Vt_B + \sqrt{D}}{q^2 - r_B' Vr_B} \leqq U_B$$

with respect to q†, results Table 1 which shows property (vii) .

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† $q = \sqrt{r_B\, Vr_B}$ cannot happen as is shown in Appendix 3 if $q > 0$ only considered.

6. Algorithm for P'

First some notations are defined.

S_N : scanned region of q

$q_m^{\dagger}$: $g(q_m) \triangleq \min\{g(q) \mid q \in \overline{I(B)}\}$
where - denotes ordinary "closure operation".

v^* : current best value

$(\tilde{x}, \tilde{q})$: current best solution

q_c : current searched q

B_{q_c} : optimal basis with respect to q_c

q_M : $q_M \triangleq \min\{q \mid q \in I(B_{q_c})\}$ for $I(B_{q_c}) \neq \phi$

The following algorithm 2 starts with $q_c = q_u$, $S_N = (q_u, +\infty)$, $v^* = +\infty$, $\tilde{x} = \phi$ and $\tilde{q} = \phi$. In Step 1, it solves P^q using Algorithm 1 and find $x(q_c)$ and the optimal basis B_{q_c}. $I(B_{q_c})$ is then calculated by solving the inequality (5.4) and $g(q)$ on $I(B_{q_c})$ is determined. Proceeding to Step 2, minimum point $q_m^{\dagger}$ of $g(q)$ is calculated. If $T(q_m) = q_m$, $g(q_m)$ is compared with current best value v^* and v^*, $\tilde{q}$, $\tilde{x}$ is updated in Step 4 if $v^* > g(q_m)$. While in case $T(q_m) \neq q_m$, if $S_N \cup I(B_{q_c}) \supseteq (0, +\infty)$, current $(x(\tilde{q}), \tilde{q}, v^*)$ is optimal and algorithm 2 terminates. Unless $S_N \cup I(B_{q_c}) \supseteq (0, +\infty)$, S_N is updated and augmented. Next q_c is selected as follows; if $T(q_M) < q_M$, q_c is set to $T(q_M)$: otherwise, q_c is set to $q_M - \varepsilon$ where ε is sufficiently small positive number. Then, returing to Step 2 and above procedure is repeated. (see also Figure 2).

[Algorithm 2]

Step 0 : Set $q_c \leftarrow q_u$, $S_N \leftarrow (q_u, +\infty)$, $v^* \leftarrow +\infty$, $\tilde{x} \leftarrow \phi$ and $\tilde{q} \leftarrow \phi$. Go to Step 1.

Step 1 : Apply Algorithm 1 to problem P^{q_c} and calculate B_{q_c}, $x(q_c)$, $I(B_{q_c})$ and determine $g(q)$ on $I(B_{q_c})$. Go to Step 2.

Step 2 : Calculate q_m. If $T(q_m)^{\dagger\dagger} = q_m$, go to Step 4; otherwise, go to Step 3.

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† If q_m is not unique, then we take the smallest one among these q_m.

†† Even if $q_m \notin I(B_{q_c})$, $T(q_m)$ can be calculated by continuity of $T(q)$ as $\lim_{q \to q_m+0} T(q)$ and continuity of $T(q)$ is assured by the continuity of $x(q)$.

Step 3: If $S_N \cup I(B_{q_c}) \supseteq (0, +\infty)$, go to Step 5; otherwise, set $S_N \leftarrow S_N \cup I(B_{q_c})$. If $T(q_M)^{\dagger} < q_M$, set $q_c \leftarrow T(q_M)$ and $S_N \leftarrow S_N \cup (T(q_M), q_M)$, and return to Step 1, otherwise, set $q_c \leftarrow q_M - \varepsilon$ (ε is sufficiently small number) and return to Step 2.

Step 4: If $g(q_m) \leftarrow v^*$, set $v^* \leftarrow g(q_m)$, $\tilde{x} \leftarrow x(q_m)$, and $\tilde{q} \leftarrow q_m$, and return to Step 3. Otherwise, return to Step 3 directly.

Step 5: Terminate. Current $(\tilde{x}, \tilde{q})$ is an optimal solution of P' and v^* is the optimal value.

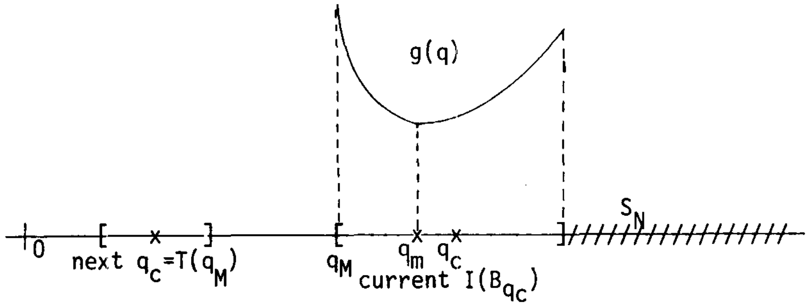

Figure 2. Illustration of computational process of Algorithm 2 at certain q_c.

Theorem 5 Algorithm 2 terminates after finite iterations and upon termination, it finds an optimal solution (x^*, q^*) of P'.

Proof: (Finiteness) It is sufficient to prove $S_N \cup I(B_{q_c}) \supseteq (0, +\infty)$ occurs after finite iterations. Whenever Step 1 is entered from Step 3, S_N is set to $S_N \cup I(B_{q_c})$, that is, augmented by $I(B_{q_c})$. This augmentation can be done only finite times, because $(0, +\infty)$ is covered by a finite number of $I(B)$. Finiteness is assured by the facts that the number of possible basis is finite. Algorithm 2 searches these $I(B)$ with jumping when $T(q_M) < q_M$ from right to left on the positive part of the real line (see Theorem 4).

Therefore, $S_N \cup I(B_{q_c}) \supseteq (0, +\infty)$ must occur after executing updation of S_N at finite times.

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† Even if $q_M \notin I(B_{q_c})$, $T(q_M)$ can be also calculated by same reasons as q_m.

(Validity) Algorithm 2 scans $(0, +\infty)$ and finds all q_m on $I(B)$ except skipped interval $(T(q_M), q_M]$ with assurance that q^* does not exist in the latter interval by Theorem 4. Moreover, it is clear by Theorem 3 that optimal q^* exists among such q_m if exists. Therefore, upon termination, algorithm 2 has scanned all possible q_m. This proves validity of Algorithm 2. $\square$

7. Example

In order to illustrate our algorithm, we consider the following example. (See Figure 3):

$$\bar{C} = \binom{3}{1}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}, \quad b = \binom{8/3}{6}.$$

and $\lambda = 2 \times 10^4$.

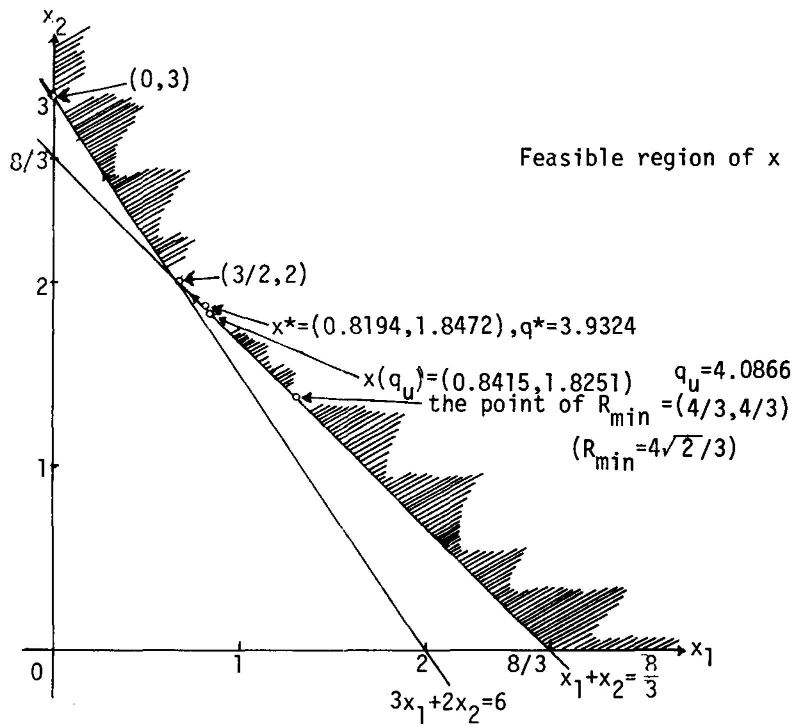

Figure 3. Illustration of computational process of the example in Section 7.

Then P' becomes as follows:

$$\text{P}': \quad \text{minimize } 3x_1 + x_2 + c\sqrt{x_1^2 + x_2^2} - 20000\,F(q)$$
$$\text{subject to} \quad x_1 + x_2 \geqq 8/3$$
$$3x_1 + 2x_2 \geqq 6$$
$$x_1,\ x_2 \geqq 0\ . \quad q > 0\ .$$

Step 0 : $q_c \leftarrow q_u (= 4.0866)$. $S_N \leftarrow (q_u,\ +\infty)$, $v^* \leftarrow +\infty$ $\tilde{x} \leftarrow \phi$ and $\tilde{q} \leftarrow \phi$.
Go to Step 1.

Step 1 : (Algorithm 1)[†]

Step 0 : Set $R \leftarrow R_1\ (=2q_u), R_u \leftarrow M$ and $R_\ell \leftarrow 0$. Go to Step 1.

Step 1 : $u_1 \quad x_1 \quad x_2 \quad e_1$

$$B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 2 & 0 \end{bmatrix} \qquad r_B = \begin{pmatrix} -6/13 \\ 9/13 \end{pmatrix} \qquad r_B = \begin{pmatrix} 18/13 \\ 12/13 \end{pmatrix}$$

$L_B = 14/9$. $U_B = 3$. Go to Step 2.

Step 2 : Since $K^q(q_c L_B) < 0$, set $R_u \leftarrow q_c L_B (= q_u \times \frac{14}{9})$ and $R \leftarrow (R_u + R_\ell)/2 (= 7/9\ q_u)$, and return to Step 1.

$u_1 \quad u_2 \quad x_1 \quad x_2$

$$r_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad r_B = \begin{matrix} 2/3 \\ 2 \end{matrix}$$

$$B = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 2 \end{bmatrix} \qquad L_B = 2/3 \qquad U_B = 14/9$$

Go to Step 2.

[†]Kuhm - Tycker condition in the example is shown in Appendix 4.

Go to Step 2.

Step 2 : $K^q(q_c L_B) < 0$, $R_u \leftarrow q_c L_B (=2/3\ q_u)$

$R \leftarrow 1/3\ q_u$. Return to Step 1.

Step 1 : $u_1 \quad x_1 \quad x_2 \quad e_2$

$$B = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 2 & -1 \end{bmatrix} \qquad r_B = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad t_B = \begin{pmatrix} 4/3 \\ 4/3 \end{pmatrix}$$

$$L_B = 0, \qquad U_B = 2/3 .$$

Go to Step 2.

Step 2 : Since $K^q(q_c L_B) = 4\sqrt{2}/3 > 0$, Go to Step 3.

Step 3 : Since $K^q(q_c U_B) = 2\sqrt{10}/3 - 2/3\ q_u < 0$,

solve Q equation ($r_B' V t_B = 0 \quad r_B' V r_B = 2 \quad t_B' V t_B = 9/32$)

$\{(q_u)^2 - 2\}R^2 - 32/9\ q_c^2 = 0$

and set $\beta \leftarrow \beta_2$ ($=0.4918\ q_u$) because of

$q_c^2 - r_B' V r_B = q_u^2 - 2 > 0$. Go to Step 4.

Step 4 : Terminate.

$$x(q_c)' = \begin{matrix} 0.8415 \\ 1.8251 \end{matrix} \qquad B_{q_c} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 2 & -1 \end{bmatrix}$$

Therefore,

$r_{B_{q_c}}' = (-1, 1)$, $t_{B_{q_c}}' = (4/3, 4/3)$, $L_{B_{q_c}} = 0$, $U_{B_{q_c}} = 2/3$

$I(B_{q_c}) = [-\sqrt{10}\ +\infty)$ and $g(q) = (4\sqrt{2}/3)\cdot\sqrt{q^2 - 2} - 20000\ F(q)$.

Go to Step 2.

Step 2 : $q_m = 3.9324 \quad q_M = \sqrt{10}$. Since $T(q_m) = q_m$, Go to Step 4.

Step 4 : Since $v^*(=\infty) > g(q_m) = -19992 \cdot 2811$, set

$v^* \leftarrow g(q_m)(= -19992.\ 2811)$,

$\tilde{x} \leftarrow x(q_m)(= (0,8194,\ 1.8472)')$ and $\tilde{q} \leftarrow q_m (= 3.9324)$,

Return to Step 3.

Step 3 : Since $S_N \cup I(B_{q_c}) = [\sqrt{10},\ +\infty) \subset (0, +\infty)$

set $S_N \leftarrow S_N \cup I(B_{q_c}) (=[\sqrt{10},\ +\infty))$

As $T(q_M) = 3.884726 > q_M = \sqrt{10} = 3.162277$, set $q \leftarrow q_M - \varepsilon$ and return to Step 1.

Since the serial computational routines are almost same as above, results only are enumerated.

$q_c = \sqrt{10} - \varepsilon$:

$$B_{q_c} = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & x_1 & x_2 \end{array} \\ \begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 2 \end{bmatrix} \end{array} \quad r_{B_{q_c}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad t_{B_{q_c}} = \begin{pmatrix} 2/3 \\ 2 \end{pmatrix},$$

$$L_{B_{q_c}} = 2/3 \qquad U_{B_{q_c}} = 14/9 .$$

$q_M = 3\sqrt{10}/7 < T(q_M) = 3.884726$ and $S_N \leftarrow [3\sqrt{10}/7, +\infty)$.

$q_c = 3\sqrt{10}/7 - \varepsilon$:

$$B_{q_c} = \begin{array}{c} \begin{array}{cccc} u_1 & x_1 & x_2 & e_1 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 2 & 0 \end{bmatrix} \end{array} \quad r_{B_q} = \begin{pmatrix} -6/13 \\ 9/13 \end{pmatrix} \quad t_{B_q} = \begin{pmatrix} 18/13 \\ 12/13 \end{pmatrix}$$

$$L_{B_{q_c}} = 14/9 \qquad U_{B_{q_c}} = 3 .$$

$I(B_{q_c}) = [1, 3\sqrt{10}/7]$, $q_M = 3\sqrt{10}/7 \neq T(q_m)$,

$T(q_M) > q_M = 1$. and $S_N \leftarrow [1, +\infty)$.

$q_c = 1 - \varepsilon$:

$$B_{q_c} = \begin{array}{c} \begin{array}{cccc} v_1 & u_2 & x_2 & e_1 \end{array} \\ \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \end{array} \quad r_{B_{q_c}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad t_{B_{q_c}} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$L_{B_{q_c}} = 3. \qquad U_{B_{q_c}} = +\infty.$$

$I(B_{q_c}) = (0, 1]$, $q_M = q_m = 0$, and $S_N \cup I(B_{q_c}) \supseteq (0, +\infty)$.

Terminate : $x^* = (0.8194, 1.8472)'$ and $q^* = 3.9324$.

8. Conclusion

The most difficult point in our algorithm is to find q_m in Step 2. Since $g(q)$ is a function of q only, we may manage to obtain q_m if the form of $g(q)$ on $I(B_q)$ is known. The second difficult point is a lack of sufficient condition about q^*. Especially in this problem, the lack of useful sufficient condition urges us to search all possible points with $T(q)=q$, among all positive q.

Note that for fixed α, our problem is equivalent to Kataoka's problem [6]. But even in such case, our method may be considered as a different approach.

Although the effect of α on the objective function was taken as $-\lambda\alpha$, there may be other ways to include the effect of α. In such cases, however, the problem may become more complicated and more difficult to solve. Admitting the linearity of the effect of α, in practical situation, the domain of α may be more restricted.

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Appendix 1 Derivation of P'

The chance constraint in (2.1) can be transformed into the following form by simplex subtrction and devision.

(A-1) $$Prob(\ c'x \leqq f) = Prob\left(\frac{c'x-\bar{c}'x}{\sqrt{x'Vx}} \leqq \frac{f-\bar{c}'x}{\sqrt{x'Vx}}\right)$$

Since c is distributed according to $N(\bar{c}, V)$,

$$\frac{c'x-\bar{c}'x}{\sqrt{x'Vx}}$$

can be considered as a normalized random variable with zero mean and unit variance (that is, standard normal distribution). Therefore (A-1) is replaced by

$$f \geqq \bar{c}'x + F^{-1}(\alpha)\sqrt{x'Vx}$$

where F is the distribution function of standard normal distribution, $N(0,1)$. Since minimum of f is attained when the equality holds. setting

$$q = F^{-1}(\alpha) \ ,$$

the objective function becomes as follows :

$$\bar{c}'x + q\sqrt{x'Vx} - \lambda F(q).$$

Appendix 2 Proofs of Properties (i) – (iv)

(i) It is clear from the strict convexity of the objective function and the fact that S is convex set.

(ii) For $q_2 > q_1 > 0$,

$$\bar{c}'x(q_2) + q_2\sqrt{x(q_2)'Vx(q_2)} > \bar{c}'x(q_2) + q_1\sqrt{x(q_2)'Vx(q_2)} \qquad (\ q_2 > q_1)$$
$$\geqq \bar{c}'x(q_1) + q_1\sqrt{x(q_1)'Vx(q_1)} \qquad \text{(optimality of } q_1 \text{ for } P^{q_1}).$$

(iii) For $q_2 > q_1 > 0$, from the optimality of $x(q_1)$, $x(q_2)$,

(A-2) $$\bar{c}'x(q_1) + q_1\sqrt{x(q_1)'Vx(q_1)} \leqq \bar{c}'x(q_2) + q_1\sqrt{x(q_2)'Vx(q_2)}$$

(A-3) $$\bar{c}'x(q_2) + q_2\sqrt{x(q_2)'Vx(q_2)} \leqq \bar{c}'x(q_1) + q_2\sqrt{x(q_1)'Vx(q_1)}$$

holds. By subtracting the right hand side of (A-2) from the left hand side of (A-3) and the left hand side of (A-2) from the right hand side of (A-3), (A-4) $(q_2-q_1)\sqrt{x(q_2)'Vx(q_2)} \leqq (q_2-q_1)\sqrt{x(q_1)'Vx(q_1)}$

holds. The assumption $(q_2-q_1)>0$ implies

$$\sqrt{x(q_1)'Vx(q_1)} \geqq \sqrt{x(q_2)'Vx(q_2)}. \qquad \Box$$

(iv) From Property (iii),

(A-5) $\bar{c}'x(q_1)+q_1\sqrt{x(q_1)'Vx(q_1)} \leqq \bar{c}'x(q_2)+q_1\sqrt{x(q_2)'Vx(q_2)}$

$$\leqq \bar{c}'x(q_2)+q_1\sqrt{x(q_1)'Vx(q_1)}$$

holds. (A-5) implies

$$\bar{c}'x(q_1) \leqq \bar{c}'x(q_2) \qquad \Box$$

Appendix 3 $q \neq \sqrt{r_B'Vr_B}$ for an optimal basis B

Assume $q=\sqrt{r_B'Vr_B}$. Then $\frac{R(q)}{q}=\frac{t_B'Vt_B}{2r_B'Vt_B}$ from Q equation, that is,

$$R(q)=\frac{t_B'Vt_B}{2r_B'Vt_B}\sqrt{r_B'Vr_B}$$

holds. Since

$$x(q)=\frac{R(q)}{q}r_B+t_B \text{ and } K^q(R(q))=0,$$

$$x(q)'Vx(q)-R(q)^2=\frac{R^2}{q^2}r_B'Vr_B+2\frac{R(q)}{q}r_B'Vt_B+t_B'Vt_B-R(q)^2$$

$$=\frac{r_B'Vr_B(t_B'Vt_B)^2}{4(r_B'Vt_B)^2}+t_B'Vt_B+t_B'Vt_B-\frac{r_B'Vr_B(t_B'Vt_B)^2}{4(r_B'Vt_B)^2}$$

$$=2t_B'Vt_B=0, \text{ or } t_B=0. \quad \text{(For } V \text{ is positive definite matrix.)}$$

This implies $\frac{R(q)}{q}=0$, or $R(q)=x(q)'Vx(q)=0$, or $x(q)=0$.

Since again V is positive definite matrix,

$$x(q)=0 \text{ and } t_B=0 \text{ together implies } r_B=0 \text{ or } q=0.$$

We considers only $q>0$ and so $q=\sqrt{r_B'Vr_B}$ cannot happen.

Appendix 4 Kuhn-Tucker condition of Problem P_R^q for the example.

$$v+A'u-Vx=\bar{c}'\frac{R}{q}$$

$$Ax-e=b, \; u'e+v'x=0, \; u, v, e, x \geqq 0, \text{ that is,}$$

$$v_1+u_1+3u_2-x_1=3\frac{R}{q}$$

$$v_2+u_1+2u_2-x_2=\frac{R}{q}$$

$$x_1 + x_2 - e_1 = 8/3$$

$$3x_1 + 2x_2 - e_2 = 6$$

$$x_1 v_1 + x_2 v_2 + e_1 u_1 + e_2 u_2 = 0, \quad x_1,\ x_2,\ v_1,\ v_2,\ e_1,\ e_2,\ u_1,\ u_2 \geqq 0.$$

Appendix 5.

Table 1. I(B).

Case	Subcase	I(B)
$r_B' V t_B \geqq 0$		$\{q \mid A_2 \leqq q \leqq A_1\}$
$r_B' V t_B < 0$	$A_4 \geqq A_5$ & $A_4 > 0$	$\{q \mid q > A_8$ & $A_2 \geqq q \geqq A_1\}$
	$A_7 < 0$, $A_5 > 0$ & $A_5 \leqq A_4$	$\{q \mid q < A_8$ & $A_2 \leqq q \leqq A_1\}$
	$A_7 \geqq 0$, $A_6 < 0$, $A_5 < 0$ & $A_3 \leqq A_2 \leqq A_1$	$\{q \mid q < A_8$ & $A_2 \leqq q \leqq A_1\}$
	$A_7 \geqq 0$, $A_6 < 0$ & $A_2 \leqq \min(A_1, A_3)$	$\{q \mid q < A_8$ & $A_1 \geqq q \geqq A_9\}$
	$A_7 \geqq 0$, $A_6 < 0$ & $A_1 \leqq \min(A_2, A_3)$	$\{q \mid q < A_8$ & $\min(A_2, A_3) \geqq q \geqq A_9\}$
	$A_6 \geqq 0$, $A_4 < 0$, $A_3 \geqq A_2$ & $A_1 \leqq \min(A_2, \sqrt{A_6})$	$\{q \mid q < A_8$ & $A_2 \geqq q \geqq A_1\}$
	$A_6 \geqq 0$ & $\max(\sqrt{A_6}, A_2)$	$\{q \mid q < A_8$ & $A_9 \leqq q \leqq A_1\}$
	$A_4 < 0$, $A_6 \geqq 0$, $A_3 \leqq A_2$ & $A_1 \leqq \sqrt{A_6}$	$\{q \mid q < A_8$ & $A_1 \leqq q \leqq A_3\}$
	$A_6 \geqq 0$ & $\sqrt{A_6} \leqq A_1 \leqq \min(A_2, A_3)$	$\{q \mid q < A_8$ & $A_9 \leqq q \leqq \min(A_2, A_3)\}$

See also next page.

In Table 1,

$$A_1 = \sqrt{r_B'Vr_B + 2r_B'Vt_B / L_B + t_B'Vt_B / L_B^2}$$

$$A_2 = \sqrt{r_B'Vr_B + 2r_B'Vt_B / U_B + t_B'Vt_B / U_B^2}$$

$$A_3 = \sqrt{r_B'Vr_B + r_B'Vt_B / U_B} \quad , \quad A_4 = \sqrt{2r_B'Vt_B / L_B + t_B'Vt_B / L_B^2}$$

$$A_5 = \sqrt{2r_B'Vt_B / U_B + t_B'Vt_B / U_B^2} \quad , \quad A_6 = \sqrt{r_B'Vr_B + r_B'Vt_B / L_B}$$

$$A_7 = \sqrt{r_B'Vr_B + r_B'Vt_B / U_B} \quad , \quad A_8 = \sqrt{r_B'Vr_B}$$

$$A_9 = \sqrt{r_B'Vr_B - (r_B'Vt_B)^2 / t_B'Vt_B} \ .$$

Note that

(i) V is a positive definite matrix implies

$$r_B'Vr_B - (r_B'Vt_B)^2 / t_B'Vt_B \geqq 0, \quad r_B'Vr_B \geqq 0,$$

$$r_B'Vr_B + 2r_B'Vt_B / L_B + t_B'Vt_B / L_B^2 = (r_B + t_B / L_B)'V(r_B + t_B / L_B) \geqq 0$$

and $r_B'Vr_B + 2r_B'Vt_B / U_B + t_B'Vt_B / U_B^2 \geqq 0$, similarly.

(ii) $r_B'Vt_B \geqq 0$ implies $A_8 < A_2 < A_1$.

(iii) $r_B'Vt_B < 0$ implies $A_1, A_2, A_8 \geqq A_9$.

$A_1 \geqq \sqrt{A_6}$ implies $A_5 \geqq \sqrt{A_6}$ in case of $r_B'Vt_B < 0$ and $A_6 > 0$.