Journal of the Operations Research
Society of Japan
Vol.21, No.1, March, 1978

NECESSARY CONDITIONS FOR THE COMPLETION OF PARTIAL LATIN SQUARES*

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(Received May 24, 1977; Revised October 11, 1977)

ABSTRACT

A latin square is an n x n square matrix each of which cells contains a symbol chosen from the set [1, 2, . . . , n]; each symbol occurs exactly once in each row or column of the matrix. A partial latin square is a latin square in which some cells are unoccupied. We consider the problem of obtaining necessary and sufficient conditions for a partial latin square to be completed to a latin square. For this problem A. B. Cruse has recently given a necessary condition associated with triply stochastic matrices. In this paper two sets of necessary condtions are given, one developed from network flow theory and another obtained from matroid theory. It is shown that the network condition is equivalent to Cruse's condition and that the matroid condition is strictly stronger than either of the former.

1. Introduction

In the 18th century a *latin square* (LS) was regarded by Euler as a square matrix with n^2 *cells* of n different elements, each of which occurs exactly once

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* This work is a part of the research the author has done in the school of operations research in Cornell University. More details and related results are referred in [9].

in any row or column of the matrix. The reference [2] contains an exhaustive study of known results and conjectures of the subjects relating to latin squares.

The elements of the latin square, also called *symbols*, are the integers 1, 2, ..., n. The integer n is called the *order* of the latin square. A *partial latin square* (PLS) is an n×n square matrix such that some cells of the matrix are left unoccupied. A (PLS) is said to be *consistent* if each integer from the set { 1, 2, ..., n } appears at most once in any row or column in the occupied cells of the (PLS). In the following sections (PLS) means consistent (PLS). An n×n (PLS) P is said to be *completed* to the n×n (LS) P' if, for all i, j, whenever the cell (i, j) of P is occupied by a symbol k, then this same symbol k also occupies the cell (i, j) of P'. Namely an n×n (PLS) P is said to be completed to the n×n (LS) P' if we can construct an n×n (LS) by putting appropriate symbols in unoccupied cells of the (PLS). In this work we focus on (PLS)'s whose occupied cells are arbitrarily chosen and the question of whether they can be completed to a corresponding (LS); i.e., necessary and sufficient conditions for the *completion of partial latin squares* (CPLS) are investigated.

For the (CPLS) problem A. B. Cruse (1975, see [1]) proposed a necessary condition associated with a characteristic matrix of the (PLS). We briefly describe his condition (CC) in this section.

Given an n×n (PLS) P there is an n×n×n three-dimensional (0,1)-matrix which we call the *characteristic matrix* for P. This matrix $C = (c_{ijk})$ is defined by putting $c_{ijk} = 1$ if the cell (i, j) of P is occupied by symbol k and $c_{ijk} = 0$ otherwise.

An n×n×n matrix $S = (s_{ijk})$ with nonnegative real elements is called *triply stochastic* if it satisfies the equations

$$\sum_{i\in N} s_{ijk} = 1, \text{ for any } j, k \in N$$

$$(1.1) \qquad \sum_{j\in N} s_{ijk} = 1, \text{ for any } i, k \in N$$

$$\sum_{k\in N} s_{ijk} = 1, \text{ for any } i, j \in N.$$

It can be easily seen that triply stochastic matrices, all of whose elements consist of integers, are precisely the characteristic matrices for (LS)'s of order n.

Let I_0, J_0, K_0 be any subsets of the set N. We denote the set of elements of the form (i, j, k), where $i\in I_0$, $j\in J_0$ and $k\in K_0$, by $I_0\times J_0\times K_0$. The element (i, j, k) of the set $I \times J \times K$ is said to be preassigned if the cell (i, j) is occupied by symbol k in the (PLS) P. Let $P(I_0 \times J_0 \times K_0)$ denote the set of preassigned elements in the set $I_0 \times J_0 \times K_0$. For the summation of the elements x_{ijk}'s of the triply stochastic matrix or the characteristic matrix of the (PLS), we denote $\sum_{i \in I_0,\ j \in J_0,\ k \in K_0} x_{ijk}$ by $\sum_{I_0,\ J_0,\ K_0} x_{ijk}$. The following Theorem 1.1, which is THEOREM 1 in [1], is used to derive (CC).

Theorem 1.1. Any triply stochastic matrix $S = (s_{ijk})$ satisfies the equations

$$(1.2) \qquad \sum_{I_0, J_0, K_0} s_{ijk} + \sum_{\overline{I}_0, \overline{J}_0, \overline{K}_0} s_{ijk} = \frac{|I_0| \cdot |J_0| \cdot |K_0| + |\overline{I}_0| \cdot |\overline{J}_0| \cdot |\overline{K}_0|}{n}$$

where I_0, J_0, K_0 are any subsets of the set N. (Here $\overline{I}_0$ and $|I_0|$ denote the complement and the cardinality of the subset I_0 respectively.)

From the above theorem we derive (CC) in the following way. Let P be a (PLS) and P' be an (LS) of order n. If $C = (c_{ijk})$ and $C' = (c'_{ijk})$ denote characteristic matrices for P and P' respectively, then P is completed to P' if and only if $c_{ijk} \leqq c'_{ijk}$ holds for all indices i, j, k $\in$ N. (We write $C \leqq C'$ in this case.) Hence the inequalities in the following theorem, which are (CC) in THEOREM 2 in [1], express necessary conditions for an $n \times n$ (PLS) to be completed to an (LS) of order n.

Theorem 1.2. In order for a given (PLS) P to be completed to an (LS) of order n, it is necessary that the preassignments in the (PLS) P satisfy the inequalities

$$(1.3) \qquad |P(I_0 \times J_0 \times K_0)| + |P(\overline{I}_0 \times \overline{J}_0 \times \overline{K}_0)| \leqq \frac{1}{n} \left(|I_0| \cdot |J_0| \cdot |K_0| + |\overline{I}_0| \cdot |\overline{J}_0| \cdot |\overline{K}_0| \right),$$

for all subsets I_0, J_0, K_0 of the set N.

2. Network Condition (NC)

The (CPLS) problem has very close relations to multi-commodity flow problem, graph coloring problem, time-tabling problem and machine scheduling problems. The details are shown in [9]. In this section we consider the relation of (CPLS) problem to multi-commodity flow problem for later use.

Given a (PLS) P we construct the corresponding network as follows. Let I, J, K be the sets of rows, columns and symbols respectively; hence we have

$I = J = K = N$, where $N = \{1, 2, \ldots, n\}$. The directed network G consists of the node sets K_1, I, J, K_2, where $K_1 = K_2 = K$, and the edge sets U, V, Y as follows.

Nodes

$K_1 = K_2 = K = \{ k \mid k \in N \}$, $I = \{ i \mid i \in N \}$, $J = \{ j \mid j \in N \}$.

Directed edges

$U = \{(k, i) \mid k \in K_1, i \in I$, symbol k does not appear in row i in P$\}$

$Y = \{(i, j) \mid i \in I, j \in J$, cell (i, j) in P is unoccupied$\}$

$V = \{(j, k) \mid j \in J, k \in K_2$, symbol k does not appear in column j in P$\}$.

Capacity functions

Let $w(i, j)$, $c(i, j)$ denote the lower, upper capacities of the edge (i, j). Then for any $(k, i) \in U$, $(i, j) \in Y$ and $(j, k) \in V$, we have

$w(k, i) = w(i, j) = w(j, k) = 0$, and

$c(k, i) = c(i, j) = c(j, k) = 1$.

Example 2.1 We give an example of a (PLS) to illustrate the network G defined above. The example is to be used in Section 4.

P =

1		2	
	2		3
		3	
			1

Figure 2.1 (PLS) P

Then the network G corresponding to the above (PLS) P is illustrated in Figure 2.2 below.

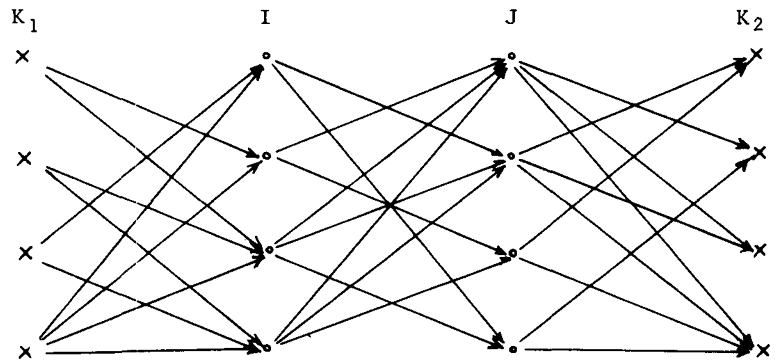

Figure 2.2 Network G

We consider the n-commodity supply-demand type flow problem in the network G. First we make the restriction that the flow of commodity k must be sent from the source $k \in K_1$ to the sink $k \in K_2$. The supply-demand conditions are given as follows. K_1, K_2 are the sets of supply, demand nodes respectively. Let $S(k_1)$ denote the amount of supply of commodity k_1 at the node $k_1 \in K_1$ and $D(k_2)$ denote the amount of demand of commodity k_2 at the node $k_2 \in K_2$. Then the conditions are

$$S(k_1) = \left| \{(k_1, i) \mid (k_1, i) \in U\} \right|, \text{ for any } k_1 \in K_1, \text{ and}$$
$$D(k_2) = \left| \{(j, k_2) \mid (j, k_2) \in V\} \right|, \text{ for any } k_2 \in K_2,$$

where $|E|$ indicates the cardinality of the edge set E. Given a network G corresponding to the (PLS) P suppose there exists an integral feasible flow satisfying the above supply-demand conditions, then $f^k_{ij} = 1$, where f^k_{ij} is the flow value of commodity k on the edge $(i, j) \in Y$, indicates that the symbol k is assigned to the unoccupied cell (i, j) in P. Thus we obtain a completion of the (PLS) P. Conversely, if the (PLS) is completed to an (LS) of order n, obviously there exists an integral feasible flow satisfying the supply-demand conditions. Therefore the existence of an integral feasible n-commodity flow satisfying the above supply-demand conditions in the network G is equivalent to the completion of the corresponding (PLS).

Given a (PLS) P we had the corresponding network G as in Figure 2.2. For a given set $K_0 \subseteq K$, we construct a subnetwork $G(K_0)$ of G consisting of the node sets K^1_0, I, J, K^2_0, where $K^1_0 = K^2_0 = K_0 \subseteq K$, and newly added source s and sink t (see Figure 2.3 below in Example 2.2). The edge sets are given as follows.

Directed edges

$$W(K_0) = \{ (s, k) \mid k \in K^1_0 \}$$
$$U(K_0) = \{ (k, i) \mid k \in K^1_0,\ i \in I,\ (k, i) \in U \}$$
$$Y \quad = \{ (i, j) \mid i \in I,\ j \in J, \text{ cell } (i, j) \text{ is unoccupied} \}$$
$$V(K_0) = \{ (j, k) \mid j \in J,\ k \in K^2_0,\ (j, k) \in V \}$$
$$Z(K_0) = \{ (k, t) \mid k \in K^2_0 \}$$

Capacity functions

For any $(s, k) \in W(K_0)$, $(k, i) \in U(K_0)$, $(i, j) \in Y$, $(j, k) \in V(K_0)$, $(k, t) \in Z(K_0)$, we have

$$w(s, k) = w(k, i) = w(i, j) = w(j, k) = w(k, t) = 0$$

and

$$c(s, k) = c(k, t) = \infty,$$
$$c(k, i) = c(i, j) = c(j, k) = 1.$$

Example 2.2 We use an example of (PLS) given in Figure 2.1, and let $K_0 = \{1, 2\}$ to illustrate the network $G(K_0)$.

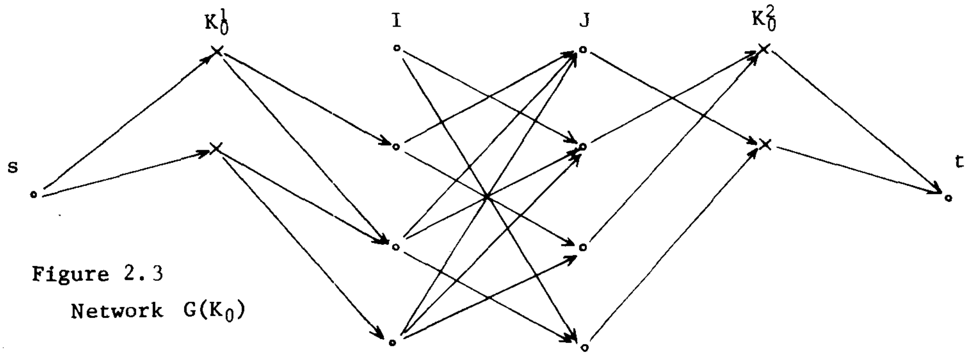

Figure 2.3
Network $G(K_0)$

For any node sets X, Y, we denote the set of all edges that emanate from nodes $x \in X$ to nodes $y \in Y$ by (X, Y), and let c(X, Y) denote the sum of upper capacities on the set of edges in (X, Y). Then our (NC) is given in the following theorem.

Theorem 2.1. In order for a given (PLS) to be completed to an (LS) of order n, it is necessary that the inequalities

$$(2.1) \qquad |(K_0, I_0)| + |(\bar{I}_0, \bar{J}_0)| + |(J_0, K_0)| \geq \sum_{k \in K_0} N_k,$$

for any $I_0 \subseteq I$ and $J_0 \subseteq J$,

should be satisfied in any subnetwork $G(K_0)$, $K_0 \subseteq K$, of G. (Here N_k denotes the number of edges emanating from node $k \in K_0^1$, or going into node $k \in K_0^2$, in $G(K_0)$.)

Proof: If the (PLS) P can be completed to an (LS) of order n, there exists an integral feasible n-commodity flow satisfying the supply-demand conditions. Hence for any subset $K_0 \subseteq K$ the subnetwork consisting of the node sets K_0^1, I, J, K_0^2, where $K_0^1 = K_0^2 = K_0$, and their adjacent edges has an integral feasible $|K_0|$-commodity flow satisfying the supply-demand conditions at the nodes in $K_0^1 \cup K_0^2$. Therefore if we relax the $|K_0|$-commodity flow problem in the subnetwork to a single-commodity flow problem in the subnetwork $G(K_0)$, it is clear that a maximal flow from the source s to the sink t must have value $\sum_{k \in K_0} N_k$. Applying the max-flow min-cut theorem (see THEOREM 5.1 in [7]) to the above network $G(K_0)$, any cut separating the source s and the sink t should have capacity greater than or equal to $\sum_{k \varepsilon K_0} N_k$. Let I_0, J_0, L_1, L_2 be arbitrary subsets of I, J, K_0^1, K_0^2 respectively and $\bar{I}_0$, $\bar{J}_0$, $\bar{L}_1$, $\bar{L}_2$ be their complements with respect to the sets I, J, K_0^1, K_0^2 respectively. We define a subset X and its complement $\bar{X}$ as follows.

$$X = s \cup L_1 \cup \bar{I}_0 \cup J_0 \cup L_2 \text{ and } \bar{X} = \bar{L}_1 \cup I_0 \cup \bar{J}_0 \cup \bar{L}_2 \cup t,$$

where we denote the set {s}, {t} simply by s, t, respectively. Then $(X, \bar{X})$ is a cut separating the source s and the sink t since $s \varepsilon X$ and $t \varepsilon \bar{X}$. In order that we have a finite capacity $\bar{L}_1 = L_2 = \phi$ is necessary since we have $c(s, k)= \infty$, for $(s, k) \varepsilon W(K_0)$, and $c(k, t) = \infty$, for $(k, t) \varepsilon Z(K_0)$. The cut capacity $c(X, \bar{X})$ can be written as

$$(2.2) \qquad c(X, \bar{X}) = c(K_0^1, I_0) + c(\bar{I}_0, \bar{J}_0) + c(J_0, K_0^2), \quad \text{for any } I_0 \subseteq I \text{ and } J_0 \subseteq J,$$

and the max-flow min-cut theorem requires that

$$(2.3) \qquad c(X, \bar{X}) \geqq \sum_{k \varepsilon K_0} N_k,$$

for any node set X such that $s \varepsilon X$ and $t \varepsilon \bar{X}$.

Since we have

$$(2.4) \qquad c(K_0^1, I_0) = \sum_{(k, i) \varepsilon (K_0^1, I_0)} c(k, i) = |(K_0^1, I_0)|,$$

$$(2.5) \qquad c(\bar{I}_0, \bar{J}_0) = \sum_{(i, j) \varepsilon (\bar{I}_0, \bar{J}_0)} c(i, j) = |(\bar{I}_0, \bar{J}_0)|,$$

$$(2.6) \qquad c(J_0, K_0^2) = \sum_{(j, k) \varepsilon (J_0, K_0^2)} c(j, k) = |(J_0, K_0^2)|,$$

substituting (2.4) - (2.6) into (2.2) and using the relation (2.3), we obtain

$$(2.7) \qquad |(K_0^1, I_0)| + |(\bar{I}_0, \bar{J}_0)| + |(J_0, K_0^2)| \geqq \sum_{k \varepsilon K_0} N_k,$$

for any $I_0 \subseteq I$ and $J_0 \subseteq J$.

We have assumed $K_0^1 = K_0^2 = K_0$ and the inequalities (2.7) must hold for any sets $K_0 \subseteq K$. Thus the theorem is proved. □

3. Network Condition (NC) and its Relation to Cruse's Condition (CC)

Now we give a theorem showing the relation between Cruse's condition in (1.3) and network condition in (2.1).

Theorem 3.1. (CC) of (1.3) and (NC) of (2.1) are equivalent.

Proof: Let I_0, J_0, K_0 be any subsets of the set N, and $\bar{I}_0$, $\bar{J}_0$, $\bar{K}_0$ be their respective complements in N. Since we have $I = J = K = N$, these sets I_0, J_0, K_0 satisfy $I_0 \subseteq I$, $J_0 \subseteq J$ and $K_0 \subseteq K$.

From the (NC) of (2.1) we get

$$|(K_0, I_0)| = |K_0|\cdot|I_0| - |P(I_0 \times N \times K_0)|, \tag{3.1}$$

$$|(\bar{I}_0, \bar{J}_0)| = |\bar{I}_0|\cdot|\bar{J}_0| - |P(\bar{I}_0 \times \bar{J}_0 \times N)|, \tag{3.2}$$

$$|(J_0, K_0)| = |J_0|\cdot|K_0| - |P(N \times J_0 \times K_0)|. \tag{3.3}$$

Since we have $N_k = n - |P(\{k\})|$, where $P(\{k\})$ indicates the set $P(N \times N \times \{k\})$,

$$\sum_{k \in K_0} N_k = \sum_{k \in K_0} \{ n - |P(\{k\})| \}. \tag{3.4}$$

Substituting the relations (3.1) - (3.4) into (2.1) shows that

$$|K_0|\cdot|I_0| + |\bar{I}_0|\cdot|\bar{J}_0| + |J_0|\cdot|K_0| - |P(I_0 \times N \times K_0)| - |P(\bar{I}_0 \times \bar{J}_0 \times N)| - |P(N \times J_0 \times K_0)| \geqq \sum_{k \in K_0} \{ n - |P(\{k\})| \}.$$

Hence

$$\begin{aligned} &|K_0|\cdot|I_0| + |\bar{I}_0|\cdot|\bar{J}_0| + |J_0|\cdot|K_0| - n\,|K_0| \\ &\quad \geqq |P(I_0 \times N \times K_0)| + |P(\bar{I}_0 \times \bar{J}_0 \times N)| \\ &\quad + |P(N \times J_0 \times K_0)| - |P(N \times N \times K_0)|. \end{aligned} \tag{3.5}$$

The left side of (3.5) can be rewritten as follows:

$$\begin{aligned} &|K_0|\cdot|I_0| + |\bar{I}_0|\cdot|\bar{J}_0| + |J_0|\cdot|K_0| - (|I_0| + |\bar{I}_0|)\cdot|K_0| \\ &\quad = |K_0|\cdot|J_0| + |\bar{I}_0|\cdot|\bar{J}_0| - |\bar{I}_0|\cdot|K_0|. \end{aligned} \tag{3.6}$$

Furthermore, each term on the right side of (3.5) may be expressed as follows:

$$\begin{aligned} |P(I_0 \times N \times K_0)| &= |P(I_0 \times J_0 \times K_0)| + |P(I_0 \times \bar{J}_0 \times K_0)|, \\ |P(\bar{I}_0 \times \bar{J}_0 \times N)| &= |P(\bar{I}_0 \times \bar{J}_0 \times K_0)| + |P(\bar{I}_0 \times \bar{J}_0 \times \bar{K}_0)|, \\ |P(N \times J_0 \times K_0)| &= |P(I_0 \times J_0 \times K_0)| + |P(\bar{I}_0 \times J_0 \times K_0)|, \\ |P(N \times N \times K_0)| &= |P(I_0 \times J_0 \times K_0)| + |P(I_0 \times \bar{J}_0 \times K_0)| + \\ &\quad |P(\bar{I}_0 \times J_0 \times K_0)| + |P(\bar{I}_0 \times \bar{J}_0 \times K_0)| \end{aligned} \tag{3.7}$$

Therefore substituting (3.6) and (3.7) into (3.5), we obtain

$$\begin{aligned} &|P(I_0 \times J_0 \times K_0)| + |P(\bar{I}_0 \times \bar{J}_0 \times \bar{K}_0)| \\ &\quad \leqq |K_0|\cdot|J_0| + |\bar{I}_0|\cdot|\bar{J}_0| - |\bar{I}_0|\cdot|K_0|. \end{aligned} \tag{3.8}$$

Thus the (NC) of (2.1) is equivalent to requiring that the inequality (3.8) holds for any subsets $I_0 \subseteq I$, $J_0 \subseteq J$ and $K_0 \subseteq K$.

On the other hand (CC) of (1.3) can be rewritten as follows:

$$|P(I_0 \times J_0 \times K_0)| + |P(\bar{I}_0 \times \bar{J}_0 \times \bar{K}_0)|$$

$$(3.9)\qquad \leq \frac{1}{n}(|I_0|\cdot|J_0|\cdot|K_0| + |\overline{I}_0|\cdot|J_0|\cdot|K_0| + |\overline{I}_0|\cdot|\overline{J}_0|\cdot|\overline{K}_0| + |\overline{I}_0|\cdot|\overline{J}_0|\cdot|K_0| - |\overline{I}_0|\cdot|J_0|\cdot|K_0| - |\overline{I}_0|\cdot|\overline{J}_0|\cdot|K_0|).$$

Using the fact that $|I_0| + |\overline{I}_0| = |J_0| + |\overline{J}_0| = |K_0| + |\overline{K}_0| = n$ in (3.9), we get the relation (3.8). Thus (CC) of (1.3) is equivalent to requiring that the inequality (3.8) holds for any subsets $I_0 \subseteq I$, $J_0 \subseteq J$ and $K_0 \subseteq K$.

Therefore we can conclude that (CC) of (1.3) is equivalent to the (NC) of (2.1). □

4. Matroid Condition (MC) and its Relation to Network Condition (NC)

A *matroid* $M = (E, F)$ is defined as a finite set E and a nonempty family F of subsets of E, called *independent* subsets of E such that

(1) every subset of an independent set is independent and

(2) for every set $A \subseteq E$, all maximal independent subsets of A have the same cardinality, called the *rank* $r(A)$ of A.

A *base* of a matroid M is a maximal independent set of M. A set is called *dependent* relative to a matroid M if the set $A \subseteq E$ is not a member of F.

We give a simple example of matroid called *p-uniform matroid* defined on a finite set E. Bases are those subsets of E which contain exactly p elements, where $p \leq |E|$, and independent sets are the subsets of E containing not more than p elements. The rank of any subset $A \subseteq E$ is given by $\min(|A|, p)$.

We define a *partition matroid* which is to be used later on. Let $P_1, P_2, \ldots, P_m$ be a partition of E (i.e., $\bigcup_i P_i = E$ and $P_i \cap P_j = \phi$ whenever $i \neq j$), and let $p_1, p_2, \ldots, p_m$ be nonnegative integers. Then the independent sets are the collection of all sets $I \subseteq E$ such that $|I \cap P_i| \leq p_i$ for $1 \leq i \leq m$. Any matroid that can be generated in such a manner (for a partition) is called a partition matroid.

Now we consider the following problem. Let $M_{1k} = (E, F_{1k})$ and $M_{2k} = (E, F_{2k})$ be matroids on E for each $k \in K = \{1, 2, \ldots, n\}$. Let $r_{1k}(A)$ and $r_{2k}(A)$ be ranks in matroids M_{1k} and M_{2k} of a subset $A \subseteq E$, respectively, for each $k \in K$. Then what are necessary and sufficient conditions for a set E to be partitioned into n sets I_k satisfying $I_k \in F_{1k} \cap F_{2k}$ for each $k \in K$? We will add that special cases of the above problem, i.e., $M_{1k} = M_{2k} = M$ for each $k \in K$, or $M_{1k} = M_1$ and $M_{2k} = M_2$ for each $k \in K$ and M_1, M_2 are strongly-base-orderable (see[8]), have been solved in [3], [4], [5] and [8]. But general case has not been solved. In the following theorem we give a necessary (but not sufficient) condition for this problem.

Theorem 4.1. Let $M_{1k} = (E, F_{1k})$ and $M_{2k} = (E, F_{2k})$ be matroids for each $k \in K$, and let r_{1k} and r_{2k} be their respective rank functions for each $k \in K$. Suppose the set E can be partitioned into n sets I_k satisfying $I_k \in F_{1k} \cap F_{2k}$ for each $k \in K$. Then we have

(4.1) $$|A| \leq \sum_{k \in K} r_{0k}(A), \text{ for all } A \subseteq E,$$

where for each $k \in K$

(4.2) $$r_{0k}(A) = \min_{H \subseteq A} \{r_{1k}(H) + r_{2k}(A \setminus H)\}.$$

Proof: Let $\{I_k, k \in K\}$ be a partition of the set E satisfying the condition $I_k \in F_{1k} \cap F_{2k}$ for each $k \in K$. Given a set $A \subseteq E$, suppose the set $H_k \subseteq A$ minimizes $r_{1k}(H) + r_{2k}(A \setminus H)$ for each $k \in K$. Then for all subsets $A \subseteq E$ we have

$$\begin{aligned} |A| &= \sum_{k \in K} |A \cap I_k| \\ &= \sum_{k \in K} \{|(A \cap I_k) \cap H_k| + |(A \cap I_k) \setminus H_k|\} \\ &\leq \sum_{k \in K} \{r_{1k}(H_k) + r_{2k}(A \setminus H_k)\} \\ &= \sum_{k \in K} \min_{H \subseteq A} \{r_{1k}(H) + r_{2k}(A \setminus H)\} \\ &= \sum_{k \in K} r_{0k}(A). \end{aligned}$$

Thus the condition (4.1) is obtained. □

We note that Edmonds has shown the following relation in [6]:

(4.3) $$\min_{H \subseteq A} \{r_{1k}(H) + r_{2k}(A \setminus H)\} = \max_{\substack{H \subseteq A \\ H \in F_{1k} \cap F_{2k}}} |H|.$$

The relation (4.3) holds for each $k \in K$. Hence from (4.2) and (4.3), the condition (4.1) can also be written as

(4.4) $$|A| \leq \sum_{k \in K} \max_{\substack{H \subseteq A \\ H \in F_{1k} \cap F_{2k}}} |H|, \text{ for all } A \subseteq E.$$

We note that there exists a case for which the condition (4.1) (or (4.4)) is both necessary and sufficient for the set E to be partitioned into n sets I_k satisfying $I_k \in F_{1k} \cap F_{2k}$ for each $k \in K$. Namely suppose for each $k \in K$ M_{1k} is a p_k-uniform matroid, where p_k is a nonnegative integer, and M_{2k} is a general matroid. Then the condition (4.1) (or (4.4)) can be necessary and sufficient for the set E to be partitioned into n sets I_k such that $I_k \in F_{1k} \cap$

F_{2k} for each $k \in K$.

Now we interpret the (CPLS) problem using the terminology of the above matroid partitioning problem. Given an $n \times n$ (PLS) P let I, J, K be the sets of rows, columns and symbols in P respectively, i.e., $I = J = K = N = \{1, 2, \ldots, n\}$, and let E be a set consisting of unoccupied cells in the (PLS). For each $i \in I$ and $j \in J$, let S_i, T_j be the subsets of E consisting of unoccupied cells in row i, column j, respectively, in the (PLS) P. Hence $\{S_i, i \in I\}$ and $\{T_j, j \in J\}$ are partitions of E respectively. For each $i \in I$, $j \in J$ and $k \in K$ let $s_{ik} = 1$ if no cell in row i in P is occupied by symbol k, and 0 otherwise, and let $t_{jk} = 1$ if no cell in column j in P is occupied by symbol k, and 0 otherwise. We define matroids $M_{1k} = (E, F_{1k})$, $M_{2k} = (E, F_{2k})$ for each $k \in K$ as follows. For each $k \in K$ let M_{1k} be the partition matroid generated by a family of subsets $\{S_i, i \in I\}$ and nonnegative integers $\{s_{ik}, i \in I\}$ and let M_{2k} be the partition matroid generated by a family of subsets $\{T_j, j \in J\}$ and nonnegative integers $\{t_{jk}, j \in J\}$. Then the problem of completing the (PLS) is equivalent to that of obtaining a partition of E into n sets I_k such that $I_k \in F_{1k} \cap F_{2k}$ for each $k \in K$. Therefore our original problem of obtaining a necessary and sufficient condition for (CPLS) is equivalent to establishing a necessary and sufficient condition for the existence of a partition of E into n sets I_k such that $I_k \in F_{1k} \cap F_{2k}$ for each $k \in K$. Consequently the condition (4.1) (or (4.4)) provides a necessary condition for the completion of a (PLS).

Condition (4.4) has an interesting interpretation in the case of (LS)'s. Since A is a subset of E, whose elements are unoccupied cells in the (PLS), $|A|$ is the number of unoccupied cells in A. On the other hand $\max_{H \subseteq A,\ H \in F_{1k} \cap F_{2k}} |H|$ is the cardinality of the maximal consistent assignment of symbol k into the set A. Thus the content of condition (4.4) is that no completion is possible when for some collection A of unoccupied cells, $|A|$ exceeds the total of the maximum assignments of individual symbols to the cells of A.

In the following theorem we show that matroid condition (MC) given in (4.1) (or equivalently (4.4)) implies network condition (NC) given in (2.1); that is, given a (PLS) P which satisfies (MC), then it satisfies (NC). But we note that the converse is not true.

Theorem 4.2. Matroid Condition (MC) of (4.1) implies Network Condition (NC) of (2.1).

Proof: Suppose we are given a (PLS) P with corresponding network G as in Section 2 and there exists a violating set to the condition (2.1). That is, suppose $I_0 \subseteq I$, $J_0 \subseteq J$ and $K_0 \subseteq K$ satisfy

$$(4.5) \qquad |(K_0, I_0)| + |(\overline{I}_0, \overline{J}_0)| + |(J_0, K_0)| < \sum_{k \in K_0} N_k.$$

Then we will show that the set

$$(4.6) \qquad A = \{(i, j) \mid \text{the cell } (i, j) \text{ is unoccupied in the (PLS) P and } i \in \overline{I}_0, j \in J_0\}$$

violates the (MC) of (4.4). From the definition of the set A in (4.6), we have

$$\begin{aligned} |A| &= |(\overline{I}_0, J_0)| \\ &= |(K, \overline{I}_0)| - |(\overline{I}_0, \overline{J}_0)| \quad (\text{since } |(K, \overline{I}_0)| = |(\overline{I}_0, \overline{J}_0)| + |(\overline{I}_0, J_0)|) \\ &= |(\overline{K}_0, \overline{I}_0)| + |(K_0, \overline{I}_0)| - |(\overline{I}_0, \overline{J}_0)| \\ &= |(\overline{K}_0, \overline{I}_0)| + \sum_{k \in K_0} N_k - |(K_0, I_0)| - |(\overline{I}_0, \overline{J}_0)|. \end{aligned} \qquad (4.7)$$

Combining (4.5) and (4.7), we obtain

$$\begin{aligned} |A| = |(\overline{K}_0, \overline{I}_0)| + |(J_0, K_0)| + \sum_{k \in K_0} N_k - |(K_0, I_0)| - |(\overline{I}_0, \overline{J}_0)| \\ - |(J_0, K_0)| > |(\overline{K}_0, \overline{I}_0)| + |(J_0, K_0)|. \end{aligned}$$

Thus

$$(4.8) \qquad |A| > |(\overline{K}_0, \overline{I}_0)| + |(J_0, K_0)|.$$

On the other hand, in the (MC)'s of (4.1) and (4.4) we can recognize that $r_{0k}(A) = \max_{H \subseteq A,\, H \in F_{1k} \subset F_{2k}} |H|$ is equal to the value of a maximal flow from the source $k \in K_1$ to the sink $k \in K_2$ in the subnetwork of G defined by nodes $k \in K_1$, $\overline{I}_0 \subseteq I$, $J_0 \subseteq J$ and $k \in K_2$. Hence applying the max-flow min-cut theorem in this subnetwork, we obtain

$$(4.9) \qquad r_{0k}(A) \leq \{|(k, X)| + |(\overline{X}, \overline{Y})| + |(Y, k)|\}$$

$$\text{for any } X \subseteq \overline{I}_0 \text{ and } Y \subseteq J_0,$$

where $\overline{X}$, $\overline{Y}$ are the complements of X, Y in the sets $\overline{I}_0$, J_0 respectively. We choose the subsets X, Y as follows,

$$(4.10) \qquad \begin{aligned} &X = \phi,\ Y = J_0, \text{ for all } k \in K_0, \\ &X = \overline{I}_0,\ Y = \phi, \text{ for all } k \in \overline{K}_0. \end{aligned}$$

Then from (4.10) we obtain

$$\begin{aligned} &\sum_{k \in K} \{|(k, X)| + |(\overline{X}, \overline{Y})| + |(Y, k)|\} \\ &\qquad = \sum_{k \in K_0} |(J_0, k)| + \sum_{k \in \overline{K}_0} |(k, \overline{I}_0)| \\ &\qquad = |(J_0, K_0)| + |(\overline{K}_0, \overline{I}_0)|. \end{aligned} \qquad (4.11)$$

From (4.9) and (4.11) we get that

(4.12) $$\sum_{k \in K} r_{0k}(A) \leq |(J_0, K_0)| + |(\bar{K}_0, \bar{I}_0)|.$$

From (4.8) and (4.12) we obtain

(4.13) $$|A| > \sum_{k \in K} r_{0k}(A).$$

(4.13) shows that the set A defined by (4.6) is a violating set to the (MC) of (4.1) or (4.4). Thus we can conclude that whenever we have a violating set to (NC), there exists a violating set to (MC); i.c., (MC) implies (NC). □

Now we give an example showing that the converse of Theorem 4.1 is not true.

Example 4.2 Consider the (PLS) illustrated in Figure 4.3 below.

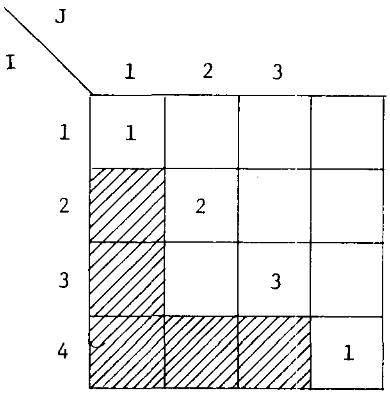

Figure 4.1 (PLS) P

The above (PLS) P cannot be completed since symbols 2 and 3 conflict in cell (4.1). It is routine to check that this P satisfies (NC) of (3.1). But there exists a violating set A to the (MC) of (4.4), which is given by

$$A = \{(2, 1), (3, 1), (4, 1), (4, 2), (4, 3)\}.$$

The above set A corresponds to the shaded area in the (PLS) P of Figure 4.1. Namely we have

$$|A| = 5 > \sum_{k=1}^{4} \{ \max_{H \subseteq A,\ H \in F_{1k} \cap F_{2k}} |H| \}$$

$$= 0 + 1 + 1 + 2 = 4$$

We noted that neither of these necessary conditions given in (3.1) and (4.1) was sufficient for the completion of (PLS)'s (an example showing this is given in [1]), but there are some special cases in which network condition of (3.1) can be both necessary and sufficient. These cases are given in [9].

Moreover we can consider another necessary condition based on the triply stochastic matrix. In [9] the relation between this condition and matroid condition is investigated and some results are shown.

Acknowledgements

The author would like to express his sincere thanks to Prof. L.E. Trotter and Dr. F. R. Giles for their invaluable suggestions and comments for this research.

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