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M+1-OUT-OF-N:G SYSTEM WITH CORRELATED FAILURE AND SINGLE REPAIR FACILITY

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ABSTRACT

This paper deals with the M + 1 - out-of-N:G system with correlated failure and single repair facility, where correlated failure means that i ($1 \leqq i \leqq N$) operating units in the system are possible to fail at the same time. We consider two repair policies, and under each repair policy Laplace transform of point-wise availability and reliability, meantime to the first system failure and stationary availability are derived. Finally some properties of stationary availability for each repair policy are given.

1. Introduction.

In this paper the simultaneous failure of i operating units may be described as SF_i.

Several authors have studied the system with correlated failure. R. HARRIS [1] has studied the two units system with bivariate exponential failure distribution which is defined by A.W.MARSHALL and I.OLKIN [5]. T.ITOI, T. MURAKAMI, M.KODAMA and T.NISHIDA [2] have defined the bivariate Erlang distribution and applied it to the system reliability analysis. About the N-unit system with correlated failure T.ITOI, T.NISHIDA, M.KODAMA and F.OHI [3] have studied, but the studied system is N-unit parallel redundant system and are comsidered only SF_2 and simultaneous failure of all the operating units. In this paper we deal with the M+1-out-of-N:G system with SF_i ($2 \leqq i \leqq N$) and single repair facility.

We consider two repair policies. One of which is that the failed unit is repaired as soon as the unit fails and after completion of repair the unit becomes to the operating unit, and the other is that the repair is begun after N-M units fail and after completion of all repairs system begins to operate. We call the model under the former repair policy as Model 1 and the one under the latter repair policy as Model 2.

For both models we derive Laplace transforms (L-T) of point-wise availability and reliability, mean time to the system failure (MTSF) and stationary availability. Finally we show that stationary availability of Model 2 is decreasing in M and N, and tends to zero as $N \to \infty$ for any fixed M ($0 \leq M \leq N-1$) when there is no correlated failure.

2. Definition of Models and Notations

The system consists of N units and initially all units are operating. In the system there are $q_i = \binom{N}{i}$ groups of size i ($1 \leq i \leq N$) which are symbolized as $G_{i1}, \dots G_{iq_i}$. Poisson process with parameter λ_{ij} governs the occurrence of shocks to G_{ij}. These Poisson processes are mutually independent. When the shocks to G_{ij} occurs, the operating units in G_{ij} fail. In this paper we assume that $\lambda_{ij} \equiv \lambda_i$ ($j = 1,\dots,q_i$). Then the rate at which occurs the shock to group of size i is $\binom{N}{i}\lambda_i$. The system is considered good when at least M+1 units are operating. When the system is down the residual operating units do not fail. There is one repair man and distribution of repair time to a failed unit is general.

Through this paper we use the following notations;

$\{i\}$	state that i units are operating
$P_i(t)$	Pr[the system is in state $\{i\}$ at time t] ($0 \leq i \leq N$)
$P_i(t,x)dx$	Pr[the system is in state $\{i\}$ at time t and elapsed repair time of unit under repair lies between x and x+dx] ($0 \leq i \leq N-1$ for Model 1, $0 \leq i \leq M$ for Model 2) $P_i(t) = \int_0^\infty P_i(t,x)dx$
$P_{(i)}(x)dx$	Pr[in equilibrium state the system is in state $\{i\}$ and elapsed repair time of unit under repair lies between x and x+dx] ($0 \leq i \leq N-1$ for Model 1, $0 \leq i \leq M$ for Model 2)
P_i	$\int_0^\infty P_{(i)}(x)dx = \lim_{t \to \infty} P_i(t)$ ($0 \leq i \leq N-1$)
P_N	$\lim_{t \to \infty} P_N(t)$
$P_A^{(j)}(t)$	point-wise availability of Model j ($j = 1,2$)
$P_A^{(j)}$	stationary availability of Model j ($j = 1,2$)

$R^{(j)}(t)$	system reliability of Model j (j = 1,2)
$MTSF^{(j)}$	mean time to the first system failure of Model j (j = 1,2)
$g_j(t)$	repair time density of Model j (j = 1,2)
$h_j(t)$	repair rate, i.e., $g_j(t) / \int_t^\infty g_j(x)dx$ (j = 1,2)
$1/\mu_j$	$\int_0^\infty x g_j(x)dx$ (j = 1,2)
ρ_j	λ_1 / μ_j (j = 1,2)
$g_2^*(t)$	the (N-M)-th convolution of $g_2(t)$
$h_2^*(t)$	$g_2^*(t) / \int_t^\infty g_2^*(x)dx$
$\hat{f}(s)$	Laplace transform of f(t), i.e., $\hat{f}(s) = \int_0^\infty e^{-st} f(t)dt$
β_i	$\sum_{n=1}^{N} \{\binom{N}{n} - \binom{N-i}{n}\}\lambda_n$ ($1 \leq i \leq N$)
α_i	$\binom{N}{i} \sum_{n=1}^{N} \binom{N-i}{n}\lambda_n$ ($1 \leq i \leq N-1$)
δ_{ij}	Kronecker's delta

3. Model 1.

3.1. Analysis for $0 \leq M \leq N-2$

Viewing the nature of this system, the following set of integro-differential equations can be set up easily:

$$\left[\frac{d}{dt} + \sum_{j=0}^{N-1} \binom{N}{N-j}\lambda_{N-j}\right]P_N(t) = \int_0^t P_{N-1}(t,x)h_1(x)dx \ , \tag{3.1.1}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \sum_{j=0}^{i-1} \binom{i}{i-j} \sum_{n=0}^{N-i} \binom{N-i}{n}\lambda_{i-j+n} + h_1(x)\right]P_i(t,x) \tag{3.1.2}$$
$$= (1-\delta_{i,N-1}) \sum_{j=i+1}^{N-1} \binom{j}{j-i} \sum_{n=0}^{N-j} \binom{N-j}{n}\lambda_{j-i+n}P_j(t,x) \quad (M+1 \leq i \leq N-1),$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + h_1(x)\right]P_i(t,x) = \sum_{j=M+1}^{N-1} \binom{j}{j-i} \sum_{n=0}^{N-j} \binom{N-j}{n}\lambda_{j-i+n}P_j(t,x) \tag{3.1.3}$$
$$(0 \leq i \leq M)$$

Equations (3.1.1)~(3.1.3) are to be solved under the following boundary and initial conditions:

$$P_i(t,0) = \binom{N}{N-i}\lambda_{N-i}P_N(t) + (1-\delta_{i,0})\int_0^t P_{i-1}(t,x)h_1(x)dx \quad (0 \leq i \leq N-1), \tag{3.1.4}$$

$$P_i(0) = \delta_{i,N} \quad (0 \leq i \leq N). \tag{3.1.5}$$

Taking Laplace transform of (3.1.1)~(3.1.4) under the initial conditions (3.1.5), we have:

$$\left[s + \sum_{j=0}^{N-1} \binom{N}{N-j}\lambda_{N-j}\right]\hat{P}_N(s) = 1 + \int_0^\infty \hat{P}_{N-1}(s,x)h_1(x)dx, \tag{3.1.6}$$

(3.1.7) $$[s+\frac{d}{dx}+\sum_{j=0}^{i-1}\binom{i}{i-j}\sum_{n=0}^{N-i}\binom{N-i}{n}\lambda_{i-j+n}+h_1(x)]\hat{P}_i(s,x)$$
$$=(1-\delta_{i,N-1})\sum_{j=i+1}^{N-1}\binom{j}{j-i}\sum_{n=0}^{N-j}\binom{N-j}{n}\lambda_{j-i+n}\hat{P}_j(s,x)\qquad(M+1\leq i\leq N-1),$$

(3.1.8) $$[s+\frac{d}{dx}+h_1(x)]\hat{P}_i(s,x)=\sum_{j=M+1}^{N-1}\binom{j}{j-i}\sum_{n=0}^{N-j}\binom{N-j}{n}\lambda_{j-i+n}\hat{P}_j(s,x)\qquad(0\leq i\leq M),$$

(3.1.9) $$\hat{P}_i(s,0)=\binom{N}{N-i}\lambda_{N-i}\hat{P}_N(s)+(1-\delta_{i,0})\int_0^\infty\hat{P}_{i-1}(s,x)h_1(x)dx\qquad(0\leq i\leq N-1).$$

Using the discrete transform [4], i.e.,

(3.1.10) $$\hat{u}_j(s,x)=\sum_{i=j}^{N-1}\binom{i}{j}\hat{P}_i(s,x)\qquad(M+1\leq j\leq N-1),$$

(3.1.11) $$\hat{P}_j(s,x)=\sum_{i=j}^{N-1}(-1)^{i-j}\binom{i}{j}\hat{u}_i(s,x)\qquad(M+1\leq j\leq N-1),$$

we can rewrite the equations (3.1.7) amd (3.1.9) for $M+2\leq i\leq N-1$ as follows. (See appendix 1 and 2 respectively.)

(3.1.12) $$[s+\frac{d}{dx}+\beta_i+h_1(x)]\hat{u}_i(s,x)=0\qquad(M+1\leq i\leq N-1),$$

(3.1.13) $$\hat{u}_i(s,0)=[\hat{g}_1(s+\beta_{i-1})\hat{u}_{i-1}(s,0)+\{\alpha_i-\hat{u}_{n-1}(s,0)\hat{g}_1(s+\beta_{N-1})\{(s+\beta_N)\binom{N}{i}$$
$$-\alpha_i\}\}/(s+\beta_N)]/\{1-\hat{g}_1(s+\beta_i)\}\qquad(M+2\leq i\leq N-1).$$

Consequently, by solving (3.1.6), (3.1.8), (3.1.9), (3.1.11), (3.1.12) and (3.1.13), we have after simple but tedious calculation:

(3.1.14) $$\hat{P}_j(s)=\int_0^\infty\hat{P}_j(s,x)dx=\sum_{i=j}^{N-1}(-1)^{i-j}\binom{i}{j}\hat{u}_i(s,0)\{1-\hat{g}(s+\beta_i)\}/(s+\beta_i)$$
$$(M+1\leq j\leq N-1),$$

(3.1.15) $$\hat{P}_N(s)=\{1+\hat{u}_{N-1}(s,0)\hat{g}(s+\beta_{N-1})\}/(s+\beta_N),$$

where

$$\hat{u}_j(s,0)=\hat{a}_j(s)\hat{u}_{M+1}(s,0)+\hat{b}_j(s)\qquad(M+1\leq j\leq N-1),$$

$$\hat{a}_j(s)=[\hat{g}_1(s+\beta_{M+1})\sum_{n=j+1}^{N}\hat{k}_{n-1,M}(s)\{(s+\beta_n)\binom{N}{n}-\alpha_n\}]/[\sum_{n=M+2}^{N}\hat{k}_{n-1,M}(s)\{(s+\beta_N)\binom{N}{n}-$$
$$-\alpha_n\}\hat{g}_1(s+\beta_j)\hat{k}_{j,M}(s)]\qquad(M+1\leq j\leq N-1),$$

$$\hat{b}_j(s)=[\sum_{n=j+1}^{N}\sum_{m=M+2}^{j}\hat{k}_{n-1,M}(s)\hat{k}_{m-1,M}(s)\{\alpha_m\binom{N}{n}-\alpha_n\binom{N}{m}\}]/[\hat{g}_1(s+\beta_j)\cdot\hat{k}_{j,M}(s)\sum_{n=M+2}^{N}$$
$$\hat{k}_{n-1,M}(s)\{(s+\beta_N)\binom{N}{n}-\alpha_n\}]\qquad(M+1\leq j\leq N-1),$$

$$\hat{k}_{j,M}(s)=\begin{cases}\prod_{l=M+2}^{j}\{1-\hat{g}_1(s+\beta_l)\}/\hat{g}_1(s+\beta_l) & (M+2\leq j\leq N-1),\\ 1 & (j=M+1),\end{cases}$$

$$\hat{u}_{M+1}(s,0) = [(s+\beta_N)\sum_{j=M+1}^{N-1}(-1)^j\hat{b}_j(s)\hat{c}(j,M,s)+\alpha_{M+1}+\sum_{j=0}^{M}\binom{N}{N-j}\lambda_{N-j}\hat{\gamma}_{2j}(s)$$
$$+\hat{b}_{N-1}(s)\hat{g}_1(s+\beta_{N-1})\{\alpha_{M+1}+\sum_{j=0}^{M}\binom{N}{N-j}\lambda_{N-j}\hat{\gamma}_{2j}(s)-(s+\beta_n)\binom{N}{M+1}\}]$$
$$/[(s+\beta_N)\{1-\hat{g}_1(s+\beta_{N-1})\}+(s+\beta_N)\sum_{j=M+1}^{N-1}(-1)^j\hat{a}_j(s)\hat{c}(j,M,s)+$$
$$\hat{a}_{N-1}(s)\hat{g}_1(s+\beta_{N-1})\{(s+\beta_N)\binom{N}{M+1}-\sum_{j=0}^{M}\binom{N}{N-j}\lambda_{N-j}\hat{\gamma}_{2j}(s)-\alpha_{M+1}\}],$$

$$\hat{c}(i,M,s) = \hat{g}_1(s+\beta_i)\sum_{j=M+1}^{i}\binom{j}{M}\binom{i}{j}(-1)^{j+1}+\{\hat{\gamma}_{1i}(s)/\beta_i\}\sum_{j=0}^{M}(\hat{g}(s))^{M-j}B(i,j,M)$$
$$(M+1\leqq i\leqq N-1),$$

$$\hat{\gamma}_{1j}(s) = \hat{g}(s) - \hat{g}(s+\beta_j) \quad (M+1\leqq j\leqq N-1),$$
$$\hat{\gamma}_{2j}(s) = (\hat{g}(s))^{M+1-j} \quad (0\leqq j\leqq M)$$

and

$$B(i,j,M) = \sum_{k=M+1}^{i}\sum_{n=0}^{N-k}\binom{k}{k-j}\binom{N-k}{n}\lambda_{k-j+n}(-1)^{k+1}\binom{i}{k}.$$

The L-T of point-wise availability $\hat{P}_A^{(1)}(s)$ is obtained as follows from (3.1.12) and (3.1.14):

(3.1.16) $$\hat{P}_A^{(1)}(s) = \sum_{i=M+1}^{N-1}(-1)^{M+1+i}\binom{i-1}{M}\{\hat{a}_i(s)\hat{u}_{M+1}(s,0)+\hat{b}_i(s)\}\{1-\hat{g}(s+\beta_i)\}/(s+\beta_i)+[1+\hat{g}(s+\beta_{N-1})\{\hat{a}_{N-1}(s)\hat{u}_{M+1}(s,0)+\hat{b}_{N-1}(s)\}]/(s+\beta_N).$$

The L-T of $R^{(1)}(t)$ can be obtained from $\hat{P}_A^{(1)}(s)$ by making suitable transformations. Putting $\hat{\gamma}_{1j}(s)=0$ $(M+1\leqq j\leqq N-1)$ and $\hat{\gamma}_{2j}(s)=0$ $(0\leqq j\leqq M)$ in (3.1.16) yields $\hat{R}^{(1)}(s)$ since the substitution is equivalent to the assertion that the probability of the system moving from down state to up state is zero. Moreover, if we set s=0 in $\hat{R}^{(1)}(s)$, we obtain the MTSF$^{(1)}$.

In order to derive stationary availability we set up the following set of differential-difference equations:

(3.1.17) $$\beta_N P_N = \int_0^\infty P_{(N-1)}(x)h_1(x)dx,$$

(3.1.18) $$[\frac{d}{dx}+\sum_{j=0}^{i-1}\binom{i}{i-j}\sum_{n=0}^{N-i}\binom{N-i}{n}\lambda_{i-j+n}+h_1(x)]P_{(i)}(x)$$
$$= (1-\delta_{i,N-1})\sum_{j=i+1}^{N-1}\binom{j}{j-i}\sum_{n=0}^{N-j}\binom{N-j}{n}\lambda_{j-i+n}P_{(j)}(x)$$
$$(M+1\leqq i\leqq N-1),$$

(3.1.19) $$[\frac{d}{dx}+h_1(x)]P_{(i)}(x) = \sum_{j=M+1}^{N-1}\binom{j}{j-i}\sum_{n=0}^{N-j}\binom{N-j}{n}\lambda_{j-i+n}P_{(j)}(x) \quad (0\leqq i\leqq M)$$

with the following boundary conditions and normalizing conditions:

(3.1.20) $$P_{(i)}(0) = \binom{N}{N-i}\lambda_{N-i}P_N+(1-\delta_{i,0})\int_0^\infty P_{(i-1)}(x)h_1(x)dx \quad (0\leqq i\leqq N-1),$$

(3.1.21) $$\sum_{i=0}^{N} P_i = 1 .$$

Thus using the similar discrete transform, stationary availability $P_A^{(1)}$ is given as

(3.1.22) $$P_A^{(1)} = u_{M+1}[\sum_{i=M+1}^{N-1} (-1)^{i+M-1}\binom{i-1}{M}\hat{a}_i(0)\{1-\hat{g}(\beta_i)\}/\beta_i + \hat{a}_{N-1}(0)\hat{g}(\beta_{N-1})/\beta_N],$$

where

$$u_{M+1} = \mu_1\beta_N/[\beta_N \sum_{m=M+1}^{N-1} (-1)^{m-1}\hat{a}_m(0)[\sum_{i=0}^{M} B(m,i,M)\{\beta_m-\mu_1+\mu_1\hat{g}_1(\beta_m)\}/\beta_m^2+\{\mu_1(-1)^M \cdot\binom{m-1}{M}+\sum_{i=1}^{M}(M-i+1)B(m,i-1,M)\}\{1-\hat{g}_1(\beta_m)\}/\beta_m+\{\mu_1+\sum_{i=0}^{M}(M-i+1)\binom{N}{N-i}\lambda_{N-i}\} \cdot\hat{g}_1(\beta_{N-1})\hat{a}_{N-1}(0)]] .$$

3.2. Analysis for M=N-1

In this case the equations describing the behavior of the system are (3.1.6), (3.1.8) and (3.1.9), and equations(3.1.7) are cut. These equations can be easily solved and we have

(3.2.1) $$\hat{P}_A^{(1)}(s) = 1/[s+\beta_N - \sum_{j=0}^{N-1}\binom{N}{N-1}\lambda_{N-j}\{\hat{g}_1(s)\}^{N-j}],$$

(3.2.2) $$\hat{R}^{(1)}(s) = 1/[s+\beta_N],$$

(3.2.3) $$\text{MTSF} = 1/\beta_N .$$

And stationary availability can be obtained easily as

(3.2.4) $$P_A^{(1)} = \mu_1/[\mu_1 + N\sum_{j=0}^{N-1}\binom{N-1}{j}\lambda_{N-j}].$$

3.3. Properties of $P_A^{(1)}$ when there is no correlated failure

We assume that M=0 and for convenience we use the notation $P_A^{(1)}(N)$ in place of $P_A^{(1)}$.

If M=0 and $\lambda_j=0$ $(2\leq j\leq N)$, $P_A^{(1)}(N)$ given by (3.1.22) is that

$$P_A^{(1)}(N)=[\sum_{n=1}^{N-1}\binom{N-1}{n}\hat{k}_n(0)\sum_{j=1}^{n}(-1)^{j-1}\{(1-\hat{g}_1(\beta_j))/j\hat{g}_1(\beta_j)\hat{k}_j(0)\}+\{1/N\}]/[\rho_1\sum_{n=1}^{N-1}\binom{N-1}{n}\hat{k}_n(0)\sum_{j=1}^{n}(-1)^{j-1}\{1/\hat{g}_1(\beta_j)\hat{k}_j(0)\}+\{1/N\}].$$

Intuitively we conceive that $P_A^{(1)}(N)$ is increasing in N for any repair time distribution. But the following example shows that this is not true. Then the problem to determine the class of repair time distribution to assure that $P_A^{(1)}(N)$ is increasing in N will arise, which remins open.

Example. Noticing that we may understand $\hat{g}_1(s)$ as the Laplace-Stieltjes transform of repair time distribution $G_1(t)$, we consider the G_1 described in Fig.1. In this case with that $\rho_1=1$, it is easily shown that $P_A^{(1)}(3) - P_A^{(1)}(2)$ is negative.

If $g_1(t)=e^{-\mu_1 t}$, $P_A^{(1)}(N)=[\sum_{j=1}^{N}(1/j!)\cdot(1/\rho_1)^j]/[\sum_{j=0}^{N}(1/j!)(1/\rho_1)^j]$. Then $P_A^{(1)}(N) \to 1-e^{-(1/\rho_1)}$ $(N \to \infty)$.

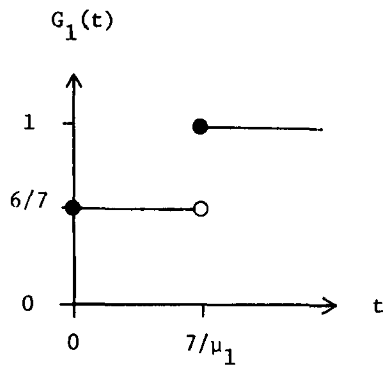

Fig.1.

4. Model 2

4.1. Analysis

When M=N-1 the results of this model are coincident with (3.2.1), (3.2.2), (3.2.3) and (3.2.4) evidently. In the case that $0 \leq M \leq N-2$ the following set of integro-differential equations can be set up easily:

(4.1.1) $$\left[\frac{d}{dt} + \sum_{j=0}^{N-1}\binom{N}{N-j}\lambda_{N-j}\right]P_N(t) = \int_0^t P_M(t,x)h_2^*(x)dx$$

(4.1.2) $$\left[\frac{d}{dt} + \sum_{j=0}^{i-1}\binom{i}{i-j}\sum_{n=0}^{N-i}\binom{N-i}{n}\lambda_{i-j+n}\right]P_i(t) = \sum_{j=i+1}^{N}\binom{i}{j-i}\sum_{n=0}^{N-j}\binom{N-j}{n}\lambda_{j-i+n}P_j(t)$$ $(M+1 \leq i \leq N-1)$,

(4.1.3) $$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + h_2^*(x)\right]P_M(t,x) = 0 ,$$

(4.1.4) $$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + h_2(x)\right]P_i(t,x) = 0 \quad (0 \leq i \leq M-1) .$$

Equations (4.1.1)∿(4.1.4) are to be solved under the following boundary and initial conditions:

(4.1.5) $$P_i(t,0) = \sum_{j=M+1}^{N}\binom{j}{j-i}\sum_{n=0}^{N-j}\binom{N-j}{n}\lambda_{j-i+n}P_j(t) + (1-\delta_{i,0})\int_0^t P_{i-1}(t,x)h_2(x)dx$$ $(0 \leq i \leq M)$,

(4.1.6) $$P_i(0) = \delta_{i,N} \quad (0 \leq i \leq N) .$$

Taking the L-T of equations(4.1.1)∿(4.1.5) under the initial conditions (4.1.6) and using the discrete transform

(4.1.7) $$\hat{u}_j(s) = \sum_{i=j}^{N-1}\binom{i}{j}\hat{P}_i(s) \quad (M+1 \leq j \leq N-1),$$

(4.1.8) $$\hat{P}_j(s) = \sum_{i=j}^{N-1}(-1)^{i-j}\binom{i}{j}\hat{u}_i(s) \quad (M+1 \leq j \leq N-1),$$

we have

(4.1.9) $$[s+\beta_N]\hat{P}_N(s) = 1+\int_0^\infty \hat{P}_M(s,x)h_2^*(x)dx ,$$

(4.1.10) $[s+\beta_j]\hat{u}_j(s) = \alpha_j P_N(s) \quad (M+1\leq j\leq N-1)$,

(4.1.11) $[s+\frac{d}{dx}+h_2^*(x)]\hat{P}_M(s,x) = 0$,

(4.1.12) $[s+\frac{d}{dx}+h_2(x)]\hat{P}_i(s,x) = 0 \quad (0\leq i\leq M-1)$,

(4.1.13) $\hat{P}_i(s,0)=\sum_{j=M+1}^{N}\binom{j}{j-i}\sum_{n=0}^{N-j}\binom{N-j}{n}\lambda_{j-i+n}\hat{P}_j(s)+(1-\delta_{i,0})\int_0^\infty \hat{P}_{i-1}(s,x)h_2(x)dx$

$(0\leq i\leq M)$.

These equations (4.1.8)∿(4.1.13) are easily solved and we obtain

(4.1.14) $\hat{P}_A^{(2)}(s)=[1+\sum_{i=M+1}^{N-1}(-1)^{M+1+i}\binom{i-1}{M}\alpha_i/(s+\beta_i)]/[s+\beta_N-\hat{g}_2^*(s)\sum_{i=0}^{M}\hat{\Gamma}_i(s)\cdot(\hat{g}_2(s))^{M-i}]$.

Putting $\hat{g}_2^*(s) = 0$ in (4.1.14), we obtain

(4.1.15) $\hat{R}^{(2)}(s) = [1+\sum_{i=M+1}^{N-1}(-1)^{M+1+i}\binom{i-1}{M}\alpha_i/(s+\beta_i)]/(s+\beta_N)$.

Taking the inverse transform of $\hat{R}^{(2)}(s)$, we obtain

(4.1.16) $R^{(2)}(t)=\sum_{i=M+1}^{N-1}(-1)^{M+1+i}\binom{i-1}{M}\alpha_i(e^{-\beta_i t}-e^{-\beta_N t})/\sum_{n=1}^{N}\binom{N-i}{n}\lambda_n+e^{-\beta_N t}$.

(4.1.17) $MTSF^{(2)} = \hat{R}^{(2)}(0)$.

In order to derive stationary availability we set up the following set of differential-difference equations:

$$\beta_N P_N = \int_0^\infty P_M(x)h_2^*(x)dx ,$$

$$P_i\sum_{j=0}^{i-1}\binom{i}{i-j}\sum_{n=0}^{N-i}\binom{N-i}{n}\lambda_{i-j+n} = \sum_{j=i+1}^{N}\binom{j}{j-i}\sum_{n=0}^{N-j}\binom{N-j}{n}\lambda_{j-i+n}P_j \quad (M+1\leq i\leq N-1) ,$$

$$[\frac{d}{dx}+h_2^*(x)]P_{(M)}(x) = 0 ,$$

$$[\frac{d}{dx}+h_2(x)]P_{(i)}(x) = 0 \quad (0\leq i\leq M-1) ,$$

with the following boundary conditions and normalizing condition:

$$P_{(i)}(0)=\sum_{j=M+1}^{N}\binom{j}{j-i}\sum_{n=0}^{N-j}\binom{N-j}{n}\lambda_{j-i+n}P_j+(1-\delta_{i,0})\int_0^\infty P_{(i-1)}(x)h_2(x)dx \quad (0\leq i\leq M),$$

$$\sum_{j=0}^{N}P_j = 1 .$$

Thus using the similar discrete transform, we obtain

(4.1.18) $P_A^{(2)}=\mu_2[1+\sum_{i=M+1}^{N-1}(-1)^{M+1+i}\binom{i-1}{M}\alpha_i/\beta_i]/[\sum_{j=0}^{M-1}\sum_{i=0}^{j}\hat{\Gamma}_i(0)+(N-M)\beta_N+\mu_2+\mu_2\sum_{i=M+1}^{N-1}(-1)^{M+1+i}\binom{i-1}{M}\alpha_i/\beta_i]$,

where

$$\hat{\Gamma}_i(s)=\sum_{j=M+1}^{N-1}\binom{j}{j-i}\sum_{i=j}^{N-1}(-1)^{i-j}\binom{i}{j}\alpha_i/(s+\beta_i)\sum_{n=0}^{N-j}\binom{N-j}{n}\lambda_{j-i+n}+\binom{N}{N-i}\lambda_{N-i}\qquad (0\leqq i\leqq M).$$

Since $\sum_{m=i}^{j}=0$ $(j<i)$, the results (4.1.14), (4.1.15), (4.1.16), (4.1.17) and (4.1.18) are valid for M=N-1.

4.2. Properties of $P_A^{(2)}$ when there is no correlated failure

In this section we show that $P_A^{(2)}$ given by (4.1.18) is decreasing in M and N respectively, and tends to zero $N\to 0$ for any fixed M $(0\leqq M\leqq N-1)$ when there is no correlated failure, i.e., $\lambda_i=0$ $(2\leqq i\leqq N)$.

For convenience we use the notation $P_A^{(2)}(N,M)$ in place of $P_A^{(2)}$ throughout this section.

When $\lambda_i=0$ $(2\leqq i\leqq N)$, $P_A^{(2)}(N,M)$ given by (4.1.18) is that

$$P_A^{(2)}(N,M)=\left[\sum_{j=M+1}^{N}(1/j)\right]/\left[(N-M)\rho_2+\sum_{j=M+1}^{N}(1/j)\right]\qquad (0\leqq M\leqq N-1)\ .$$

Theorem 4.2.1. (i) $P_A^{(2)}(N,M)$ is increasing in M $(0\leqq M\leqq N-1)$ for any fixed $N\geqq 1$.

(ii) $P_A^{(2)}(N,M)$ is decreasing in N for any fixed $M\geqq 0$, where $N>M$.

(iii) $P_A^{(2)}(N,M)\to 0$ as $N\to\infty$ for any fixed $M\geqq 0$.

Proof: (i) amd (ii) are easily proved.

(iii) We may assume $N>M$.

$$[1/(N-M)]\sum_{j=M+1}^{N}(1/j)\to 0 \text{ as } N\to\infty\ . \qquad (4.2.1)$$

Since

$$[1/(N-M)]\sum_{j=M+1}^{N}(1/j)<[1/(N-M)]\int_{M+1}^{N}(1/x)dx=[\log\{N/(M+1)\}]/(N-M)\ ,$$

$$[\log\{N/(M+1)\}]/(N-M)\to 0 \text{ as } N\to\infty\ ,$$

and

$$1/[(N-M)(M+1)]\to 0 \text{ as } N\to\infty\ .$$

Then (4.2.1) is valid. ||

5. Concluding Remarks.

We studied two models and derived several measures for each model. Some properties of stationary availability were discussed in uncorrelated failure case.

The repair policy of Model 1 has been discussed by many authors. In the practical case, however, it is more troublesome and expensive than that of

Model 2. When $g_1=g_2$ it may be conceived that $P_A^{(1)} \geqq P_A^{(2)}$, but F.OHI, M.KODAMA and T.NISHIDA [6] shows that it does not necessarily hold. On the other hand, it is easily shown from (3.1.22) and (4.1.18) that $P_A^{(2)} \geqq P_A^{(1)}$ for sufficiently large μ_2 when $g_1 \neq g_2$. Thus if μ_2 is sufficiently large we had better used the repair policy of Model 2. But we must notice that from theorem 4.2.1. as increases the number of units, μ_2 must be increased to assure that $P_A^{(2)} \geqq P_A^{(1)}$.

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Appendix 1.

If $b_j = \sum_{i=j}^{N-1}\binom{i}{j}a_i \quad (1\ j\ N-1)$

then

$$(1)\quad -\sum_{i=j}^{N-1}\binom{i}{j}a_i\sum_{m=0}^{i-1}\binom{i}{i-m}\sum_{n=0}^{N-i}\binom{N-i}{n}\lambda_{i-m+n}+\sum_{i=j}^{N-2}\binom{i}{j}\sum_{m=i+1}^{N-1}\binom{m}{m-i}a_m\sum_{n=0}^{N-m}\binom{N-m}{n}\lambda_{m-i+n} = -\beta_j b_j .$$

Proof: (the first step)

$$\sum_{i=j}^{N-2}\binom{i}{j}\sum_{m=i+1}^{N-1}\binom{m}{m-i}a_m\sum_{n=0}^{N-m}\binom{N-m}{n}\lambda_{m-i+n} = \sum_{n=1}^{N-j-1}\lambda_n\sum_{i=j}^{N-1-n}\binom{i}{j}\sum_{m=i+1}^{i+n}\binom{m}{m-i}\binom{N-m}{n+i-m}a_m + \sum_{n=2}^{N-j}\binom{N-n}{j}\lambda_n\sum_{m=N-n+1}^{N-1}\binom{m}{m-N+n}a_m ,$$

$$\sum_{i=j}^{N-1}\binom{i}{j}a_i\sum_{m=0}^{i-1}\binom{i}{i-m}\sum_{n=0}^{N-i}\binom{N-i}{n}\lambda_{i-m+n} = \sum_{n=1}^{N}\lambda_n\sum_{i=j}^{N-1}\binom{i}{j}a_i\sum_{m=i-n}^{i-1}\binom{i}{i-m}\binom{N-i}{n-i+m} .$$

Then

(2) the left hand side of (1)

$$= -\sum_{n=1}^{N}\lambda_n\sum_{i=j}^{N-1}\binom{i}{j}a_i\sum_{m=i-n}^{i-1}\binom{i}{i-m}\binom{N-i}{n-i+m} + \sum_{n=1}^{N-j-1}\lambda_n\sum_{i=j}^{N-1-n}\binom{i}{j}\sum_{m=i+1}^{i+n}\binom{m}{m-i}\binom{N-m}{n+i-m}a_m + \sum_{n=2}^{N-j}\lambda_n\sum_{m=N-n+1}^{N-1}\binom{N-n}{j}\binom{m}{m-N+n}a_m .$$

(the second step)

(3) the right hand side of (2)

$$= -\sum_{n=N-j-1}^{N}\lambda_n\sum_{i=j}^{N-1}\binom{i}{j}a\sum_{m=i-n}^{i-1}\binom{i}{i-m}\binom{N-i}{n-i+m} + \Big[-\sum_{n=1}^{N-j}\lambda_n\sum_{i=j}^{N-1}\binom{i}{j}a_i\sum_{m=i-n}^{i-1}\binom{i}{i-m}\binom{N-i}{n-i+m}$$
$$+\sum_{n=1}^{N-j-1}\lambda_n\sum_{i=j}^{N-1-n}\binom{i}{j}\sum_{m=i+1}^{i+n}\binom{m}{m-i}\binom{N-m}{n+i-m}a_m + \sum_{n=2}^{N-j}\lambda_n\sum_{m=N-n+1}^{N-1}\binom{N-n}{j}\binom{m}{m-N+n}a_m\Big] .$$

Noticing that

$$\sum_{m=0}^{n-1}\binom{i}{n-m}\binom{N-i}{m} = \binom{N}{n} - \binom{N-i}{n} ,$$

the first term of the left hand side of (3) $= \sum_{n=N-j+1}^{N}\binom{N}{n}\lambda_n b$.

the bracketed passage of the left hand side of (3)

$$= \lambda_1\Big[-\sum_{i=j}^{N-1}\binom{i}{j}a_i\sum_{m=i-1}^{i-1}\binom{i}{i-m}\binom{N-i}{1-i+m}+\sum_{i=j}^{N-1-1}\binom{i}{j}\sum_{m=i+1}^{i+1}\binom{m}{m-i}\binom{N-m}{1+i-m}a_m\Big]$$
$$+\lambda_{N-j}\Big[-\sum_{i=j}^{N-1}\binom{i}{j}a_i\sum_{m=i-N+j}^{i-1}\binom{i}{i-m}\binom{N-i}{N-j-i+m} + \sum_{m=N-N+j+1}^{N-1}\binom{N-N+j}{j}\binom{m}{m-N+N-j}a_m\Big]$$
$$+\sum_{n=2}^{N-j-1}\lambda_n\Big[-\sum_{i=j}^{N-1}\binom{i}{j}a_i\sum_{m=i-n}^{i-1}\binom{i}{i-m}\binom{N-i}{n-i+m} + \sum_{i=j}^{N-1-n}\binom{i}{j}\sum_{m=i+1}^{i+n}\binom{m}{m-i}\binom{N-m}{n+i-m}a_m$$

$$+ \sum_{m=N-n+1}^{N-1} \binom{N-n}{j}\binom{m}{m-N+n} a_m]$$

$$= -\lambda_1 j b_j + \lambda_{N-j}[1-\binom{N}{N-j}]b_j + \sum_{n=2}^{N-j-1} \lambda_n [\binom{N-j}{n} - \binom{N}{n}]b_j .$$

Then

$$\text{the left hand side of (1)} = -[j\lambda_1 + \sum_{n=2}^{N} \{\binom{N}{n}-\binom{N-j}{n}\}\lambda_n]b_j$$

$$= -\sum_{n=1}^{N} \{\binom{N}{n} - \binom{N-j}{n}\}\lambda_n b_j$$

$$= -\beta_j b_j . \qquad ||$$

Appendix 2.

For $M+2 \leq i \leq N-1$ using (3.1.9),

$$\hat{u}_i(s,0) = \sum_{j=i}^{N-1} \binom{j}{i} \hat{P}_j(s,0)$$

$$= \alpha_i \hat{P}_N(s)+\int_0^\infty \hat{u}_i(s,x)h_1(x)dx+\int_0^\infty \hat{u}_{i-1}(s.x)h_1(x)dx-\binom{N}{i}\int_0^\infty \hat{u}_{N-1}(s,x)h_1(x)dx .$$

Substituting (3.1.15) and

$$\hat{u}_i(s,x) = \hat{u}_i(s,o)e^{-(s+\beta_i)x-\int_0^x h_1(x)dx} \quad \text{(resulted from (3.1.12))},$$

we have

$$\hat{u}_i(s,0) = \hat{g}_1(s+\beta_i)\hat{u}_i(s,0) + \hat{g}_1(s+\beta_{i-1})\hat{u}_{i-1}(s,0)$$
$$+[\{\alpha_i - (s+\beta_N)\binom{N}{i}\}\hat{g}_1(s+\beta_{N-1})\hat{u}_{N-1}(s,0) + \alpha_i]/(s+\beta_N) .$$

Then (3.1.13) is concluded. ||