

WHEN TO STOP: RANDOMLY APPEARING BIVARIATE TARGET VALUES

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ABSTRACT

This paper studies, as a continuation of previous work [8], an optimal stopping problem without recall in which the decision-maker observes a sequence of *iid* bivariate random variables which appear sequentially one by one in a Poisson manner. The problem can be interpreted as deciding to buy a house which has the two-dimensional worth, for example, the values for husband and for his wife. The concept of equilibrium neutral functions is introduced, and by using it an explicit solution of the problem is derived by means of finding a unique solution of some simultaneous differential equations. Some examples are included to illustrate the computations required by an "equilibrium neutral strategy".

1. Introduction and Summary

Let (X_i, Y_i) , $i = 1, 2, \dots, n$, be independently and identically distributed bivariate random variables that can be observed sequentially. The common distribution function $H(x, y)$ of each of the observation (X_i, Y_i) is assumed to be known by the decision-maker. When a random variable is observed it is either accepted, or rejected never to be accepted later. Only one observation, can be accepted, and if the player has not accepted until the final observation, then he is obliged to accept this one.

For univariate random variables X_i , many authors have studied the problem of finding the stopping policy which maximizes the expected value of the observation accepted. (See, for reference DeGroot [2, Chapter 13]). There

are quite few works for this problem for bivariate random variables, and as far as the present author knows his study [8] is the only one beginning work. In [8] the concept of equilibrium neutral values was introduced, and by using it a stopping policy was derived which "optimizes", in some appropriate sense, the expected values of the observation accepted.

In the present paper, we shall investigate some consequences of deleting the requirement that the number of the offers (i.e., length of the planning period) is deterministically known and fixed. We will consider the optimal stopping problem in which the offers are presented sequentially one by one and randomly in a Poisson manner during some given time interval. Associated with the offer newly presented at time τ is a bivariate random variable (X_τ, Y_τ) , which takes on the values (x_τ, y_τ) . Whenever an offer is presented with values (x_τ, y_τ) the decision-maker is asked to decide whether he accepts the offer and terminates the process, or rejects the offer and continues his search process. We assume that any decision must be made immediately after the arrival time of an offer — hesitation is not permitted. Suppose that the offers arrive in a Poisson manner with arrival rate λ . It is assumed that if τ_1, τ_2, \dots be the realized arrival times of offers, (X_{τ_i}, Y_{τ_i}) , $i = 1, 2, \dots$, are iid non-negative valued bivariate random variables each with the common cdf $H(x, y)$. Let $F(x)$ and $G(y)$ be the marginal cdf of X and Y , respectively, such that $0 < E(X) = \mu < \infty$ and $0 < E(Y) = \nu < \infty$. Extension of the theory to the three-or-more dimensional random variables is immediate at least conceptually.

If one acts in disregard of the values of Y_i 's and wants to maximize the expected value of the observation X accepted, then his problem is solved as follows (See, Karlin [6], Albright [1] and Sakaguchi [9]): Let

$u^0(t)$ = the expected payoff obtainable by following an optimal policy under the condition that t units of time remain before the deadline and any offer has not been accepted previously.

Then $u^0(t)$ satisfies the differential equation

$$(1) \quad u^{0'}(t) = \lambda T_F(u^0(t)), \quad u^0(0) = 0$$

where $T_F(z)$ is the mean-shortage function defined by

$$(2) \quad T_F(z) \equiv E[(X - z)^+] \equiv \int_z^\infty (x - z)dF(x) = \int_z^\infty (1 - F(x))dx.$$

An optimal policy is as follows: Whenever an offer has just arrived with values (x, y) at the instant with time t remaining,

$$(3) \quad \left\{ \begin{array}{l} \text{Accept} \\ \text{Reject} \end{array} \right\} \text{ the offer, if } x \left\{ \begin{array}{l} \geq \\ < \end{array} \right\} u^0(t) .$$

We find that (1) has a unique solution, which is non-negative, concave and non-decreasing.

Similarly if one acts in disregard of the observations of X_i 's and wants to maximize the observation of the Y accepted, define $v^0(t)$, similarly as in $u^0(t)$, interchanging the roles of X and Y . Then $v^0(t)$ satisfies

$$(1') \quad v^{0'}(t) = \lambda T_G(v^0(t)) , \quad v^0(0) = 0 ,$$

and an optimal policy is described by (3), with x and $u^0(t)$, replaced by y and $v^0(t)$, respectively.

Our main concern in the problem we want to discuss in the present paper is to find how to stop optimally, in some appropriate sense, if we cannot be in disregard of any one variable and have a think of both of X and Y with equal importance. An outline of the paper is as follows : In Section 2 the concept of equilibrium neutral functions is introduced, and by using it an explicit solution of the problem is derived through finding a unique solution of some simultaneous differential equations. The reduction to a non-cooperative non-zero-sum differential game is suggested. In Section 3 some examples are included to illustrate the computations required by an "equilibrium neutral strategy".

2. Equilibrium Neutral Strategy.

We shall consider a class of stopping policies in which the decision-maker has a pair of "neutral" functions $u(t)$ and $v(t)$, in the sense that his search process is terminated by accepting the first offer such that $X_\tau \geq u(t)$ and $Y_\tau \geq v(t)$, where t is the time remaining before the deadline at the instant of the arrival of the τ -th offer.

In what follows in this paper, we occasionally use the term "time", which means the future time remaining before the deadline. No confusion will occur by this. Let $M_1[t|u(\cdot), v(\cdot)]$ be the expected payoff from the observations of X 's under the condition that any offer has not been accepted previously by time t , and a pair of neutral functions $u(\cdot)$ and $v(\cdot)$ is employed thereafter. Let $M_2[t|u(\cdot), v(\cdot)]$ be defined similarly for Y .

Now borrowing the concept of the equilibrium from non-cooperative game theory, we want to find a pair of neutral functions $u^*(\cdot)$ and $v^*(\cdot)$, such that a unilateral departure from this pair by either $u(\cdot)$ or $v(\cdot)$ will result in a lower payoff. More precisely, for any $t > 0$.

$$(4) \quad \begin{aligned} M_1[t|u^*(\cdot), v^*(\cdot)] &= \max_{u(\cdot) \text{ on } [0,t]} M_1[t|u(\cdot), v^*(\cdot)] , \\ M_2[t|u^*(\cdot), v^*(\cdot)] &= \max_{v(\cdot) \text{ on } [0,t]} M_2[t|u^*(\cdot), v(\cdot)] . \end{aligned}$$

A pair of functions $u^*(\cdot)$ and $v^*(\cdot)$ is said to constitute an equilibrium neutral strategy if it satisfies (4).

For any $t > 0$, let $F[Y \geq v(t)]$ be the conditional cdf of X given that $Y \geq v(t)$, and similarly $G[X \geq u(t)]$, the conditional cdf of Y given that $X \geq u(t)$. Consider the simultaneous differential equations

$$(5) \quad \begin{aligned} u'(t) &= \text{Pr}\{Y \geq v(t)\} T_{F[Y \geq v(t)]}(u(t)) , \\ v'(t) &= \text{Pr}\{X \geq u(t)\} T_{G[X \geq u(t)]}(v(t)) , \end{aligned}$$

with the initial conditions $u(0) = v(0) = 0$, and we shall assume that our bivariate distribution $H(x, y)$ is such that the equations have a unique solution. Then we prove the following :

Theorem Under the above mentioned assumption, let $(u^*(\cdot), v^*(\cdot))$ be the unique solution of Eqs (5) with $u(0) = v(0) = 0$. Then this is an equilibrium neutral strategy. Moreover we have, for any $t \geq 0$,

$$(6) \quad \begin{aligned} M_1[t|u^*(\cdot), v^*(\cdot)] &= u^*(t) , \\ M_2[t|u^*(\cdot), v^*(\cdot)] &= v^*(t) . \end{aligned}$$

Proof For any pair of neutral functions $u(\cdot)$ and $v(\cdot)$, define

$$(7) \quad \begin{aligned} M_1^*[t|v(\cdot)] &\equiv \max_{\bar{u}(\cdot) \text{ on } [0,t]} M_1[t|\bar{u}(\cdot), v(\cdot)] , \\ M_2^*[t|u(\cdot)] &\equiv \max_{\bar{v}(\cdot) \text{ on } [0,t]} M_2[t|u(\cdot), \bar{v}(\cdot)] . \end{aligned}$$

Then considering what can happen in some small time interval Δt and employing the Principle of Optimality in dynamic programming, we have the expression

$$\begin{aligned}
 M_1^*[t|v(\cdot)] &= \lambda \Delta t \max_{u_0} \left[\int_{u_0}^{\infty} x \, dx \int_{v(t-\Delta t)}^{\infty} h(x, y) \, dy \right. \\
 &\quad \left. + \left\{ 1 - \int_{u_0}^{\infty} dx \int_{v(t-\Delta t)}^{\infty} h(x, y) \, dy \right\} M_1^*[t - \Delta t | v(\cdot)] \right] \\
 &\quad + (1 - \lambda \Delta t) M_1^*[t - \Delta t | v(\cdot)] + o(\Delta t) ,
 \end{aligned}$$

where $h(x, y)$ is the pdf of the cdf $H(x, y)$. If $H(x, y)$ is discontinuous, a slight modification will be needed.

Rearranging terms, dividing both sides by Δt , and taking the limit as $\Delta t \rightarrow 0$, we obtain

$$\begin{aligned}
 (8) \quad \frac{d}{dt} M_1^*[t|v(\cdot)] &= \lambda \max_{u_0} \int_{u_0}^{\infty} \{x - M_1^*[t|v(\cdot)]\} dx \int_{v(t)}^{\infty} h(x, y) \, dy \\
 &= \lambda \int_{M_1^*[t|v(\cdot)]}^{\infty} \{x - M_1^*[t|v(\cdot)]\} dx \int_{v(t)}^{\infty} h(x, y) \, dy.
 \end{aligned}$$

Note that $M_1^*[t|v(\cdot)]$ is equal to the optimal choice of the neutral value for X at time t , provided that the neutral function $v(\cdot)$ for Y will be employed thereafter.

Also a similar argument as in above gives

$$(8') \quad \frac{d}{dt} M_2^*[t|u(\cdot)] = \lambda \int_{M_2^*[t|u(\cdot)]}^{\infty} \{y - M_2^*[t|u(\cdot)]\} dy \int_{u(t)}^{\infty} h(x, y) \, dx,$$

and that $M_2^*[t|u(\cdot)]$ equals the optimal choice of the neutral value for Y at time t , provided that the neutral function $u(\cdot)$ for X will be employed thereafter.

Let $(u^*(\cdot), v^*(\cdot))$ be a unique solution of the simultaneous differential equations

$$\begin{aligned}
 (9) \quad u'(t) &= \lambda \int_{u(t)}^{\infty} (x - u(t)) \, dx \int_{v(t)}^{\infty} h(x, y) \, dy , \\
 v'(t) &= \lambda \int_{v(t)}^{\infty} (y - v(t)) \, dy \int_{u(t)}^{\infty} h(x, y) \, dx .
 \end{aligned}$$

with $u(0) = v(0) = 0$. This is equivalent to (5) by interchanging the order of the integrations. Then, by (8) and (8'), we find that

$$M_1^*[t \mid v^*(\cdot)] = u^*(t)$$

$$M_2^*[t \mid u^*(\cdot)] = v^*(t) .$$

Since a pair of neutral functions $u^*(\cdot)$ and $v^*(\cdot)$ satisfies (4), if and only if

$$M_1^*[t \mid v^*(\cdot)] = M_1[t \mid u^*(\cdot), v^*(\cdot)] ,$$

$$M_2^*[t \mid u^*(\cdot)] = M_2[t \mid u^*(\cdot), v^*(\cdot)] ,$$

it follows that $(u^*(\cdot), v^*(\cdot))$ is an equilibrium neutral strategy and (6) is true. This completes the proof of the theorem. \square

The above theorem implies the following important fact : the pair of functions $u(\cdot)$ and $v(\cdot)$ defined by the simultaneous differential equations (5), or equivalently, (9), plays two roles. (Hereafter we shall omit the asterisks in $u^*(t)$ and $v^*(t)$.) First it constitutes an equilibrium neutral strategy for the search process, and secondly, $(u(t), v(t))$ is the equilibrium expected payoffs for a play of the remaining period t . In a later section we shall show that some elementary bivariate distributions give, relatively easily, explicit solutions of the simultaneous differential equations (5).

Remark Our derivation of the equations (8) and (8') suggests that the problem we are considering is nothing but a non-cooperative non-zero-sum differential game, in which the payoffs to player 1 and player 2 are $M_1(T)$ and $M_2(T)$, respectively, the differential equations are

$$M_1'(t) = \lambda \int_{u(t)}^{\infty} (x - M_1(t)) dx \int_{v(t)}^{\infty} h(x, y) dy , \quad M_1(0) = 0 ,$$

$$M_2'(t) = \lambda \int_{v(t)}^{\infty} (y - M_2(t)) dy \int_{u(t)}^{\infty} h(x, y) dx , \quad M_2(0) = 0 .$$

and the controls satisfy, in $0 \leq t \leq T$,

$$0 \leq u(t) \leq \sup \{x \mid F(x) < 1\} ,$$

$$0 \leq v(t) \leq \sup \{y \mid G(y) < 1\} .$$

For the detailed expository discussions about the differential games with a special emphasis on their applications to economic analysis, see, for example, Intriligator's book [5; Chapter 15]. Also, one of the earliest and hence the easiest example of zero-sum differential games is, as far as the present author knows, Zachrisson's tank duel [10]. Here the problem has

the form :

$$\left\{ \begin{array}{l} M(T) \rightarrow \max_{u(\cdot)} \min_{v(\cdot)} \\ \text{subject to} \\ M'(t) = \Phi(t, M(t), u(t), v(t)), M(0) = c \\ \text{each of the controls being restricted in some given interval of} \\ \text{real numbers during } 0 \leq t \leq T. \end{array} \right.$$

To prove the following corollaries, it is worth stating the properties of the function $T_F(z)$ defined by (2). It is continuous, non-negative, convex and strictly decreasing on the set where it is positive. Furthermore $T_F(z) \geq \mu - z$, $\lim_{z \rightarrow -\infty} \{T_F(z) - (\mu - z)\} = 0$ (where $\mu \equiv EX \equiv \int_0^\infty x dF(x)$), $\lim_{z \rightarrow \infty} T_F(z) = 0$, and $T_F'(z) = -(1-F(z))$.

Corollary 1 Each of $u(t)$ and $v(t)$ is non-decreasing and concave in $t \geq 0$.

Proof. The proof will be shown for $u(t)$ only. Non-decreasing property is evident from (5) and non-negativity of the mean-shortage function. Also (9) gives

$$\begin{aligned} \lambda^{-1} u''(t) &= \{u'(t) \frac{\partial}{\partial u} + v'(t) \frac{\partial}{\partial v}\} \cdot \int_u^\infty (x-u) dx \int_v^\infty h(x,y) dy \\ &= -u'(t) \int_{u(t)}^\infty dx \int_{v(t)}^\infty h(x,y) dy - v'(t) \int_{u(t)}^\infty (x-u(t)) h(x,v(t)) dx, \end{aligned}$$

in which all of the two integrals and the two derivatives are non-negative. This proves the concavity of $u(t)$.

Corollary 2 Assume that X and Y are mutually independent. Let $u^0(t)$ denote the expected payoff in an optimal play for the planning period t based on the observations of X_i 's only, in disregard of Y_i 's. Define $v^0(t)$ similarly for Y_i 's only, in disregard of X_i 's. Then we have $u(t) \leq u^0(t)$ and $v(t) \leq v^0(t)$, for any $t \geq 0$.

Proof. If X is independent of Y , then (5) becomes

$$(10) \quad \begin{aligned} u'(t) &= \lambda \Pr. \{Y \geq v(t)\} T_F(u(t)), \\ v'(t) &= \lambda \Pr. \{X \geq u(t)\} T_G(v(t)). \end{aligned}$$

Since $u^0(t)$ and $v^0(t)$ satisfies the differential equations (1) and (1') respectively, the assertion of the corollary follows.

3. Examples

The behaviors of the equilibrium neutral strategy $(u(t), v(t))$ depend, on the form of the distribution of the observation, especially on the shapes of upper tails. In this section we show four simple examples which may help understanding the theory in cases of three distributions: Bernoulli, uniform and exponential.

Let $F[y]$ denote, the conditional cdf of X given y , and $G[x]$, that of Y given x . Sometimes it is convenient to rewrite (5) as

$$(5') \quad \begin{aligned} u'(t) &= \lambda \int_{v(t)}^{\infty} T_{F[y]}(u(t)) dG(y) , \\ V'(t) &= \lambda \int_{u(t)}^{\infty} T_{G[x]}(v(t)) dF(x) . \end{aligned}$$

These expressions will be used in the following examples.

Example 1 Bivariate Bernoulli distribution. Let $0 \leq a_1 < a_2$. If (X,Y) is a bivariate Bernoulli random variable with marginal parameters p, q , where $0 < p, q < 1$, and coefficient of correlation ρ , then the probability function on (X,Y) is given by:

	$Y = a_1$	$Y = a_2$	
$X = a_1$	$\bar{p} \bar{q} + d$	$\bar{p} q - d$	$\bar{p} (\equiv 1-p)$
$X = a_2$	$p \bar{q} - d$	$p q + d$	p
	$\bar{q} (\equiv 1-q)$	q	

where $d = \rho(\bar{p}\bar{q}\bar{q})^{1/2}$ and $|\rho| \leq 1$. (Hamdan and Martinson [4]) For this distribution we have

$$(11) \quad \begin{aligned} T_{F[y=a_1]}(u) &= \begin{cases} (\bar{p}+d/\bar{q})a_1 + (p-d/\bar{q})a_2 - u , & 0 \leq u \leq a_1 \\ (p-d/\bar{q})(a_2-u) , & a_1 \leq u \leq a_2 \\ 0 , & u > a_2 \end{cases} \\ T_{F[y=a_2]}(u) &= \begin{cases} (\bar{p}-d/q)a_1 + (p+d/q)a_2 - u , & 0 \leq u \leq a_1 \\ (p+d/q)(a_2-u) , & a_1 \leq u \leq a_2 \\ 0 , & u > a_2 \end{cases} \end{aligned}$$

and $\int_v^\infty T_{F[y]}(u)dG(y)$ is, as a function of u and v ,

	$0 \leq u \leq a_1$	$a_1 < u \leq a_2$	$u > a_2$
$0 \leq v \leq a_1$	$\bar{p}a_1 + pa_2 - u$	$p(a_2 - u)$	0
$a_1 < v \leq a_2$	$(\bar{p}q - d)a_1 + (pq + d)a_2 - qu$	$(pq + d)(a_2 - u)$	0
$v > a_2$	0	0	0

which is continuous in u , but discontinuous in v .

For simplicity we set $p=q$. Then, by symmetry, $u(t) = v(t)$, and the pair of differential equations (5) reduces to a single

$$(12) \quad \lambda^{-1}u'(t) = \begin{cases} \bar{p}a_1 + pa_2 - u(t), & \text{if } 0 < u(t) < a_1 \\ (p^2 + d)(a_2 - u(t)), & \text{if } a_1 \leq u(t) \leq a_2 \\ 0, & \text{if } u(t) > a_2. \end{cases}$$

Integration gives

$$(13) \quad u(t) = \begin{cases} (\bar{p}a_1 + pa_2)(1 - e^{-\lambda t}), & 0 \leq t \leq t_1 \\ a_2 - (a_2 - a_1)e^{-\lambda(p^2 + d)(t - t_1)}, & t > t_1 \end{cases}$$

where t_1 is given by a unique root of $u(t) = a_1$, i.e.,

$$(14) \quad t_1 = \frac{1}{\lambda} \log \frac{\bar{p}a_1 + pa_2}{(a_2 - a_1)p}.$$

The equilibrium neutral strategy at the decision instant $(t; x, y)$ is now apparant. If the observation is other than $x=y=a_2$, then the decision is:

$$\left\{ \begin{array}{l} \text{Accept} \\ \text{Reject} \end{array} \right\} \text{ the observation, if } t \left\{ \begin{array}{l} \leq \\ > \end{array} \right\} t_1.$$

If the observation is $x=y=a_2$, then accept it independently of t .

Furthermore note that, if we disregard one variable of the bivariate (X, Y) , we have $T_F(u)$ given by (11) with $d=0$, and Eq. (1) becomes

$$\lambda^{-1} u^{0'}(t) = \begin{cases} \bar{p}a_1 + pa_2 - u^0(t), & \text{if } 0 < u(t) < a_1 \\ p(a_2 - u^0(t)), & \text{if } a_1 \leq u(t) \leq a_2 \\ 0, & \text{if } u(t) > a_2, \end{cases}$$

giving, after integration,

$$u^0(t) = \begin{cases} (\bar{p}a_1 + pa_2)(1 - e^{-\lambda t}), & 0 \leq t \leq t_1 \\ a_2 - (a_2 - a_1)e^{-\lambda p(t-t_1)}, & t > t_1 \end{cases}$$

with t_1 defined by (14). Since $|d| \leq p\bar{p}$ one easily find that $u(t) \leq u^0(t)$, although X and Y are dependent each other.

Example 2. Bivariate uniform distribution. A class of bivariate pdf's with given marginal pdf's $f(x)$ and $g(y)$ is given by

$$(15) \quad h(x,y) = f(x)g(y) \{1 + \gamma(1-2F(x))(1-2G(y))\}$$

where F and G are the corresponding marginal cdf's and γ is an arbitrary constant with $-1 \leq \gamma \leq 1$ (Gumbel [2]). It is easy to check that the bivariate cdf is

$$(16) \quad H(x,y) = F(x)G(y) \{1 + \gamma(1-F(x))(1-G(y))\}$$

and that x is independent of y if and only if $\gamma=0$. This class of bivariate distributions is theoretically important because of its simple form and the fact that the constant γ actually measures the degree of dependency between the component variables, independently of $f(\cdot)$ and $g(\cdot)$ (Sakaguchi [7]).

For this class of bivariate distributions it is easy (Sakaguchi [8]) to obtain

$$(17) \quad \int_v^\infty T_{F[y]}(u) dG(y) = (1-G(v)) [T_F(u) - \gamma G(v) \{T_F(u) - T_{F^2}(u)\}],$$

where F^2 is the cdf of the maximum of the two iid r.v., each with cdf F .

We also have the similar expression for $\int_u^\infty T_{G[x]}(v) dF(x)$.

Now for bivariate uniform distribution put $F(x) = G(x) = x$ for $0 \leq x \leq 1$. Substituting this into (17) and considering symmetry, we find that the simultaneous differential equations (5') reduce to a single

$$(18) \quad u'(t) = \lambda(1-u)^3 \left\{ \frac{1}{2} + \gamma \left(\frac{1}{6}u + \frac{1}{3}u^2 \right) \right\}, \quad u(0) = 0.$$

and, of course, $u(t) \equiv v(t)$.

The differential equation (18) has, if $\gamma=0$ (independence), a unique solution

$$(19) \quad u(t) = 1 - (1 + \lambda t)^{-1/2}, \quad t \geq 0.$$

This function is concavely increasing from 0 at $t=0$ and approaches to 1 as $t \rightarrow \infty$. One easily find that $u(t) \leq u^0(t)$, where $u^0(t) = \lambda t / (\lambda t + 2)$ is a unique solution of Eq. (1), with $T_F(z) = (1-z)^2/2$.

Example 3. Bivariate exponential distribution. Let

$$H(x,y) = F(x)G(y), \quad F(x) = 1 - e^{-x/\mu}, \quad G(y) = 1 - e^{-y/\nu}.$$

Then, since $T_F(u) = \mu e^{-u/\mu}$ etc., Eq. (10) in this case becomes

$$u'(t) = \lambda \mu e^{-\{u(t)/\mu + v(t)/\nu\}},$$

$$v'(t) = \lambda \nu e^{-\{u(t)/\mu + v(t)/\nu\}},$$

Integrating we get

$$u(t)/\mu = v(t)/\nu = \frac{1}{2} \log(1+2\lambda t).$$

One easily find that $u(t) \leq u^0(t)$ and $v(t) \leq v^0(t)$, where $u^0(t) = u \log(1+\lambda t)$ is a unique solution of Eq. (1), and $v^0(t) = v \log(1+\lambda t)$, that of Eq. (1').

Example 4. Mixed-type bivariate distribution. Let

$$h(x,y) = \begin{cases} \mu^{-1} e^{-x/\mu}, & \text{if } 0 \leq y \leq 1 \\ 0, & \text{if otherwise,} \end{cases}$$

that is, X is exponentially distributed with mean μ , and Y, which is independent of X, is uniformly distributed over the unit interval. Then Eq. (10) becomes

$$(20) \quad u'(t) = \begin{cases} \lambda(1-v(t)) \mu e^{-u(t)/\mu}, & \text{if } 0 \leq v(t) \leq 1 \\ 0, & \text{if } v(t) > 1 \end{cases}$$

$$(21) \quad v'(t) = \begin{cases} \frac{\lambda}{2} e^{-u(t)/\mu} (1-v(t))^2, & \text{if } 0 \leq v(t) \leq 1 \\ 0, & \text{if } v(t) > 1. \end{cases}$$

Integrating (20) we obtain

$$e^{u(t)/\mu} = 1 + \lambda \int_0^t (1-v(t_1)) dt_1,$$

which, combined with (21), gives an integral-differential equation

$$v'(t) = \frac{\lambda(1-v(t))^2}{2\{1 + \lambda \int_0^t (1-v(t_1))dt_1\}}.$$

With $V(t) \equiv \int_0^t (1-v(t_1))dt_1$, this becomes a second-order differential equation

$$v''(t) = -\frac{\lambda}{2} (v'(t))^2(1+V(t))^{-1}, \quad v(0) = 0, \quad v'(0) = 1.$$

Integrating twice we get

$$V'(t) = (1 + \lambda V(t))^{-1/2}, \quad \text{and} \quad (1 + \lambda V(t))^{3/2} = 1 + \frac{3}{2} \lambda t .$$

Hence we obtain

$$u(t)/\mu = \log(1 + \lambda V(t)) = \frac{2}{3} \log(1 + \frac{3}{2} \lambda t) ,$$

$$v(t) = 1 - V'(t) = 1 - (1 + \frac{3}{2} \lambda t)^{-1/3} .$$

4. Concluding Remark

As with any model, the model presented here is merely one abstraction of reality. There may be various way of treating the problem differently. One situation, indeed of interest, is the one where the objective of the decision-maker is to maximize the probability of "win" with a given bivariate distribution function. (The problem in the univariate case was already solved by Sakaguchi [9].) One must describe the model as follows : We refer to an observation which is the efficient one so far, that is, there is no observation greater than (in the bivariate sense) the present one among the previous observations, as a candidate. The event in which we accept a candidate which happens to be the efficient one through the whole planning horizon is called a "win". We are asked to find a stopping policy which maximizes the probability of win. We can derive some dynamic programming equation which will determine the optimal strategy, but the critical difficulty comes from the explosive nature of the underlying state space.

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