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AN APPROXIMATION METHOD USING CONTINUOUS MODELS FOR QUEUEING PROBLEMS

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ABSTRACT

Typical queueing problems of general arrival and general service systems are solved by an approximation method. That is, approximate formulae are given for the mean customer number in the G/G/1(∞) system, the mean customer number in the the G/G/1(m) system and the mean queue length in the G/G/s(∞) system.

The routine of the proposed method is as follows. The sequences of arrival and departure of customers are approximately replaced by normal stochastic processes. That is, the sequences are characterized by mean and variance only. Next, the diffusion equation is constructed as to the probability distribution of the customer number in the system. Using the solution of the equation, the aimed statistical value for the queue is calculated. Finally, the calculated value may be revised to agree with the exact solution on the area where it is known.

1. Introduction

In the analysis of queueing problems the tendency is to get exact solutions for simple models only. But it is questionable from the viewpoint of practical application. On the other hand, several approximation methods have been proposed (for example, [1], [3]). Recently, in the case of long waiting lines, as for example rush hour analysis, an approximation method applying continuous models has been developed [5]. In this paper, a more general method for queueing problems which have not-necessarily long waiting lines is proposed.

2. Mean Number of Customers in the G/G/1(∞) System

For the G/G/1(∞) system, let the mean of arrival numbers per unit time or the arrival rate be λ and its variance be Δ_i, the mean of departure numbers per unit time under the condition that the system is not empty or the service rate be μ and its variance be Δ_o. That is, considering $A(t)$ as the cumulative arrival in time t, which is considered as large, we assume that

$$\frac{E\{A(t)\}}{t} \simeq \lambda \quad \text{and} \quad \frac{Var\{A(t)\}}{t} \simeq \Delta_i$$

Similarly, for the departure process, under the same condition as that of arrival process, we assume that

$$\frac{E\{B(t)\}}{t} \simeq \mu \quad \text{and} \quad \frac{Var\{B(t)\}}{t} \simeq \Delta_o$$

where $B(t)$ is the cumulative departure in time t.

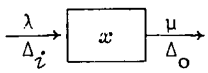

Fig. 1 The G/G/1(∞) System

We consider here the arrival and departure number as continuous quantities, and assume that the customer number in the system varies as stationary normal stochastic process (see [5]). So, for the probability density function $f(x)$ of the customer number x in the system, the following diffusion equation exists

(2.1) $$0 = -(\lambda - \mu)f(x) + \frac{1}{2}\{\Delta f(x)\} ,$$

here

$$\Delta \equiv \Delta_i + \Delta_o$$

Taking

$$a \equiv \frac{-2(\lambda - \mu)}{\Delta} ,$$

we get from equation (2.1)

$$0 = af(x) + f'(x)$$

and its solution is

(2.3) $$f(x) = ae^{-ax}.$$

For the Erlangian system $E_\ell/E_k/1(\infty)$, $\Delta_i = \frac{\lambda}{\ell}$ and $\Delta_o = \frac{\ell}{k}$.
So, we write $G_\ell/G_k/1(\infty)$ for the general arrival and general service system when

$$\Delta_i = \frac{\lambda}{\ell} \quad \text{and} \quad \Delta_o = \frac{\mu}{k}$$

Using $\rho \equiv \frac{\lambda}{\mu}$, we can write

(2.4) $$a = \frac{-2(\lambda - \mu)}{\frac{\lambda}{\ell} + \frac{\mu}{k}} = \frac{-2(\rho - 1)}{\frac{\rho}{\ell} + \frac{1}{k}}$$

The first approximation of the mean customer number L in the system becomes

(2.5) $$L_1 = \int_0^\infty xae^{-ax}dx = \frac{1}{a}$$

$$= \frac{\frac{\rho}{\ell} + \frac{1}{k}}{2(1 - \rho)},$$

which is effective for the case of $\rho \simeq 1$, i.e., a long waiting line, because the continuous state model is used instead of the integer state model and the relative error of L_1 is supposed to be of the order of $\frac{1}{L}$.

On the other hand, the exact value L has the characterisitics that

(2.6) $$L \simeq \rho \qquad \text{for } \rho \ll 1.$$

Hence, let us add supplementary terms of the order 1 to (2.5) and propose the second approximate formula, which is effective for $0 \leqq \rho < 1$, such as

(2.7) $$L_2 = \frac{\frac{\rho}{\ell} + \frac{1}{k}}{2(1 - \rho)} + C_0 + C_1 \rho$$

so that formula (2.7) has the property of (2.6). Expanding formula (2.5) in terms of ρ, we get for $\rho \ll 1$

$$L_1 = \frac{1}{a} = \frac{\frac{\rho}{\ell}+\frac{1}{k}}{2(1-\rho)} \simeq \frac{1}{2k} + \frac{\rho}{2}\left(\frac{1}{\ell}+\frac{1}{k}\right)$$

Comparing this expansion with formula (2.7), we get

$$C_0 = -\frac{1}{2k}$$
$$C_1 = -\frac{1}{2}\left(\frac{1}{\ell}+\frac{1}{k}\right) + 1 .$$

Therefore formula (2.7) becomes

$$(2.8) \qquad L_2 = \frac{\frac{\rho}{\ell}+\frac{1}{k}}{2(1-\rho)} - \frac{1}{2k} - \frac{\rho}{2}\left(\frac{1}{\ell}+\frac{1}{k}\right) + \rho$$
$$= \frac{\rho^2\left(\frac{1}{\ell}+\frac{1}{k}\right)}{2(1-\rho)} + \rho .$$

When $\ell = 1$, this result coincides with the exact formula, i.e., the Pollaczek-Khintchine's formula [4]

$$(2.9) \qquad L = \frac{\rho^2\left(1+\frac{1}{k}\right)}{2(1-\rho)} + \rho .$$

In the following Tables, the approximate values for two other systems, which are calculated from (2.8), are also compared with the corresponding values obtained in Page's book [6]. In the book the mean waiting time in units of mean service time is tabulated. So, we calculated the mean number in the system from it.

Table 1. Mean number in the system for $E_2/E_2/1(\infty)$

ρ	L_2	L (Page)	error
0.1	0.11	0.10	0.01
0.2	0.23	0.21	0.02
0.3	0.36	0.34	0.02
0.4	0.53	0.49	0.04
0.5	0.75	0.70	0.05
0.6	1.05	0.98	0.07
0.7	1.52	1.43	0.09
0.8	2.40	2.29	0.11
0.9	4.95	4.82	0.13

Table 2. Mean number in the system for $E_{10}/M/1(\infty)$

ρ	L_2	L (Page)	error
0.1	0.11	0.10	0.01
0.2	0.23	0.20	0.03
0.3	0.37	0.32	0.05
0.4	0.55	0.47	0.08
0.5	0.78	0.66	0.12
0.6	1.10	0.95	0.15
0.7	1.60	1.41	0.19
0.8	2.56	2.34	0.22
0.9	5.35	5.09	0.26

3. Mean Number of Customers in the G/G/1(m) System

In this case the customer number L in the system is limited to m or the length of queue cannot be longer than m-1.

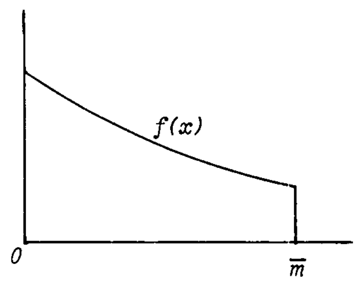

Fig. 2 Continuous distribution for G/G/1(m)

We take here the boundary condition

$$0 \leqq x \leqq \bar{m} \ . \tag{3.1}$$

Although the customer number is limited upto m, but for the continuous approximation we take the boundary value to be $\bar{m}$ which is a real number near to the integer m, so that we get a better approximate solution.

In this case, the diffusion equation is the same as (2.1), so we get the first approximate formula

(3.2)
$$L_1 = \frac{\int_0^{\bar{m}} xae^{-ax}dx}{\int_0^{\bar{m}} ae^{-ax}dx} = \frac{-\bar{m}e^{-a\bar{m}}}{1-e^{-a\bar{m}}} + \frac{1}{a}$$

where $\bar{m} = m$. The error of L_1 is supposed to be of the order 1.

On the other hand, the exact value L has the characteristics that

(3.3)
$$L \simeq \rho \quad \text{for} \quad \rho \ll 1,$$

(3.4)
$$L \simeq m \quad \text{for} \quad \rho \gg 1.$$

But for the two extreme values of ρ, i.e., $\rho \ll 1$ and $\rho \gg 1$, we have the approximate values for the two terms in the right hand side of the equation (3.2), which are given as bellow:

(3.5)
$$\frac{1}{a} \simeq \begin{cases} \frac{1}{2k} + \frac{\rho}{2}\left(\frac{1}{\ell} + \frac{1}{k}\right) & \text{for} \quad \rho \ll 1 \\ -\frac{1}{2\ell} & \text{for} \quad \rho \gg 1, \end{cases}$$

(3.6)
$$\frac{-\bar{m}e^{-a\bar{m}}}{1-e^{-a\bar{m}}} \simeq \begin{cases} 0 & \text{for} \quad \rho \ll 1 \\ \bar{m} & \text{for} \quad \rho \gg 1. \end{cases}$$

In the equation (3.6) we have considered $\bar{m}e^{-2k\bar{m}} \ll 1$ and $e^{2\ell\bar{m}} \gg 1$.

In our system, ρ is to be considered to vary from zero to ∞, hence, for the improvement of (3.2) the supplementary term cannot be $C_0 + C_1\rho$ as considered in (2.7), because $C_0 + C_1\rho \gg 1$ for $\rho \gg 1$. Instead we use the supplementary term of the order 1 as $C_0 + C_1\frac{\rho}{1+\rho}$, the second term of which has the property

$$\frac{\rho}{1+\rho} \begin{cases} < 1 & \text{for} \quad 0 \leqq \rho < \infty \\ \simeq \rho & \text{for} \quad \rho \ll 1 \\ \simeq 1 & \text{for} \quad \rho \gg 1. \end{cases}$$

So, we propose the second approximate formula as

(3.7)
$$L_2 = \frac{-\bar{m}e^{-a\bar{m}}}{1-e^{-a\bar{m}}} + \frac{1}{a} + C_0 + C_1\frac{\rho}{1+\rho}$$

and in order to satisfy the properties of (3.3) and (3.4) we get

(3.8) $$\overline{m} = m + \frac{1}{\ell} + \frac{1}{k} - 1\,,$$

(3.9) $$C_0 = -\frac{1}{2k}\,,$$

(3.10) $$C_1 = -\frac{1}{2\ell} - \frac{1}{2k}\,.$$

To examine the degree of approximation, we compare the value from the equation (3.7) with the exact value for the case M/M/1(m) [4] . In this case, for $\ell = 1$ and $k = 1$, we get

$$\overline{m} + m = 1, \quad C_0 = -\frac{1}{2} \quad \text{and} \quad C_1 = 1.$$

The comparison is shown in Fig. 3.

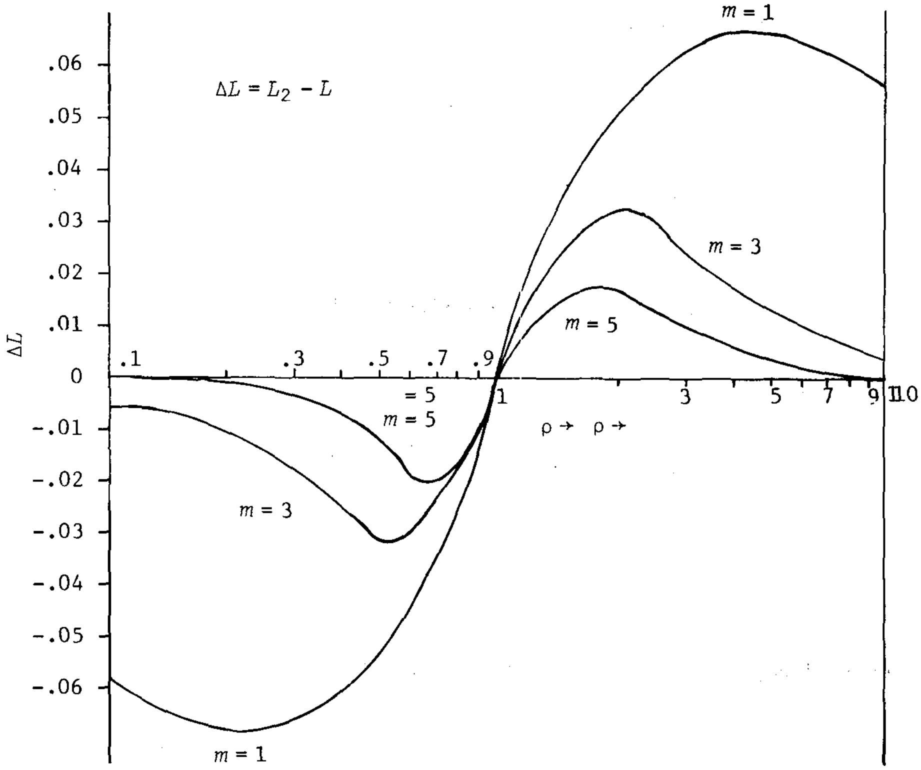

Fig. 3 Error of the approximate formula for the mean customer number in M/M/1(m) system

4. Mean Queue Length in the G/G/s(∞) System

Let us consider the queueing system with s parallel servers and no restriction for the waiting line. When $s \geqq 2$, the characteristics of the exact value, as that of (2.6), do not give enough information for the construction of the approximate formula. So, we propose a different method.

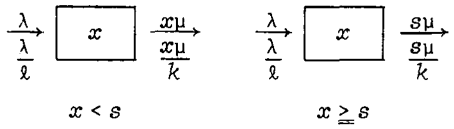

Fig. 4 G/G/s(∞) system

The mean value of departure is $x\mu$ and its variance is $\frac{x\mu}{k}$ when $x < s$ They are $s\mu$ and $\frac{s\mu}{k}$ when $x \geqq s$. (See Fig. 4.) The mean value of arrival is considered the same as earlier. For the case of $x < s$, the mean increasing rate of the customer in the system and its variance are

$$\lambda - x\mu \quad \text{and} \quad \frac{\lambda}{\ell} + \frac{x\mu}{k}$$

per unit time. So, for the equilibrium state we get the following diffusion equation [2].

$$(4.1) \qquad 0 = -(\lambda - x\mu)f(x) + \frac{1}{2}\{(\frac{\lambda}{\ell} + \frac{x\mu}{k})f(x)\}$$

and its solution

$$(4.2) \qquad f(x) = c(x + \frac{\alpha k}{\ell})^{2\alpha k(\frac{k}{\ell}+1) - 1} e^{-2kx}$$

$$\equiv cg(x),$$

where

$$\alpha = \frac{\lambda}{\mu}.$$

For the case of $x \geqq s$, the mean increasing rate of x and its variance are

$$\lambda - s\mu \quad \text{and} \quad \frac{\lambda}{\ell} + \frac{s\mu}{k}$$

So, we get the diffusion equation

$$(4.3) \qquad 0 = -(\lambda - s\mu)f(x) + \frac{1}{2}\{(\frac{\lambda}{\ell} + \frac{s\mu}{k})f(x)\}'$$

and its solution

(4.4) $$f(x) = cg(s)e^{-a(x-s)},$$

where $$a \equiv \frac{-2(\lambda - s\mu)}{\frac{\lambda}{\ell} + \frac{s\mu}{k}} = \frac{-2(\rho - 1)}{\frac{\rho}{\ell} + \frac{1}{k}}$$

and $$\rho \equiv \frac{\lambda}{s\mu} = \frac{\alpha}{s}.$$

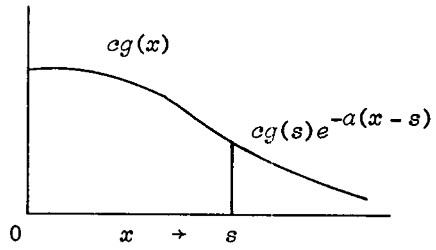

Fig. 5 Connection of the two solutions

In the above analysis the two solutions (4.2) and (4.4) are continuously connected at s. (See Fig. 5.) The constant c can be derived from the condition

$$1 = \int_0^s cg(x)dx + \int_s^\infty cg(s)e^{-a(x-s)}dx,$$

so $$c = \frac{1}{\int_0^s g(\xi)d\xi + \frac{g(s)}{a}}$$

Hence, the probability density function $\psi(y)$ of the waiting line y is

(4.5) $$\psi(y) = \frac{ae^{-ay}}{\frac{a}{g(s)}\int_0^s g(\xi)d\xi + 1}$$

and the approximate formula for the mean of the queue length Lq_1 is

(4.6) $$Lq_1 = \frac{\frac{1}{a}}{\frac{a}{g(s)}\int_0^s g(\xi)d\xi + 1}.$$

We compare the approximate value from equation (4.6) with its exact value for the case of M/M/s(∞) [4]. For this case $\ell = k = 1$, we get

(4.7) $$g(x) = (x+\alpha)^{4\alpha-1}e^{-2x}$$

and

(4.8) $$a = \frac{-2(\rho-1)}{1+\rho}$$

Their comparison is shown in Table 3.

Table 3. Mean queue length for M/M/s(∞)

	s = 2			s = 5		
ρ	Lq_1	exact	error	Lq_1	exact	error
0.1	0.01	0.00	0.01	0.00	0.00	0.00
0.2	0.04	0.02	0.02	0.00	0.00	0.00
0.3	0.10	0.06	0.04	0.01	0.01	0.00
0.4	0.21	0.15	0.06	0.05	0.04	0.01
0.5	0.41	0.33	0.08	0.15	0.13	0.02
0.6	0.78	0.68	0.10	0.38	0.35	0.03
0.7	1.47	1.34	0.13	0.91	0.88	0.03
0.8	3.00	2.84	0.16	2.25	2.22	0.03
0.9	7.86	7.67	0.19	6.90	6.86	0.04

The approximate values of queue length obtained by using equation (4.6) for M/D/s(∞), D/M/s(∞) and $E_2/E_2/s(\infty)$ are compared with the corresponding values which are tabulated in Page's book [6]. The formula of $g(x)$ and a for each case are mentioned at the top of each of the following Tables.

Table 4. Mean queue length for M/D/s(∞)

$$g(x) = e^{2(1 - \frac{x}{2\alpha})x}{}^{*},$$

$$\alpha = \frac{-2(\rho - 1)}{\rho}.$$

	$s = 2$			$s = 5$		
ρ	Lq_1	Lq(Page)	error	Lq_1	Lq(Page)	error
0.1	0.00	0.00	0.00	0.00	0.00	0.00
0.2	0.00	0.01	-0.01	0.00	0.00	0.00
0.3	0.00	0.03	-0.03	0.00	0.00	0.00
0.4	0.01	0.08	-0.07	0.00	0.02	-0.02
0.5	0.05	0.18	-0.13	0.01	0.08	-0.07
0.6	0.17	0.35	-0.18	0.05	0.20	-0.15
0.7	0.44	0.69	-0.25	0.21	0.47	-0.26
0.8	1.12	1.44	-0.32	0.73	1.16	-0.43
0.9	3.46	3.62	-0.16	2.89	3.26	-0.37

* It is calculated from equation (4.1) when $\ell = 1$ and $k = \infty$.

Table 5. Mean queue length for D/M/s(∞)

$$g(x) = x^{2\alpha-1}e^{-2x},$$

$$\alpha = -2(\rho - 1).$$

	$s = 2$			$s = 5$		
ρ	Lq_1	Lq(Page)	error	Lq_1	Lq(Page)	error
0.1	0.00	0.00	0.00	0.00	0.00	0.00
0.2	0.01	0.00	0.01	0.00	0.00	0.00
0.3	0.03	0.00	0.03	0.00	0.00	0.00
0.4	0.06	0.02	0.04	0.01	0.00	0.01
0.5	0.14	0.06	0.08	0.04	0.01	0.03
0.6	0.29	0.18	0.11	0.12	0.06	0.06
0.7	0.59	0.46	0.13	0.32	0.23	0.09
0.8	1.31	1.14	0.17	0.92	0.78	0.14
0.9	3.68	3.48	0.20	3.14	2.96	0.18

Table 6. Mean queue length for $E_2/E_2/s(\infty)$

$$g(x) = (x+\alpha)^{8\alpha-1} e^{-4x},$$

$$a = \frac{-4(\rho - 1)}{\rho + 1} .$$

	$s = 2$			$s = 5$		
ρ	Lq_1	Lq(Page)	error	Lq_1	Lq(Page)	error
0.1	0.00	0.00	0.00	0.00	0.00	0.00
0.2	0.00	0.00	0.00	0.00	0.00	0.00
0.3	0.01	0.01	0.00	0.00	0.00	0.00
0.4	0.04	0.05	-0.01	0.00	0.01	-0.01
0.5	0.10	0.12	-0.02	0.02	0.04	-0.02
0.6	0.23	0.27	-0.04	0.08	0.12	-0.04
0.7	0.52	0.58	-0.06	0.26	0.35	-0.09
0.8	1.22	1.33	-0.11	0.83	0.92	-0.09
0.9	3.58	3.60	-0.02	3.02	2.93	0.09

For $G/G/1(\infty)$, the approximate formula (4.6) can also be used. On the other hand, another approximate formula Lq_2 can be calculated from (2.8), which is as follows.

(4.9)
$$Lq_2 = L_2 - \rho = \frac{\rho^2(\frac{1}{\ell} + \frac{1}{k})}{2(1-\rho)} .$$

The comparisons among Lq_1, Lq_2 and the exact value are shown for $M/M/1(\infty)$ and $E_2/E_2/1(\infty)$ in the following Table.

Table 7. Mean queue length for $M/M/1(\infty)$ and $E_2/E_2/1(\infty)$

	$M/M/1(\infty)$			$E_2/E_2/1(\infty)$		
ρ	Lq_1	Lq_2	exact	Lq_1	Lq_2	Lq(Page)
0.1	0.05	the same as exact	0.01	0.01	0.01	0.00
0.2	0.12		0.05	0.02	0.03	0.01
0.3	0.23		0.13	0.05	0.06	0.04
0.4	0.41		0.27	0.10	0.13	0.09
0.5	0.68		0.50	0.21	0.25	0.20
0.6	1.12		0.90	0.39	0.45	0.38
0.7	1.90		1.63	0.74	0.82	0.73
0.8	3.52		3.20	1.50	1.60	1.49
0.9	8.47		8.10	3.93	4.05	3.92

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