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COMMUTATIVE TANDEM QUEUE WITH FINITE WAITING ROOM

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Abstract This paper considers tandem queuing system in which the order of performing service at two serial service stations can be changed. Interarrival distribution and service distribution of each station are assumed to be exponential.

For the above system, we derive the mean queue length and the mean availability per station for finite queue case. And some numerical results will be attached.

1. Introduction

In some assembly lines, there are many cases in which empty stations can be used regardless of ordered sequence of stations in order to increase efficiency of system. Thus in this paper, we shall consider commutative tandem queue in which the waiting room allowed ahead of the first station is finite. We assume that customers arrive according to a Poisson stream with parameter λ and each service time of two stations is exponentially distributed with parameter μ. In the case of which each service rate of two stations is different, the detailed balance equations for steady states are easily derived, but its analysis is complicated. So, for simplicity, we only concern with the case of the same service rate.

For this system, we derive the mean queue length in the queue and the availability per station. And we can compare the characteristic value of commutative tandem queue with those of ordinary one. We can also derive the values of the infinite case by tending the capacity of waiting room to infinity. These results are already obtained in our paper [7]. And in final part, some numerical results are attached.

We concern with the case of finite possible queue ahead of the first station and no queue between two stations. Arriving customers enter the first station if both stations are free, and they join the queue if both are occupied. They can first enter the second station if this is free and the first station is busy. The capacity of waiting-room is N. A customer who, upon his arrival, finds the system full departs never to return. If a customer has already completed service of two stations, then he emerges from this system. But if he has not completed by the other station and it is not free, he has to stay there, that is to say, this station is blocked, and when the other station has completed service, he is able to enter it.

It is also assumed that customers can transfer between stations instantaneously. The queuing discipline is first-come-first-served.

2. Mean Queue Length for Finite Waiting Room

The particular state of the system is labeled by the states of the queue length ahead of the first station and the states of the two stations. The state of the queue length is represented by the number of customers in queue. Each station can be empty (0), serving a customer who has not received (unfinished) service at the other station (u), serving a customer who finished service already at the other station (f), or blocked (b) when it has completed own service but the other is still occupied. It is convenient to express the probability for this system by the form P(.,.,.), where the first dot is the state of queue, the second is that of the first station and the third is that of the second station. For simplicity, the probability of no customers in the system is denoted by P(0). The detailed balance equations for steady states are as follows.

$$
\begin{aligned}
\lambda P(0) &= \mu P(0,f,0) + \mu P(0,0,f) \\
(\lambda+\mu)P(0,u,0) &= \lambda P(0) + \mu P(0,u,f) \\
(\lambda+\mu)P(0,f,0) &= \mu P(0,0,u) + \mu P(0,f,f) + \mu P(0,f,b) \\
(\lambda+\mu)P(0,0,u) &= \mu P(0,f,u) \\
(\lambda+\mu)P(0,0,f) &= \mu P(0,u,0) + \mu P(0,f,f) + \mu P(0,b,f) \\
(\lambda+2\mu)P(0,u,u) &= \lambda P(0,0,u) + \lambda P(0,u,0) + \mu P(1,f,u) + \mu P(1,u,f)
\end{aligned}
$$

$$(\lambda+2\mu)P(0,f,f) = \mu P(0,u,b) + \mu P(0,b,u)$$
$$(\lambda+2\mu)P(0,u,f) = \lambda P(0,0,f) + \mu P(1,f,f) + \mu P(1,b,f)$$
$$(\lambda+2\mu)P(0,f,u) = \lambda P(0,f,0) + \mu P(1,f,f) + \mu P(1,f,b)$$
$$(\lambda+\mu)P(0,b,u) = \mu P(0,u,u)$$
$$(\lambda+\mu)P(0,b,f) = \mu P(0,u,f)$$
$$(\lambda+\mu)P(0,u,b) = \mu P(0,u,u)$$
$$(\lambda+\mu)P(0,f,b) = \mu P(0,f,u)$$

(1)

$$(\lambda+2\mu)P(n,u,u) = \lambda P(n-1,u,u) + \mu P(n+1,f,u) + \mu P(n+1,u,f)$$
$$(\lambda+2\mu)P(n,f,f) = \lambda P(n-1,f,f) + \mu P(n,b,u) + \mu P(n,u,b)$$
$$(\lambda+2\mu)P(n,u,f) = \lambda P(n-1,u,f) + \mu P(n+1,f,f) + \mu P(n+1,b,f)$$
$$(\lambda+2\mu)P(n,f,u) = \lambda P(n-1,f,u) + \mu P(n+1,f,f) + \mu P(n+1,f,b)$$
$$(\lambda+\mu)P(n,b,u) = \lambda P(n-1,b,u) + \mu P(n,u,u)$$
$$(\lambda+\mu)P(n,b,f) = \lambda P(n-1,b,f) + \mu P(n,u,f)$$
$$(\lambda+\mu)P(n,u,b) = \lambda P(n-1,u,b) + \mu P(n,u,u) \qquad (1 \leqq n \leqq N-1)$$
$$(\lambda+\mu)P(n,f,b) = \lambda P(n-1,f,b) + \mu P(n,f,u)$$

$$2\mu P(N,u,u) = \lambda P(N-1,u,u)$$
$$2\mu P(N,f,f) = \lambda P(N-1,f,f) + \mu P(N,b,u) + \mu P(N,u,b)$$
$$2\mu P(N,u,f) = \lambda P(N-1,u,f)$$
$$2\mu P(N,f,u) = \lambda P(N-1,f,u)$$
$$\mu P(N,b,u) = \lambda P(N-1,b,u) + \mu P(N,u,u)$$
$$\mu P(N,b,f) = \lambda P(N-1,b,f) + \mu P(N,u,f)$$
$$\mu P(N,u,b) = \lambda P(N-1,u,b) + \mu P(N,u,u)$$
$$\mu P(N,f,b) = \lambda P(N-1,f,b) + \mu P(N,f,u) \quad .$$

Setting

(2)

$$P(n,1) = P(n,u,u)$$
$$P(n,2) = P(n,f,u) + P(n,u,f)$$
$$P(n,3) = P(n,f,f)$$
$$P(n,4) = P(n,u,b) + P(n,b,u)$$
$$P(n,5) = P(n,f,b) + P(n,b,f) \quad .$$

We get :

(3)

$$(2+\rho)P(n,1) = \rho P(n-1,1) + P(n+1,2)$$
$$(2+\rho)P(n,2) = \rho P(n-1,2) + 2P(n+1,3) + P(n+1,5)$$
$$(2+\rho)P(n,3) = \rho P(n-1,3) + P(n,4)$$
$$(1+\rho)P(n,4) = \rho P(n-1,4) + 2P(n,1)$$
$$(1+\rho)P(n,5) = \rho P(n-1,5) + P(n,2) \qquad (0 \leqq n \leqq N-1)$$

$$2P(N,1) = \rho P(N-1,1)$$
$$2P(N,2) = \rho P(N-1,2)$$
$$2P(N,3) = \rho P(N-1,3) + P(N,4)$$
$$P(N,4) = \rho P(N-1,4) + 2P(N,1)$$
$$P(N,5) = \rho P(N-1,5) + P(N,2) \quad ,$$

where $\rho = \lambda/\mu$, and

$$P(-1,1) = (\rho P(0) + P(0,2))/(\rho+1)$$
$$P(-1,2) = \rho P(0)$$
$$P(-1,3) = P(-1,4) = P(-1,5) = 0 \quad .$$

Now, we define the following five generating functions :

(4) $$G_i(z) = \sum_{n=0}^{N} z^n P(n,i) \qquad (i=1,2,\cdots\cdots,5) \ .$$

From the equation (3), we have

$$(2+\rho-\rho z)G_1(z) - \frac{1}{z} G_2(z) = \rho(1-z)z^N P(N,1) + \frac{\rho^2 P(0)+\rho P(0,2)}{\rho + 1} - \frac{P(0,2)}{z}$$

$$(2+\rho-\rho z)G_2(z) - \frac{2}{z} G_3(z) - \frac{1}{z} G_5(z) = \rho(1-z)z^N P(N,2) + \rho^2 P(0) - \frac{(\rho^3+2\rho^2)P(0)-P(0,2)}{z(\rho + 1)}$$

(5) $$(2+\rho-\rho z)G_3(z) - G_4(z) = \rho(1-z)z^N P(N,3)$$
$$2G_1(z) - (1+\rho-\rho z)G_4(z) = -\rho(1-z)z^N P(N,4)$$
$$G_2(z) - (1+\rho-\rho z)G_5(z) = -\rho(1-z)z^N P(N,5) \quad .$$

From (5), we can solve $G_i(z)$ as a function of $P(0)$, $P(0,2)$, $P(N,i)$ $(i=1,\cdots,5)$. To determine these probabilities, we may use the following :

i) The normalization equation is

(6) $$\sum_{i=1}^{5} G_i(1) + \sum_{j=1}^{5} P(-1,j) + P(0) = 1 \ .$$

Solving (5) with z=1 and substituting these results into (6), we get

(7) $$4\rho(\rho+1)\left(\sum_{i=1}^{5} P(N,i)\right) - P(0,2) - (\rho^2+5\rho+3)P(0) = (\rho+1)(4\rho-3) \quad .$$

ii) The generating functions $G_i(z)$ $(i=1,2,\cdots,5)$ are regular on z-plane.

Let $F(z)$ be the denominator of $G_i(z)$ $(i=1,2)$, then $(1+\rho-\rho z)F(z)$ is the denominator of $G_i(z)$ $(i=3,4,5)$. Now, $F(-1)>0$, $F(0)<0$, $F(1)<0$, $F(2/\rho)>0$, $F(5/\rho)<0$. Therefore, there exists at least one real root of $F(z)=0$ in each of intervals $(-1,0)$, $(1,2/\rho)$, $(2/\rho,5/\rho)$. Let z_0, z_1 and z_2 be one of roots in each interval such that $z_0<z_1<z_2$. Thus, we can write

(8) $$F(z) = -(1+\rho)\rho^4(z-z_0)(z-z_1)(z-z_2)(z^2+pz+q) \ .$$

Let z_3, z_4 be the roots of the equation $z^2+pz+q = 0$.

Then,

(9) $$F(z) = -(1+\rho)\rho^4(z-z_0)(z-z_1)(z-z_2)(z-z_3)(z-z_4) \ .$$

And $z_5 = \frac{1+\rho}{\rho}$ is also zero of the denominator of $G_i(z)$ $(i=3,4,5)$. If we denote the numerator of $G_5(z)$ as $H_5(z)$, we must have

(10) $$H_5(z_i) = 0 \quad (i=0,1,\cdots,5).$$

From (7), (10) and eliminating $P(0,2)$, we obtain

(11) $$A_iP(N,1)+B_iP(N,2)+C_iP(N,3)+D_iP(N,4)+E_iP(N,5)+F_iP(0)+I_i = 0 \qquad (i=0,1,\cdots,5) \ ,$$

where

$$A_i = 4\rho(\rho+1)(z_i^{N+1}-4\rho+(1+\rho-\rho z_i)(4\rho+\rho^2-\rho^2 z_i)z_i+4(\rho z_i-1))$$

$$B_i = \rho(\rho+1)(4(z_i^{N+1}-4\rho)+((1+\rho-\rho z_i)(4\rho+\rho^2-\rho^2 z_i)z_i+4(\rho z_i-1))((1-z_i)z_i^{N+1}+4))$$

$$C_i = \rho(\rho+1)(4(z_i^{N+1}-4\rho)+4((1+\rho-\rho z_i)(4\rho+\rho^2-\rho^2 z_i)z_i+4(\rho z_i-1)) +2\rho z_i(3+\rho-\rho z_i)(1-z_i)z_i^N)$$

$$D_i = 2\rho(\rho+1)(2(z_i^{N+1}-4\rho)+2((1+\rho-\rho z_i)(4\rho+\rho^2-\rho^2 z_i)z_i+4(\rho z_i-1))+\rho(1-z_i)Z_i^N)$$

$$E_i = \rho(\rho+1)(4(z_i^{N+1}-4\rho)+4((1+\rho-\rho z_i)(4\rho+\rho^2-\rho^2 z_i)z_i+4(\rho z_i-1)) +(\rho z_i(2+\rho-\rho z_i)(2z_i+\rho z_i-\rho z_i^2)(3+\rho-\rho z_i)+4(\rho z_i-1) +2(\rho z_i-2)(2z_i+\rho z_i-\rho z_i^2))(1-z_i)z_i^N)$$

$$F_i = (\rho+1)(12\rho+((1+\rho-\rho z_i)(4\rho+\rho^2-\rho^2 z_i)z_i+4(\rho z_i-1))(\rho^2 z_i-\rho^2-2\rho-3))$$

$$I_i = (\rho+1)(4\rho-3)(4\rho-(1+\rho-\rho z_i)(4\rho+\rho^2-\rho^2 z_i)z_i-4(\rho z_i-1)) \qquad (i=0,1,\cdots,5) \ .$$

We can solve for $P(0)$, $P(N,i)$ $(i=1,2,\cdots,5)$, from the system of linear equations (11). And if $H_5(z_i) = 0$ $(i=0,1,\cdots,5)$, it is easily seen from (5) that $H_j(z_i) = 0$ $(i=0,1,\cdots,4,\ j=1,2)$, and $H_j(z_i) = 0$ $(i=0,1,\cdots,5,\ j=3,4)$. Therefore (10) is necessary and sufficient for the regularities of $G_i(z)$.

Using these conditions, we can determine P(0), P(0,2) and P(N,i) (i=1,···,5), and we can obtain

$$\text{(12)} \qquad G_i(z) = \frac{H_i(z)}{F(z)} \quad (i=1,2), \qquad G_i(z) = \frac{H_i(z)}{(1+\rho-\rho z)F(z)} \quad (i=3,4,5),$$

where

$$F(z) = (\rho+1)(-\rho^4 z^5+(3\rho^4+7\rho^3)z^4-(3\rho^4+14\rho^3+18\rho^2)z^3$$
$$+(\rho^4+7\rho^3+19\rho^2+20\rho)z^2-(\rho^2+4\rho+8)z-4).$$

$H_i(z)$, (i=1,···,5) can be also expressed explicitly, but these expressions are lengthy. So we shall omit here these one.

The mean length in the queue L_c is

$$\text{(13)} \qquad L_c = G_1'(1) + G_2'(1) + G_3'(1) + G_4'(1) + G_5'(1)$$
$$= \left(\frac{7\rho+16N+24}{16(4\rho-3)(\rho+1)}\right)P(0,2) + \left(\frac{19\rho^3+(16N+47)\rho^2+(80N+181)\rho+(48N+136)}{16(4\rho-3)(\rho+1)}\right)P(0)$$
$$+ \left(\frac{128\rho^2-(64N+227)\rho+(48N+136)}{16(3-4\rho)}\right) + 2\rho P(N,1) + \frac{8\rho(1-\rho)}{(3-4\rho)}P(N,2)$$
$$+ \frac{21\rho-16\rho^2}{6-8\rho}P(N,3) + \frac{5\rho-16\rho^2}{6-8\rho}P(N,4) + \frac{9\rho-8\rho^2}{3-4\rho}P(N,5) \quad ,$$

and the mean availability per station A_c is

$$\text{(14)} \qquad A_c = \frac{1}{2}\left(2\sum_{i=1}^{3} G_i(1) + \sum_{i=4}^{5} G_i(1) + \sum_{j=1}^{2} P(-1,j)\right)$$
$$= \frac{3}{4} - \frac{P(0,2)}{4(\rho+1)} - \frac{\rho^2+5\rho+3}{4(\rho+1)} \quad .$$

The blocking probability P_{BC} is derived by the sum of $G_4(1)$ and $G_5(1)$, and we have

$$\text{(15)} \qquad P_{BC} = \frac{1}{2} - \frac{P(0,2)}{2(\rho+1)} - \frac{\rho^2+3\rho+1}{2(\rho+1)} \quad .$$

In the ordinary tandem queuing system, modifying the results by P.M. Morse [6] for the case of infinite queue, the mean length in queue L_0 is

$$\text{(16)} \qquad L_0 = \frac{1}{(2-3\rho)^2}((8\rho^3+4\rho^2)P(0)+(-16\rho^2+16\rho-(2\rho-3\rho^2)(3N+5))P(N,1,0)$$
$$+ (-22\rho^2+10\rho-(2\rho-3\rho^2)(3N+8))P(N,1,1)$$
$$+ (-25\rho^2+12\rho-(2\rho-3\rho^2)(3N+8))P(N,b,1) \quad ,$$

where,

(0) ; the both stations are empty ,

(N,1,0) ; the state of queue is N, the first station is serving and the second is empty,

(N,1,1) ; the state of queue is N, the first and the second stations are serving,

(N,b,1) ; the state of queue is N, the first is blocked and the second is serving.

The mean availability per station A_0 is

(17) $$A_0 = \frac{2}{3} - \frac{\rho+2}{3}P(0) \quad ,$$

and the blocking probabilty P_{BO} is

(18) $$P_{BO} = \frac{(2-3\rho)+(6\rho^2-\rho-2)P(0)}{3(2-3\rho)} \quad .$$

3. Special Case

In this section, we derive the value of queue length for the infinite case by tending N to ∞ .

By elementary calculation for (11), we have for $N\to\infty$, $P(N,1),\cdots,$ $P(N,5) \longrightarrow 0$. And it is easily seen from the normalization condition's results of finite and infinite case that $\sum_{i=1}^{5} P(N,1) \longrightarrow 0$ for $N \to \infty$.

Therefore,

(19) $$L_C \xrightarrow{N\to\infty} \frac{16+39\rho^2-38\rho}{4(3-4\rho)} - \frac{16+3\rho^2-6\rho}{4(3-4\rho)}P(0) \quad .$$

We can get this result directly from the equations of infinite queue case [7]. Then, in the infinite case, allowable utilization factor ρ of each queuing system are as follows;

Ordinary Case,
$0<\rho_0<2/3$,

Commutative Case,
$0<\rho_C<3/4$,

therefore, it follows that compared with ordinary case, the maximum utilization of commutative's is notably increased.

4. Some Numerical Results

We shall display the mean length in queue and the availability per station in order to compare commutative with ordinary tandem queue. For N=0, N=1, N=2 and $N \to \infty$, numerical values for the mean length in queue and the availability per station are given below.

TABLE 1.

ρ \ N	1		2		∞	
	L_C	L_O	L_C	L_O	L_C	L_O
0.1	0.003	0.010	0.004	0.013	0.004	0.013
0.2	0.018	0.040	0.027	0.061	0.031	0.073
0.3	0.048	0.084	0.082	0.151	0.116	0.228
0.4	0.090	0.136	0.172	0.278	0.324	0.600
0.5	0.139	0.192	0.290	0.426	0.821	1.600
0.6	0.191	0.248	0.424	0.582	2.208	6.092
0.7	0.242	0.301	0.563	0.731	9.878	

TABLE 2.

ρ \ N	0		1		2		∞	
	A_C	A_O	A_C	A_O	A_C	A_O	A_C	A_O
0.1	0.098	0.091	0.100	0.099	0.100	0.100	0.100	0.100
0.2	0.185	0.164	0.196	0.192	0.199	0.198	0.200	0.200
0.3	0.259	0.225	0.285	0.275	0.295	0.290	0.300	0.300
0.4	0.320	0.275	0.364	0.346	0.384	0.374	0.400	0.400
0.5	0.371	0.316	0.430	0.404	0.461	0.443	0.500	0.500
0.6	0.413	0.350	0.486	0.451	0.526	0.499	0.600	0.600
0.7	0.448	0.380	0.531	0.489	0.578	0.541	0.700	

From these tables, it is seen that the mean length of commutative system is smaller than the ordinary one for each ρ and the availability per station of commutative system is greater than or equal the ordinary's. Therefore, the efficiency of commutative system is better than the ordinary's.

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