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A SEQUENTIAL ALLOCATION GAME FOR TARGETS WITH VARYING VALUES

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Abstract The two-sided competitive situation where the players allocate ordnance, subject to resource constraints and limited mission time, to attack (or defense, for the opponent player) for the targets carrying some cargo of varying values is formulated as a two-sided time-sequential zero-sum game. A system of difference equations is derived which is in theory solvable recursively and determines the optimal strategies of the players. A special case of the game is completely solved. The continuous-time version of the problem is also discussed.

1. Introduction and Summary

This paper describes an application of two-person zero-sum sequential game in examining logistics allocation decisions in a combat setting as follows. Suppose that an Attacker (Player I) and a Defender (Player II), carrying k and ℓ weapons, respectively, are attemping to engage in a succesion of n battles or contests in obtaining n targets. The n targets arrive in sequential order, i.e., first target 1 appears, followed by target 2, etc. Associated with the j-th ($j = 1, \ldots, n$) target is a non-negative random variable X_j which takes on the value x_j. This j-th target is then referred to as a "type x_j" target. We assume that the X's are iid random variables with a known cdf $F(x)$. At each time a target appears, the players have to decide whether to expend a weapon or not in order to attack the target for player I, and to defend for II. We assume that both players know the values of k, ℓ, n and x, the realized type of the target. For each appearing targets with type x, the payoff to I,

$$\begin{array}{ccc} & & \text{II} \\ & & \begin{array}{cc} \text{Defend} & \text{Not defend} \end{array} \\ \text{I} & \begin{array}{l} \text{Attack} \\ \text{Not attack} \end{array} & \begin{bmatrix} \alpha\beta x & \alpha x \\ 0 & 0 \end{bmatrix} \end{array}$$

is paid by II, where $0 < \alpha,\beta < 1$ are known constants. α is the probability that a I's attack is successful, if he attacks the target. β is the probability of failure of II's defense if he defends the target I attacks. Since the number of weapons in hand is restricted for each player and the mission time is also limited, if a target arrives with relatively small x and relatively large n (mission time) remaining, the attacker, (and hence the defender too), will postpone expending a weapon and wait, expecting some more favorable opportunity may arrive in the future. Thus each player has to find an allocation policy which will indicate him what type targets he should attack (or defend) as a function of the mission time and number of weapons he has remaining on station. The problem belongs to a type of zero-sum-game version of one studied in some earlier works Albright [1], Donis and Pollock [4], Mastran and Thomas [6] and Sakaguchi [9, 10]. Moreover, our problem generalizes in a certain way the card game Goofspiel discussed by Ross [8].

There are, in fact, many situations both in and out of warfare that have similar nature of "opportunity analysis" in common.

An outline of the paper is as follows. In Section 2 we shall derive, by a dynamic programming formulation of the problem, a fundamental system of difference equations in time n to go and amount of ordnance k and ℓ, for each player at hand. This system of difference equations, which is in theory solvable recursively, determines the optimal strategies for the players. In Section 3 the game is completely solved in the case of the deterministic target value. In Section 4 the continuous-time version of the problem is discussed. A system of differential equations is given, which characterizes the optimal strategies of the players. Some concluding remarks are given in the final section.

2. Solution of the Sequential Game.

Let $\Gamma(n, k, \ell)$ denote the game described in the previous section. (n, k, ℓ) denotes the state of the system in which Attacker and Defender possess k and ℓ weapons, respectively, and they have mission time n to go. Then the normalized form of the game $\Gamma(n, k, \ell)$ has the matrix

$$\begin{array}{ccc} & & \text{II} \\ & \text{Defend} & \text{Not defend} \\ \text{I} \begin{array}{l} \text{Attack} \\ \text{Not attack} \end{array} & \multicolumn{2}{c}{\begin{bmatrix} \alpha\beta x \& \Gamma(n-1, k-1, \ell-1) & \alpha x \& \Gamma(n-1, k-1, \ell) \\ 0 \& \Gamma(n-1, k, \ell-1) & 0 \& \Gamma(n-1, k, \ell) \end{bmatrix}} \end{array}$$

if the first target appears with type x. The interpretation of $\alpha\beta x\&\Gamma(n-1, k-1, \ell-1)$ is that the system yields an immediate payoff $\alpha\beta x$ and then moves to the state $(n-1, k-1, \ell-1)$, given that the both players use their first pure strategy.

Let $V_n(k, \ell)$ represent the value of the game $\Gamma(n, k, \ell)$. Then $V_n(k, \ell)$ satisfies

$$(1)\quad V_n(k, \ell) = \int_0^\infty \mathrm{Val} \begin{bmatrix} \alpha\beta x + V_{n-1}(k-1, \ell-1) & \alpha x + V_{n-1}(k-1, \ell) \\ V_{n-1}(k, \ell-1) & V_{n-1}(k, \ell) \end{bmatrix} dF(x)$$

$$(1 \leqq k, \ell \leqq n-1 \; ; \; n \geqq 1)$$

with the initial condition $V_0(0, 0) = 0$. The notation val A for a matrix A is used for the value of the matrix game A.

In order to obtain the solution of the game, it is convenient to define the mean shortage function

$$(2)\qquad T_F(z) \equiv \int_z^\infty (x-z)dF(x) = \int_z^\infty (1-F(x))dx \; .$$

We assume that $0 < \mu \equiv E(X) = \int_0^\infty x dF(x) < \infty$, and hence $T_F(z)$ exists. This function has been known to play an important role in the area of opportunity analysis (see, for example, DeGroot [3; Chapter 13]), and is non-negative, convex and strictly decreasing on the set where it is positive. Furthermore $T_F(0) = \mu > 0$, $T_F(z) \geqq \mu - z$, $(0 \leqq z < \infty)$, and $\lim_{z\to\infty} T_F(z) = 0$.

Define

(3) $$S_F(z) = z + T_F(z) \; .$$

Then this function is convex, increasing, $\max(z, \mu) \leqq S_F(z) \leqq z + \mu$, and $\lim_{z\to\infty} (S_F(z) - z) = 0$.

Some immediate examples of $T_F(z)$ are : $T_F(z) = (a-z)^+$, for deterministic target values $X = a (> 0)$, w.p. 1. ; $T_F(z) = (z-a)^2/(2a)$, $(0 \leqq z \leqq a)$, for uniform distribution $F(x) = x/a$ $(0 \leqq x \leqq a)$; $T_F(z) = a^{-1}e^{-az}$, for exponential distribution $F(x) = 1-e^{-ax}$ $(x \geqq 0)$.

Now let $\{g_{n,i}\}_{1\leqq i\leqq n}$ be a triangular array of possitive numbers defined by the recurrence relations

(4)
$$g_{n,1} = S_F(g_{n-1,1}) \; , \quad (n \geqq 2 \; ; \; g_{1,1} = \mu)$$
$$g_{n,i} = S_F(g_{n-1,i}) - \sum_{j=1}^{i-1} (g_{n,j} - g_{n-1,j}) \; ,$$
$$(2 \leqq i \leqq n-1, \; n \geqq i+1, \; g_{n,n} = n\mu - \sum_{j=1}^{n-1} g_{n,j}) .$$

We shall note, after the following Proposition is proved, that g_{ii}'s are given in the definition (4) such that the evident conditions $V_n(n, 0) = n\alpha\mu$ and $V_n(n, n) = n\alpha\beta\mu$ are consistent with the relations (5b) $\sim$ (5d).

An immediate example of $g_{n,i}$'s is : For uniform distribution $F(x) = x$ $(0 \leqq x \leqq 1)$, $S_F(z) = \frac{1}{2}(1 + z^2)$, $(0 \leqq z \leqq 1)$ and hence

$$g_{n,1} = \frac{1}{2}(1 + g_{n-1,1}^2) \qquad (n \geqq 2 \; ; \; g_{1,1} = 1/2) \; ,$$
$$g_{n,i} = \frac{1}{2}(1 + g_{n-1,i}^2) - \frac{1}{2}(1 - g_{n-1,i-1})^2$$
$$(i \geqq 2, \; n \geqq i+1, \; g_{n,n} = (1/2)n - \sum_{j=1}^{n-1} g_{n,j}).$$

In obtaining $V_n(k, \ell)$ from (1) we need the following

Proposition 1. The boundary conditions for the system of recurrence relations (1) are given by

(5a) $$V_n(0, \ell) \equiv 0 \; , \qquad 0 \leqq \ell \leqq n$$

$$\text{(5b)}\qquad V_n(k,\ 0) = \alpha \sum_{i=1}^{k} g_{n,i}\ , \qquad 0 \leqq k \leqq n$$

$$\text{(5c)}\qquad V_n(n,\ \ell) = n\alpha\mu - \alpha(1-\beta) \sum_{i=1}^{\ell} g_{n,i}\ , \qquad 0 \leqq \ell \leqq n$$

$$\text{(5d)}\qquad V_n(k,\ n) = \alpha\beta \sum_{i=1}^{k} g_{n,i}\ , \qquad 0 \leqq k \leqq n\ .$$

<u>Proof.</u> (5a) is evident. For $0 < k < n$ and $\ell = 0$, we have

$$\text{(6b)}\qquad V_n(k,\ 0) = \int_0^\infty \max\{\alpha x + V_{n-1}(k-1,\ 0),\ V_{n-1}(k,\ 0)\}dF(x)$$

$$= V_{n-1}(k,\ 0) + \alpha T_F[\alpha^{-1}\{V_{n-1}(k,\ 0) - V_{n-1}(k-1,\ 0)\}]\ .$$

This difference equation has the solution (5b). In fact, substituting (5b) into the right-hand side of (6b), we get, by (4),

$$\alpha \sum_{i=1}^{k} g_{n-1,i} + \alpha T_F(g_{n-1,k}) = \alpha\{\sum_{i=1}^{k-1} g_{n-1,i} + S_F(g_{n-1,k})\} = \alpha \sum_{i=1}^{k} g_{n,i}\ ,$$

which is $V_n(k,\ 0)$.

For $0 < k < n$ and $\ell = n$, we have

$$\text{(6d)}\qquad V_n(k,\ n) = \int_0^\infty \max\{\alpha\beta x + V_{n-1}(k-1,\ n-1),\ V_{n-1}(k,\ n-1)\}dF(x)$$

$$= V_{n-1}(k,\ n-1) + \alpha\beta T_F[(\alpha\beta)^{-1}\{V_{n-1}(k,\ n-1) - V_{n-1}(k-1,\ n-1)\}].$$

Comparing (6d) with (6b), we find that (6d) has the solution (5d).

Finally, for $k = n$ and $0 < \ell < n$, we have

$$V_n(n,\ \ell) = \int_0^\infty \min\{\alpha\beta x + V_{n-1}(n-1,\ \ell-1),\ \alpha x + V_{n-1}(n-1,\ \ell)\}dF(x)$$

$$= \int_0^\infty [\alpha x + V_{n-1}(n-1,\ \ell) - \{\alpha(1-\beta)x + V_{n-1}(n-1,\ \ell) - V_{n-1}(n-1,\ \ell-1)\}^+]dF(x)$$

$$= \alpha\mu + V_{n-1}(n-1,\ \ell) - \alpha(1-\beta)T_F[\alpha^{-1}(1-\beta)^{-1}\{V_{n-1}(n-1,\ \ell-1) - V_{n-1}(n-1,\ \ell)\}]\ .$$

With $W_n(n,\ \ell) \equiv (1-\beta)^{-1}\{n\alpha\mu - V_n(n,\ \ell)\}$, this becomes

$$W_n(n, \ell) = W_{n-1}(n-1, \ell) + \alpha T_F[\alpha^{-1} W_{n-1}(n-1, \ell) - W_{n-1}(n-1, \ell-1)] .$$

Comparing this again with (6b), we find that $W_n(n, \ell) = \alpha \sum_{i=1}^{\ell} g_{n,i}$, and hence we obtain the solution (5c). This completes all the proof.

Thus we have reached to

Proposition 2. The value of the game, $V_n(k, \ell)$, satisfies the system of difference equations (1) with the boundary conditions (5a) $\sim$ (5d), where $g_{n,i}$'s are defined by (4). The optimal strategy for each player is that of the matrix game in the right-hand side of (1), if a target with type x arrives in state (n, k, ℓ).

The system of difference equations (1) with the boundary conditions (5a) $\sim$ (5d), which is in theory solvable recursively, determines the optimal strategies for the players. We shall see, however, that finding the solution of the game explicitly even for the most elementary cdf F(x)'s, seems to be hopeless. From (1) and (5a) $\sim$ (5d), we find, for n = 1,

$$V_1(0, \ell) = 0, \quad (\ell = 0, 1)$$

$$V_1(1, 0) = \alpha\mu, \quad V_1(1, 1) = \alpha\beta\mu ;$$

and, for n = 2,

$$V_2(0, \ell) = 0 , \quad (\ell = 0, 1, 2)$$

$$V_2(1, 0) = \alpha S_F(\mu), \quad V_2(1, 2) = \alpha\beta S_F(\mu)$$

$$V_2(2, 0) = 2\alpha\mu, \quad V_2(2, 1) = 2\alpha\mu - \alpha(1-\beta)S_F(\mu)$$

and finally

$$V_2(1, 1) = \int_0^\infty \mathrm{val} \begin{bmatrix} \alpha\beta x + V_1(0, 0) & \alpha x + V_1(0, 1) \\ V_1(1, 0) & V_1(1, 1) \end{bmatrix} dF(x)$$

$$= \alpha \int_0^\infty \mathrm{val} \begin{bmatrix} \beta x & x \\ \mu & \beta\mu \end{bmatrix} dF(x) .$$

Since

$$\mathrm{val} \begin{bmatrix} \beta x & x \\ \mu & \beta\mu \end{bmatrix} = \begin{cases} \beta\mu , & \text{if } 0 < x < \beta\mu \\ (1+\beta)\mu x/(\mu+x) , & \text{if } \beta\mu < x < \mu/\beta \\ \beta x , & \text{if } x > \mu/\beta \end{cases}$$

it follows that

$$V_2(1,1) = \alpha\beta\mu F(\beta\mu) + \alpha(1+\beta)\int_{\beta\mu}^{\mu/\beta} \frac{\mu x}{\mu+x} dF(x) + \alpha\beta\int_{\mu/\beta}^{\infty} x dF(x)$$

and, the optimal strategies for the players at state (2, 1, 1) are

Condition	For I	For II
$0 < x \leqq \beta\mu$	Not attack	Not defend
$\beta\mu < x < \mu/\beta$	Use the mixed strategy $\left\langle \frac{\mu}{\mu+x}, \frac{x}{\mu+x} \right\rangle$	Use the mixed strategy $\left\langle \frac{(x-\beta\mu)}{(1-\beta)(\mu+x)}, \frac{(\mu-\beta x)}{(1-\beta)(\mu+x)} \right\rangle$
$x \geqq \mu/\beta$	Attack	Defend

To go on further to n = 3 is almost prohibitive, even for the most elementary cdf's.

3. Deterministic Target Value.

Our problem gives an explicit and easy solution in the case of the deterministic target value, i.e., X = 1, with probability 1. The problem in this case is clearly an extension of the inspection game in Owen [7], and reduces to a variant of the infiltration game discussed in Thomas and Nisgav [13]. The fundamental recursive relation is given, from (1), by

$$(7) \qquad V_n(k, \ell) = \mathrm{val} \begin{bmatrix} \alpha\beta+V_{n-1}(k-1, \ell-1) & \alpha+V_{n-1}(k-1, \ell) \\ V_{n-1}(k, \ell-1) & V_{n-1}(k, \ell) \end{bmatrix},$$

$$(1 \leqq k, \ell \leqq n-1;\ n \geqq 1)$$

with the boundary conditions

$$V_n(0, \ell) \equiv 0, \qquad V_n(k, 0) = \alpha k,$$

$$V_n(k, n) = k\alpha\beta, \qquad V_n(n, \ell) = \ell\alpha\beta + (n-\ell)\alpha.$$

We note that, for each player, if he has n(or more) available weapons to expend in a n-days mission, then clearly he will use them all.

Proposition 3. The soluti to the difference equation (7) for the game $\Gamma_n(k, \ell)$, is

$$(8) \qquad V_n(k, \ell) = \alpha k\{1-(1-\beta)\ell/n\}.$$

The optimal mixed strategies $x_n^*(k, \ell)$ and $y_n^*(k, \ell)$, for player I and II , respectively, are given by

(9) $x_n^*(k, \ell) = \langle k/n,\ 1-k/n \rangle$,

(10) $y_n^*(k, \ell) = \langle \ell/n,\ 1-\ell/n \rangle$.

Proof Let $W_n(k, \ell) \equiv \{V_n(k, \ell) - \alpha k\}/\alpha(1-\beta)$. Then, from (7), we obtain

(11) $$W_n(k, \ell) = \text{val} \begin{bmatrix} -1+W_{n-1}(k-1, \ell-1) & W_{n-1}(k-1, \ell) \\ W_{n-1}(k, \ell-1) & W_{n-1}(k, \ell) \end{bmatrix} .$$

The proof of (8) $\sim$ (10) follows directly by substituting $W_n(k, \ell) = -k\ell/n$ into (11) and finding the value and the optimal strategies of the matrix game in the right-hand side.

This result generalizes Theorem 2.1 in [13], which is a special case where $k = 1$. Owen's inspection game in [7; Section V. 2] is also a special case where $k = \ell = 1$.

We conclude that in order to obtain the value of the game, each player must allocate his available weapons such that his probability of performing his action (i.e., attack for I, and defense for II) is equal to the ratio of the number of his weapons in hand to the total number of the remaining periods. That is, for each player, a uniform distribution over the remainder of the time period will provide him the value of the game. Note that the optimal strategy for each player does not depend on α, β, and the number of weapons available to his opponent.

Finally in this section we shall note that an interesting work by Maschler [4] also studied this type of sequential games in non-zero-sum-game version.

4. Random Arrival Times.

In this section, we shall derive the consequences of deleting the requirement that the number of targets, and their arrival times also, are deterministically known and fixed. That is, we will consider the sequential game, investigated in previous sections, in the situation where the targets arrive sequentially one by one and randomly in a Poisson manner during some given time period. Whenever a target arrives with value x each player is asked to decide whether he attacks (or defends) the target expending one unit of his weapon, or does not attack preserving his ordnance for his future opportunities. We shall assume that the targets arrive in a Poisson manner with arrival rate λ, and that any decision must be made immediately after the arrival time of a target — hesitation is not permitted. Thus the problem belongs to a type of two-sided-game version of one studied in some earlier works Albright [1], Donis and Pollock [4] Mastran and Thomas [6], and Sakaguchi [9, 10]. Also a related work is found in Sweat [12].

Let (t, k, ℓ) denote the state of the system in which Attacker and Defender possess k and ℓ weapons, respectively, and they have mission time t remaining. Let $V_{k,\ell}(t)$ denote the conditional value of the game, given any state (t, k, ℓ). Then considering what can happen in some small time interval Δt, we have the expression

$$V_{k,\ell}(t) = \lambda\Delta t \int_0^\infty \mathrm{val} \begin{bmatrix} \alpha\beta x+V_{k-1,\ell-1}(t-\Delta t) & \alpha x+V_{k-1,\ell}(t-\Delta t) \\ V_{k,\ell-1}(t-\Delta t) & V_{k,\ell}(t-\Delta t) \end{bmatrix} dF(x)$$
$$+ (1-\lambda\Delta t)V_{k,\ell}(t-\Delta t)+o(\Delta t),$$

if $k,\ell \geqq 1$. Rearranging terms, deviding both sides by Δt, and taking the limit as $\Delta t \to 0$, we obtain a system of differential equations

$$(12) \qquad V'_{k,\ell}(t) = \lambda \int_0^\infty \mathrm{val} \begin{bmatrix} \alpha\beta x+V_{k-1,\ell-1}(t) & \alpha x+V_{k-1,\ell}(t) \\ V_{k,\ell-1}(t) & V_{k,\ell}(t) \end{bmatrix} dF(x) - \lambda V_{k,\ell}(t).$$

with the initial conditions $V_{k,\ell}(0) = 0$, $(k,\ell = 0,1,2, \dots)$.

Define the sequence of functions $g_r(t)$'s by a system of differential equations

$$(13) \qquad \begin{aligned} g_1'(t) &= \lambda T_F(g_1(t)), \\ g_r'(t) &= \lambda\{T_F(g_r(t)) - T_F(g_{r-1}(t))\}, \end{aligned}$$

with the initial conditions $g_r(0) = 0$, $(r = 1,2, \dots)$. Clearly, (13) is a continuous-time analogue of (4). This sequence of functions was first introduced in this field of problems by Sakaguchi in [9] and exploited in some phases of applications in [10, 11]. In [9], the functions $g_r(t)$ are determined explicitly for the case when X has exponential distribution, and remarks are made for several other distributions.

Proposition 4. The boundary conditions for the system of differential equations (12) are given by

$$(14a) \qquad V_{0,\ell}(t) \equiv 0 ,$$

$$(14b) \qquad V_{k,0}(t) = \alpha \sum_{i=1}^{k} g_i(t).$$

Proof. (14a) is evident from the definition of $V_{0,\ell}(t)$. For $k > 0$ and $\ell = 0$, we have

$$\begin{aligned} V'_{k,0}(t) &= \lambda \int_0^\infty \max\{\alpha x + V_{k-1,0}(t),\ V_{k,0}(t)\}\, dF(x) - \lambda V_{k,0}(t) \\ &= \lambda\alpha T_F[\alpha^{-1}(V_{k,0}(t) - V_{k-1,0}(t))]. \end{aligned}$$

Hence letting

$$h_k(t) \equiv \alpha^{-1}(V_{k,0}(t) - V_{k-1,0}(t)), \qquad k \geq 1 ,$$

we obtain

$$\sum_{i=1}^{k} h_i'(t) = \lambda T_F(h_k(t)) ,$$

or equivalently (13) with $g_i(t)$'s replaced by $h_i(t)$'s. This completes the proof of (14b).
Thus we can also state

Proposition 5. The value of the game, $V_{k,\ell}(t)$, satisfies the system of differential equation (12) with the boundary conditions (14a) and (14b), where $g_i(t)$'s are defined by (13). The optimal strategy for each player is that of the matrix game in the right-hand side of (12), if a target with type x arrives in state (t, k, ℓ).

The system (12) $\sim$ (14) is in theory solovable recursively. But again we shall find that obtaining the solution explicitly seems to be hopeless even for the most elementary cdf $F(x)$'s.

5. Concluding Remark.

As with any model, the models presented here are mere abstractions of reality. Thus, the major gains provided are through insights from identifying and examining various relationships among operational parameters. There are a host of extensions that could be made to the present study. In principle, the situation where both sides can expend more than one weapon in salvo can be treated in a similar fashion as in Sections 2 and 4. One must describle the model such that, if i and j torpedos are expended by Attacker and Defender, respectively, for a target of value x, then the expected payoff to Attacker is $(1-q^i)\beta^j x$, where $p = 1-q$, $0 < p < 1$, is the single shot probability of hit. One must treat the fundamental recursive relationships (1) and (12) which involve $(k+1)\times(\ell+1)$ matrix games.

A much more difficult problem, but indeed one of interest, is the case where both sides can expend their ordnance in continuous amounts. In this case, if $0 \leqq \psi \leqq \phi$ and $0 \leqq \psi \leqq \Psi$ of ordnance are expended by Attacker and Defender, respectively, for a target of value x, then the expected payoff to attacker is $(1-e^{-r\phi})e^{-s\psi}x$, where r and s are given positive constants. Some partial differential equation describing the system will determine the optimal strategies, and will contain "continuous games on the square" (see, for example, Owen [7; Chapter IV]), in place of matrix games in (1) and (12). The problem begins to take the form of a two-sided time-sequential game of optimal allocation of search efforts. See Croucher [2], for the two-sided non-sequential deterministic-target-value version of this problem.

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