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MARKOV MAINTENANCE MODELS WITH CONTROL OF QUEUE

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Abstract Discrete time Markov maintenance models are coupled with the theory of control of queues. Each system has an operating machine, spare machines and a repair facility. A decision maker has the option of opening or closing the repair shop when there are machines waiting for repair service, as well as the option of repairing or leaving an operating machine. A two-dimensional control limit policy is defined, and sufficient conditions for the optimality of a two-dimensional control limit policy are obtained for each model.

1. Introduction

In this paper discrete time maintenance models are treated in the context of control of queues. Because of their wide applicability in the practical world, a number of authors have studied optimization problems for discrete time machine maintenance models. Derman [1] introduced the basic model of this type, and showed the optimality of a simple rule, called a control limit policy. Kolesar [5] and Kalymon [3] generalized the cost structure without changing the basic conclusion of the model. In 1973 Kao [4] introduced a semi-Markovian approach to Derman's model. According to its semi-Markovian nature, the repair time of a machine is no longer instantaneous but takes some random time, while the supply of new spare machines is unlimited. A joint replacement and stocking problem was considered by Derman and Lieberman [2], which was generalized by Ross[8].

Aside from Markov maintenance features, the models treated here necessarily become discrete time closed queueing systems since the supply of spares is limited and machines to be repaired form a queue in front of a repair facility. Torbett [9] investigated the optimal control of closed queueing systems, but his analysis is time continuous. Discrete time open queueing systems have been vaguely discussed (see Magazine [6], [7]). There appears to have been almost no research in optimization of discrete time closed queueing systems.

Consider the following discrete time machine maintenance model, whose mechanism is illustrated in Fig. 1. There is an operating machine and S ($S \geqq 1$) spare machines in the system. At the beginning of each period, an operating machine is classified as being in one of $I+1$ ($I \geqq 1$) states showing the degree of deterioration. 0 represents the best state, while I represents the failed state. A repair shop is in the system, and an operating machine can be sent to the repair shop for the repair work at any period. A machine sent to the repair shop must wait until all the machines which have already arrived at the repair shop are completely repaired.

At the beginning of each period, the decision maker has the option of opening or closing the repair shop, as well as the option of repairing or leaving an operating machine. Therefore, at each period, four actions are available. They are denoted as a_{LC}, a_{LO}, a_{RC} and a_{RO} respectively, where L, R, C and O mean to leave an operating machine in operation, to repair an operating machine, to close the repair service gate, and to open the repair service gate respectively. If there is no operating machine, only the option of opening (a_O) or closing (a_C) the service gate is available. Closing the repair service gate implies doing nothing if it has been closed. Similarly for the case of opening the repair service gate. The repair work can

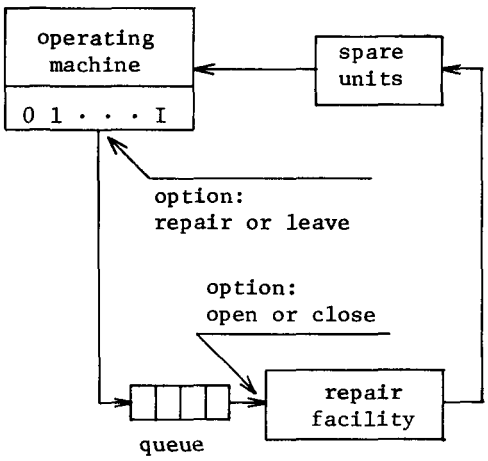

Figure 1. A machine maintenance system with control of queue

be performed only when the service gate is open. If a repair job on a machine is interrupted by the decision to close the gate, the rest of the repair work is postponed until the gate reopens. Repaired machines are available as spare machines.

If leaving an operating machine in operation is chosen, its state evolves from i to j in one period according to the transition probability $p_{ij} \geqq 0$. If repairing an operating machine is selected, it is immediately sent to the repair system, and is instantly replaced by a spare unit, if any are available. The new operating machine begins to operate just after replacement in its best condition.

The costs associated with the system are:

$A(i)$: operating cost for a machine of state i $(0 \leqq i \leqq I)$ per period.

$C(i)$: material cost for repairing a machine of state i $(0 \leqq i \leqq I)$.

$K(s,k)$: holding cost per period of s $(0 \leqq s \leqq S+1)$ machines in the repair system when the gate is closed $(k=0)$ or open $(k=1)$ at the beginning of the period just after the decision.

E : set up cost of opening a closed repair shop.

F : shut down cost of closing an open repair shop.

G : service cost of operating an open repair shop per period.

P : penalty cost assessed per period while no operating machine is available.

The objective function is the total expected α-discounted cost, and the structure of an optimal policy minimizing such a criterion is studied.

Before proceeding further, we give a couple of examples to clarify the applicability of this model in the practical world. Consider the following airplane repair problem for a privately owned flying school. Suppose the instructor owns two airplanes and a repair shop for repairing them. He teaches flying using one airplane at a time. He inspects the condition of the airplane in service periodically, and he classifies it as being in one of a finite number of states. He continues using the same airplane, until he judges that it should be repaired. Then the other airplane, if available, replaces it and begins to operate in its best condition. The previously used plane is sent to the repair shop, which is either open or closed. Repair work can be performed only when the repair shop is open, and a repaired plane will be ready for future use in its best condition. Keeping the repair shop always open may not be economical since he must pay salary to repairmen, and other costs to keep it open, even when no repair work is needed. Keeping

the repair shop closed too long is also undesirable since it may prevent planes from being available, thereby resulting in a loss of revenue. The problem of finding the best schedule for hiring repairmen and for replacing an operating airplane can be formulated as a model to be studied here.

Another example pertains to cleaning suits. Suppose a person has several suits. He wears one suit continuously until he finds that it requires cleaning, and then he changes the suit for a clean one in his wardrobe. Assume the degree of cleanliness of a suit is observable, and that the cost of wearing a suit is associated with its cleanliness. When he decides to change the suit to the clean one, it does not necessarily follow that he immediately sends the used suit for dry-cleaning since such an action may be laborious and time consuming. However, too infrequent visits to the cleaners may lead to the situation where he has no clean suits available. Assuming that a suit is available in its cleanest condition after dry-cleaning, the problem of determining the best schedule for suit changes and laundry visits can be formulated as a model to be studied.

2. Control Limit Policy with Respect to Operating Machine

When the repair service gate is open, we specify the repair time of a machine as follows: Let $q_{ss'}$ be the probability that s' machines are still in the repair system at the end of the period, given that s machines are in the repair system at the beginning of a period. Here the repair system consists of the repair shop and the queue.

Let $V_\alpha^k(i,s;n)$ be the minimum expected n period α-discounted cost given that the operating machine is in the i-th operating condition, the number of machines in the repair system is s, and the state of the repair service gate is k ($k=1$ means the gate is open, and $k=0$ means it is closed) at the beginning. Then by letting $V_\alpha^k(i,s;0) = 0$ for any feasible i, s, and k, $V_\alpha^k(i,s;n)$ ($n \geqq 1$) satisfies a set of recursive equations:

For $0 \leqq i \leqq I$, $0 \leqq s \leqq S$,

$$\begin{aligned} V_\alpha^0(i,s;n) = \min \{ & A(i) + K(s,0) + R_\alpha(i,s;n-1), \\ & A(i) + K(s,1) + E + G + Q_\alpha(i,s;n-1), \\ & C(i) + K(s+1,0) + R_\alpha(0,s+1;n-1), \\ & C(i) + K(s+1,1) + E + G + Q_\alpha(0,s+1;n-1)\}, \end{aligned}$$

$$
\begin{aligned}
V_\alpha^1(i,s;n) = \min\ \{&A(i) + K(s,0) + F + R_\alpha(i,s;n-1),\\
&A(i) + K(s,1) + G + Q_\alpha(i,s;n-1),\\
&C(i) + K(s+1,0) + F + R_\alpha(0,s+1;n-1),\\
&C(i) + K(s+1,1) + G + Q_\alpha(0,s+1;n-1)\},
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
V_\alpha^0(i,S+1;n) = \min\ \{&P + K(S+1,0) + R_\alpha(0,S+1;n-1),\\
&P + K(S+1,1) + E + G + Q_\alpha(0,S+1;n-1)\},\\
V_\alpha^1(i,S+1;n) = \min\ \{&P + K(S+1,0) + F + R_\alpha(0,S+1;n-1),\\
&P + K(S+1,1) + G + Q_\alpha(0,S+1;n-1)\},
\end{aligned}
$$

where

$$
\begin{aligned}
R_\alpha(i,s;n) &= \alpha \textstyle\sum_{j=0}^{I} p_{ij}\ V_\alpha^0(j,s;n)\\
Q_\alpha(i,s;n) &= \alpha \textstyle\sum_{j=0}^{I} p_{ij} \sum_{s'=0}^{s} q_{ss'}\ V_\alpha^1(j,s';n).
\end{aligned}
\tag{2.2}
$$

Note that $[V_\alpha^k(i,s;n)]_1$, $[V_\alpha^k(i,s;n)]_2$, $[V_\alpha^k(i,s;n)]_3$ and $[V_\alpha^k(i,s;n)]_4$ are the n period costs of taking a_{LC}, a_{LO}, a_{RC} and a_{RO} respectively at the beginning followed by the best policy, where $[V]_i$ denotes the i-th term of the right hand side of V. When $s=S+1$, a_C and a_O are the only available actions. In that case note that i in the expression $V_\alpha^k(i,S+1;n)$ is artificial and has no meaning since no machine is operating then.

As the system is a Markov decision process with discount factor $0 \leqq \alpha < 1$, the existence of a stationary policy minimizing the total α-discounted cost is guaranteed. The problem is now to find the structure of an optimal policy. It is conceivable that an optimal policy has the form that the repair decision is taken if and only if the condition of an operating machine becomes worse than some critical value, and that the decision to open the repair service gate is taken if and only if the number of machines in the repair system exceeds some critical value.

Definition. A *control limit policy with respect to operating machine* is a nonrandomized policy where, as the option of repairing or leaving an operating machine is concerned, there is an i for each k, s and n, say $i_{k,s,n}$, called the control limit, such that for all (i,k,s) with $i < i_{k,s,n}$, the decision at period n is to leave it in operation, and for all (i,k,s) with $i \geqq i_{k,s,n}$, the decision is to repair it. A *control limit policy with respect to repair shop* is a nonrandomized policy where, as the option of opening or closing the repair service gate is concerned, there is an s for each k, i and n, say $s_{k,i,n}$, called the control limit, such that for all (i,k,s) with

$s < s_{k,i,n}$, the decision at period n is to close it, and for all (i,k,s) with $s \geqq s_{k,i,n}$, the decision is to open it. A *two-dimensional control limit policy* is a control limit policy with respect to both operating machine and repair shop.

Theorem 1. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leqq i \leqq I$.
2. $A(i) - C(i)$ is nondecreasing in i for $0 \leqq i \leqq I$.
3. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leqq i \leqq I-1$
   where $P_i(k) = \sum_{j \leqq k} p_{ij}$
   and $P_i(\cdot) \subset P_{i+1}(\cdot)$ if and only if $P_i(t) \geqq P_{i+1}(t)$ for any t.

Then there exists a stationary control limit policy with respect to operating machine which minimizes the total expected α-discounted cost of the maintenance model with control of queue.

Proof: We first consider the n-stage problem. For $n \geqq 1$, $0 \leqq i \leqq I$, $0 \leqq s \leqq S$, and $k = 0,1$, let

$$\begin{aligned} f^k_{L,n}(i,s) &= \min\{[V^k_\alpha(i,s;n)]_1,\ [V^k_\alpha(i,s;n)]_2\} \\ f^k_{R,n}(i,s) &= \min\{[V^k_\alpha(i,s;n)]_3,\ [V^k_\alpha(i,s;n)]_4\}. \end{aligned} \tag{2.3}$$

Then $f^k_{L,n}(i,s)$ can be interpreted as the minimum n-stage α-discounted cost given that a machine is in (i,s,k) at the beginning, and only the decision to keep a machine is allowed at the beginning. If only the decision to repair an operating machine is allowed at the beginning, we have $f^k_{R,n}(i,s)$.

Now, for $0 \leqq i \leqq I$, $0 \leqq s \leqq S$, and $n \geqq 0$,

$$\begin{aligned} & f^0_{L,n+1}(i,s) - f^0_{R,n+1}(i,s) \\ &= A(i) + \min\{K(s,0) + R_\alpha(i,s;n),\ K(s,1) + E + G + Q_\alpha(i,s;n)\} \\ &\quad - C(i) - \min\{K(s+1,0) + R_\alpha(0,s+1;n), \\ &\qquad K(s+1,1) + E + G + Q_\alpha(0,s+1;n)\}. \end{aligned} \tag{2.4}$$

Using all the conditions of this theorem, we can easily show that $V^k_\alpha(i,s;n)$ and hence both $R_\alpha(i,s;n)$ and $Q_\alpha(i,s;n)$ are nondecreasing in i $(0 \leqq i \leqq I)$. With this, and by 2, we have that $f^0_{L,n}(i,s) - f^0_{R,n}(i,s)$ is nondecreasing in i $(0 \leqq i \leqq I)$ for each s and n. Similarly for $f^1_{L,n}(i,s) - f^1_{R,n}(i,s)$. Hence, there exists a set of critical numbers $i^n_{k,s}$ $(k = 0,1,\ 0 \leqq s \leqq S)$ for each $n \geqq 1$ such that, as far as the option of repairing or leaving an operating machine is concerned, at the beginning of each n-stage problem, if the state of the system is (i,k,s), to repair a machine is optimal if and only if its operating condition i is no less than $i^n_{k,s}$, which is a control limit policy

with respect to operating machine. By the usual technique of expanding the n-stage problem to the infinite horizon problem, the optimality of a stationary control limit policy with respect to operating machine can be shown. □

At the end of the section, we make a few remarks on the relations between optimal decisions when the repair gate is closed and the corresponding optimal decisions when the gate is open. For this discussion, both E and F are nonnegative.

Lemma 1. When the repair service gate is closed at the beginning of a period, if a_{LO} (a_{RO}) is optimal for some (i,s), then a_{LO} (a_{RO}, respectively) is also optimal for the same (i,s) when the gate is open.

Proof: We prove the statement for a_{LO}. In a similar fashion, the statement for a_{RO} can be proved. If we let $R_\alpha(i,s) = \lim_{n\to\infty} R_\alpha(i,s;n)$ and $Q_\alpha(i,s) = \lim_{n\to\infty} Q_\alpha(i,s;n)$, a_{LO} being better than a_{LC} for a fixed (i,s), and $k=0$ implies

$$A(i) + K(s,1) + E + G + Q_\alpha(i,s) \leqq A(i) + K(s,0) + R_\alpha(i,s).$$

Hence,

$$A(i) + K(s,1) + G + Q_\alpha(i,s) \leqq A(i) + K(s,0) + F + R_\alpha(i,s).$$

Thus, for (i,s) and $k=1$, a_{LO} is better than a_{LC}. Similarly, a_{LO} is better than a_{RC} (a_{RO}) for $k=0$ implies a_{LO} is better than a_{RC} (a_{RO}, respectively) for $k=1$. Therefore for (i,s), a_{LO} is optimal when the gate is open. □

Lemma 2. When the repair service gate is open at the beginning of a period, if a_{RC} (a_{LC}) is optimal for some (i,s), then a_{RC} (a_{LC}, respectively) is also optimal for the same (i,s) when the gate is closed.

Proof: Similar to Lemma 1, and hence can be omitted. □

3. Case where Repair Time is Negligible

In this section sufficient conditions to ensure the existence of a control limit policy with respect to repair shop minimizing the total α-discounted cost are of interest. The following assumption is made throughout this section. For $0 \leqq s \leqq S+1$,

$$q_{ss'} = \begin{cases} 1 & \text{if } s' = 0 \\ 0 & \text{if } s' \neq 0. \end{cases} \tag{3.1}$$

The above assumption implies that the repair time of each machine is negli-

gible compared with the length of a period. This will be reasonable if purchasing or ordering machines takes place instead of repairing when "the gate is open." Then the following lemma is shown.

Lemma 3. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leqq i \leqq I$.
2. $K(s,k)$ is nondecreasing in s $(0 \leqq s \leqq S+1)$ for $k=0,1$.
3. $P \geqq C(I)$.

Then $V_\alpha^k(i,s;n)$ is nondecreasing in s $(0 \leqq s \leqq S+1)$ for each $0 \leqq i \leqq I$, $k=0,1$, and $n \geqq 0$.

Proof: Proof is by mathematical induction. The claim trivially holds for $n=0$. Suppose it holds for $n=m-1 \geqq 0$. Then for $0 \leqq s \leqq S$,

$$\begin{aligned} V_\alpha^0(i,s;m) = \min\{&A(i) + K(s,0) + R_\alpha(i,s;m-1), \\ &A(i) + K(s,1) + E + G + Q_\alpha(i,s;m-1), \\ &C(i) + K(s+1,0) + R_\alpha(0,s+1;m-1), \\ &C(i) + K(s+1,1) + E + G + Q_\alpha(0,s+1;m-1)\}. \end{aligned}$$

Now R_α's in the above expression are nondecreasing in s by the induction hypothesis, and Q_α's are constant in s since in fact,

$$Q_\alpha(i,s;m-1) = \alpha \sum_{j=0}^{I} p_{ij} V_\alpha^1(j,0;m-1). \tag{3.2}$$

Also $K(s,k)$ is nondecreasing in s $(0 \leqq s \leqq S)$, yielding that $V_\alpha^0(i,s;m)$ is nondecreasing in s $(0 \leqq s \leqq S)$. Also,

$$\begin{aligned} &V_\alpha^0(i,S+1;m) - V_\alpha^0(i,S;m) \\ &\geqq P + \min\{K(S+1,0) + R_\alpha(0,S+1;m-1), \\ &\qquad E + G + K(S+1,1) + Q_\alpha(0,S+1;m-1)\} \\ &\quad - (C(i) + \min\{K(S+1,0) + R_\alpha(0,S+1;m-1), \\ &\qquad E + G + K(S+1,1) + Q_\alpha(0,S+1;m-1)\}) \\ &= P - C(i) \geqq 0, \text{ by 1 and 3.} \end{aligned}$$

Thus, $V_\alpha^0(i,s;m)$ is nondecreasing in s $(0 \leqq s \leqq S+1)$ for each fixed $0 \leqq i \leqq I$. Similarly, we can show that $V_\alpha^1(i,s;m)$ is nondecreasing in s $(0 \leqq s \leqq S+1)$, completing the mathematical induction and the proof. □

Using the above lemma we can prove the following theorem, which gives sufficient conditions for the optimality of a control limit policy with respect to repair shop.

Theorem 2. If all the conditions in Lemma 3 hold, and in addition if,

4. $\min_{r=0,1}\{K(s+r+1,0) - K(s+r,0)\} \geqq \max_{r=0,1}\{K(s+r+1,1) - K(s+r,1)\}, \quad 0 \leqq s \leqq S-1,$

holds, then there exists a stationary control limit policy with respect to repair shop which minimizes the total expected α-discounted cost of the simplified maintenance model with control of queue.

Proof: For $n \geqq 1$, $0 \leqq i \leqq I$, $0 \leqq s \leqq S$, and $k=0,1$, let

$$(3.3)\qquad \begin{aligned} f^k_{C,n}(i,s) &= \min\{[V^k_\alpha(i,s;n)]_1,\ [V^k_\alpha(i,s;n)]_3\} \\ f^k_{O,n}(i,s) &= \min\{[V^k_\alpha(i,s;n)]_2,\ [V^k_\alpha(i,s;n)]_4\}. \end{aligned}$$

$f^k_{C,n}(i,s)$ is the minimum n-stage α-discounted cost given that the state of the system is (i,k,s) and only the decision to close the repair shop is allowed at the beginning. If only the decision to open the repair shop is allowed at the beginning, we have $f^k_{O,n}(i,s)$.

Now as in the proof of Theorem 1, it is sufficient to verify that $f^k_{C,n}(i,s) - f^k_{O,n}(i,s)$ is nondecreasing in s $(0 \leqq s \leqq S)$ for each fixed i, k and n. But for $n \geqq 0$, $0 \leqq i \leqq I$,

$$(3.4)\qquad \begin{aligned} &f^0_{C,n+1}(i,s) - f^0_{O,n+1}(i,s) \\ &\quad = \min\{A(i) + K(s,0) + R_\alpha(i,s;n),\ C(i) + K(s+1,0) + R_\alpha(0,s+1;n)\} \\ &\qquad - E - G - \min\{A(i) + K(s,1) + Q_\alpha(i,s;n), \\ &\qquad\qquad C(i) + K(s+1,1) + Q_\alpha(0,s+1;n)\}. \end{aligned}$$

Here, the rate of increase of $f^0_{O,n+1}(i,s)$ w.r.t. s is bounded above by $\max_{r=0,1}\{K(s+r+1,1) - K(s+r,1)\}$ as Q_α's are constant in s, and that of $f^0_{C,n+1}(i,s)$ w.r.t. s is bounded below by $\min_{r=0,1}\{K(s+r+1,0) - K(s+r,0)\}$ as R_α's are nondecreasing in s by Lemma 3. Hence, if 4 holds, the difference $f^0_{C,n+1}(i,s) - f^0_{O,n+1}(i,s)$ becomes nondecreasing in s $(0 \leqq s \leqq S)$ for $n \geqq 1$ and $0 \leqq i \leqq I$. In a similar manner, $f^1_{C,n}(i,s) - f^1_{O,n}(i,s)$ is shown to be nondecreasing in s $(0 \leqq s \leqq S)$, which is what we want. □

Condition 4 gives the relation between the holding cost when the gate is closed and that when the gate is open. In particular, if $K(s,k)$ can be represented as linear functions in s, i.e., if $K(s,k) = h_k s + l_k$ $(k=0,1)$, then this condition holds when $h_0 \geqq h_1$, which seems to be a reasonable assumption.

Combining the previous two theorems gives sufficient conditions under which a two-dimensional control limit policy is optimal.

Theorem 3. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leqq i \leqq I$.

2. $A(i) - C(i)$ is nondecreasing in i for $0 \leqq i \leqq I$.
3. $K(s,k)$ is nondecreasing in s $(0 \leqq s \leqq S+1)$ for $k=0,1$.
4. $P \geqq C(I)$.
5. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leqq i \leqq I-1$.
6. $\min_{r=0,1}\{K(s+r+1,0) - K(s+r,0)\} \geqq \max_{r=0,1}\{K(s+r+1,1) - K(s+r,1)\}$, $0 \leqq s \leqq S-1$.

Then there exists a stationary two-dimensional control limit policy minimizing the total expected α-discounted cost of the simplified maintenance model with control of queue.

One realization of an optimal stationary two-dimensional control limit policy is illustrated in Fig. 2. As previously pointed out in Lemmas 1 and 2, the region where a_{LO} (a_{RO}) is optimal, called the optimal region of a_{LO} $(a_{RO}$, respectively), when the gate is open covers the optimal region of a_{LO} $(a_{RO}$, respectively) when the gate is closed. Further, the optimal region of a_{LC} (a_{RC}) when the gate is closed covers that of a_{LC} $(a_{RC}$, respectively) when the gate is open. Thus, if we keep the condition i of an operating machine fixed, an optimal policy has the following form: keep the gate closed (a_{LC} or a_{RC} is taken) if the number of machines waiting for repair service is m_i or less, and when the number of machines waiting for repair service increases to M_i $(M_i \geqq m_i)$, open the gate (a_{LO} or a_{RO} is taken), and keep it open until the number of machines to be repaired again drops to m_i. This is called a hysteresis loop policy, which often appears in the theory of control of the service process.

Notice also that the boundary of optimal regions of a_{RO} and a_{RC} is vertical. This can be easily seen by comparing the appropriate terms in (2.1). Consider the boundary of optimal regions of a_{RO} and a_{LO}. If the holding cost $K(s,1)$ is concave in s, then for each fixed i,

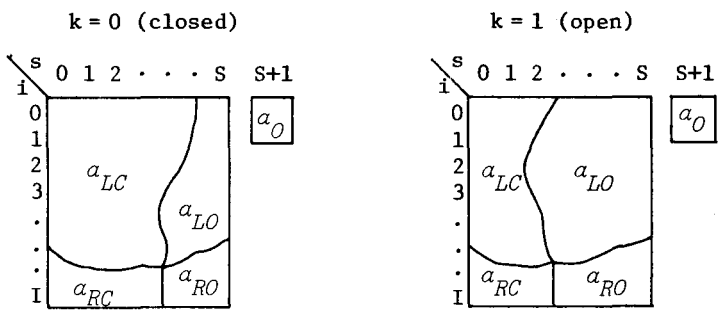

Figure 2. A typical optimal two-dimensional control limit policy

$$C(i) + K(s+1,1) + Q_\alpha(0,s+1) \leqq A(i) + K(s,1) + Q_\alpha(i,s)$$

implies

$$C(i) + K(s+2,1) + Q_\alpha(0,s+2) \leqq A(i) + K(s+1,1) + Q_\alpha(i,s+1),$$

since Q_α's are constant in s. That means, if a_{RO} is better than a_{LO} for (i,k,s), so is for $(i,k,s+1)$, which yields that the boundary curve is nondecreasing as is shown in Fig. 2. If $K(s,1)$ is convex in s, the curve becomes nonincreasing.

4. General Case

The simplified assumption on the repair time is relaxed in this section at the cost of optimality of a two-dimensional control limit policy in the strict sense. Here we assume that the reparability of the repair facility does not depend on the number of machines waiting for repair service. Let $q(r)$ be the probability that r machines are repaired in a period supposing there are infinite number of machines to be repaired. Then,

$$q_{ss'} = \begin{cases} q(s-s') & \text{if } 1 \leqq s' \leqq s \\ \sum_{r=s}^{\infty} q(r) & \text{if } s'=0. \end{cases} \tag{4.1}$$

Consider a stationary control limit policy with respect to operating machine. The existence of such a policy minimizing the total expected α—discounted cost is guaranteed if the conditions in Theorem 1 are all satisfied. In the case of a stationary control limit policy with respect to repair shop, the analysis becomes much complicated. The analysis must be performed without assuming a nice structure on the cost criterion. A bounding technique which follows next then seems appropriate for the analysis of this type of model. For the future use, let

$$\begin{aligned} \bar{K} &= \max_{s,k} \{K(s,k) - K(s-1,k)\} \\ \underline{K} &= \min_{s,k} \{K(s,k) - K(s-1,k)\}. \end{aligned} \tag{4.2}$$

Lemma 4. Assume the following conditions hold:

1. $A(i)$ is nondecreasing in i for $0 \leqq i \leqq I$.
2. $C(i)$ is nondecreasing in i for $0 \leqq i \leqq I$.
3. $K(s,k)$ is nondecreasing in s $(0 \leqq s \leqq S+1)$ for $k=0,1$.
4. $P \geqq \min\{A(0),C(0)\}$.
5. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leqq i \leqq I-1$.

Then, for $1 \leqq s \leqq S+1$, $0 \leqq i \leqq I$, $k=0,1$, and $n \geqq 0$,

$$V_\alpha^k(i,s;n) - V_\alpha^k(i,s-1;n) \leqq \bar{M}_n, \tag{4.3}$$

where

$$\bar{M}_n = \frac{1-\alpha^n}{1-\alpha}\,(P - \min\{A(0),C(0)\} + \bar{K}). \tag{4.4}$$

Proof: Mathematical induction is applied. The claim trivially holds for $n=0$. Suppose the argument holds for $n=m-1 \geqq 0$, and consider the case for $n=m$. For $k=0$ and $1 \leqq s \leqq S$, we compare the corresponding terms of the right hand side of (2.1).

$$\begin{aligned}
&[V_\alpha^0(i,s;m)]_1 - [V_\alpha^0(i,s-1;m)]_1 \\
&\quad = K(s,0) - K(s-1,0) + \alpha \textstyle\sum_{j=0}^I p_{ij}(V_\alpha^0(j,s;m-1) - V_\alpha^0(j,s-1;m-1)) \\
&\quad \leqq K(s,0) - K(s-1,0) + \alpha \textstyle\sum_{j=0}^I p_{ij}\,\bar{M}_{m-1} \leqq \bar{K} + \alpha\bar{M}_{m-1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&[V_\alpha^0(i,s;m)]_2 - [V_\alpha^0(i,s-1;m)]_2 \\
&\quad = A(i) + K(s,1) + E + G + Q_\alpha(i,s;m-1) \\
&\qquad\qquad - (A(i) + K(s-1,1) + E + G + Q_\alpha(i,s-1;m-1)) \\
&\quad = K(s,1) - K(s-1,1) + \alpha \textstyle\sum_{j=0}^I p_{ij}\{q(0)(V_\alpha^1(j,s;m-1) - V_\alpha^1(j,s-1;m-1)) \\
&\qquad\qquad + q(1)(V_\alpha^1(j,s-1;m-1) - V_\alpha^1(j,s-2;m-1)) + \cdots \\
&\qquad\qquad\qquad + q(s-1)(V_\alpha^1(j,1;m-1) - V_\alpha^1(j,0;m-1))\} \\
&\quad \leqq \bar{K} + \alpha \textstyle\sum_{j=0}^I p_{ij}(q(0) + q(1) + \cdots + q(s-1))\,\bar{M}_{m-1} \\
&\quad \leqq \bar{K} + \alpha \textstyle\sum_{j=0}^I p_{ij}\,\bar{M}_{m-1} = \bar{K} + \alpha\bar{M}_{m-1}.
\end{aligned}$$

In a similar manner, the comparison of the corresponding third terms and that of the fourth terms yield the same upper bound $\bar{K} + \alpha\bar{M}_{m-1}$. Hence, for $1 \leqq s \leqq S$, $0 \leqq i \leqq I$,

$$V_\alpha^0(i,s;m) - V_\alpha^0(i,s-1;m) \leqq \bar{K} + \alpha\bar{M}_{m-1}.$$

For $s = S+1$, and $0 \leqq i \leqq I$,

$$\begin{aligned}
&[V_\alpha^0(i,S+1;m)]_1 - [V_\alpha^0(i,S;m)]_1 \\
&\quad = P - A(i) + K(S+1,0) - K(S,0)
\end{aligned}$$

$$+ \alpha \{ \textstyle\sum_{j=0}^{I} p_{0j} \; V_\alpha^0(j,S+1;m-1) - \sum_{j=0}^{I} p_{ij} \; V_\alpha^0(j,S;m-1)\}$$

$$\leqq P - A(i) + K(S+1,0) - K(S,0)$$

$$+ \alpha \textstyle\sum_{j=0}^{I} p_{0j}(V_\alpha^0(j,S+1;m-1) - V_\alpha^0(j,S;m-1)) \quad \text{by 5}$$

$$\leqq P - min\{A(0),C(0)\} + \bar{K} + \alpha\bar{M}_{m-1}.$$

Similarly, using $P_0(\cdot) \subset P_i(\cdot)$ and that $V_\alpha^1(j,s;m-1)$ is nondecreasing in j $(0 \leqq j \leqq I)$, $[V_\alpha^0(i,S+1;m)]_2 - [V_\alpha^0(i,S;m)]_2$ can be shown to have the same upper bound. Also,

$$[V_\alpha^0(i,S+1;m)]_1 - [V_\alpha^0(i,S;m)]_3 = P - C(i) \leqq P - min\{A(0),C(0)\}.$$

We can show the same upper bound also on $[V_\alpha^0(i,S+1;m)]_2 - [V_\alpha^0(i,S;m)]_4$, yielding that for $0 \leqq i \leqq I$,

$$V_\alpha^0(i,S+1;m) - V_\alpha^0(i,S;m) \leqq P - min\{A(0),C(0)\} + \bar{K} + \alpha\bar{M}_{m-1}.$$

As $P \geqq min\{A(0),C(0)\}$ from 4, for $0 \leqq i \leqq I$, and $1 \leqq s \leqq S+1$,

$$V_\alpha^0(i,s;m) - V_\alpha^0(i,s-1;m) \leqq P - min\{A(0),C(0)\} + \bar{K} + \alpha\bar{M}_{m-1} = \bar{M}_m.$$

A similar argument indicates that for $0 \leqq i \leqq I$, and $1 \leqq s \leqq S+1$,

$$V_\alpha^1(i,s;m) - V_\alpha^1(i,s-1;m) \leqq \bar{M}_m \; ,$$

completing the mathematical induction, and hence the proof. □

The above lemma gives the upper bound on $V_\alpha^k(i,s;n) - V_\alpha^k(i,s-1;n)$. The lower bound on the same expression is given in the following lemma, whose proof is omitted since the result can be obtained by mathematical induction where its inductive step can be performed by comparing the corresponding terms for each case as in the previous lemma.

Lemma 5. If conditions 2 and 3 of Lemma 4 hold, then for $1 \leqq s \leqq S+1$, $0 \leqq i \leqq I$, $k=0,1$, and $n \geqq 1$,

$$V_\alpha^k(i,s;n) - V_\alpha^k(i,s-1;n) \geqq \underline{M}_n \; , \tag{4.5}$$

where

$$\underline{M}_n = min\{ \frac{1-(\alpha q(0))^n}{1-\alpha q(0)} \underline{K} \; , \; P - C(I)\}. \tag{4.6}$$

Let $\bar{M}_\infty = \lim_{n\to\infty} \bar{M}_n$ and $\underline{M}_\infty = \lim_{n\to\infty} \underline{M}_n$. Then it is easy to see that

$$\bar{M}_n - \underline{M}_n \leqq \bar{M}_{n+1} - \underline{M}_{n+1} \leqq \bar{M}_\infty - \underline{M}_\infty. \tag{4.7}$$

In this section sufficient conditions for the optimality of a control

limit policy with respect to repair shop in the strict sense are not derived. Instead, sufficient conditions are obtained under which a control limit type of property holds between two actions a_{LC} and a_{LO}, and between a_{RC} and a_{RO}. The next lemma gives that property.

Lemma 6. Assume all the conditions of Lemma 4 hold, and furthermore assume the following condition holds:

6. $K(s+1,0) - K(s,0) \geqq K(s+1,1) - K(s,1) + \alpha(\bar{M}_\infty - \underline{M}_\infty)$ for $0 \leqq s \leqq S$.

Then if a_{LO} is better than a_{LC} for (i,k,s) as an infinite horizon problem, so is for $(i,k,s+1)$. Similarly, if a_{RO} is better than a_{RC} for (i,k,s), so is for $(i,k,s+1)$ $(0 \leqq i \leqq I,\ 0 \leqq s \leqq S-1,\ k=0,1)$.

Proof: Consider the case where $k=0$. The proof of the case where $k=1$ is similar, and can be omitted.

Suppose a_{LO} is better than a_{LC} for $(i,k=0,s)$. That is, the total cost of choosing a_{LO} at the beginning followed by the best policy is smaller than or equal to that of choosing a_{LC} at the beginning followed by the best policy when the state of the system at the beginning is $(i,k=0,s)$. Equivalently,

$$A(i) + K(s,1) + E + G + Q_\alpha(i,s) \leqq A(i) + K(s,0) + R_\alpha(i,s).$$

Now by Lemmas 4 and 5, and by the definitions of R_α and Q_α,

$$R_\alpha(i,s+1) - R_\alpha(i,s) \geqq \alpha\underline{M}_\infty$$

$$Q_\alpha(i,s+1) - Q_\alpha(i,s) \leqq \alpha\bar{M}_\infty.$$

Hence,

$$\begin{aligned}
&A(i) + K(s+1,0) + R_\alpha(i,s+1) - (A(i) + K(s+1,1) + E + G + Q_\alpha(i,s+1)) \\
&\quad = A(i) + K(s,0) + R_\alpha(i,s) - (A(i) + K(s,1) + E + G + Q_\alpha(i,s)) \\
&\qquad + (K(s+1,0) - K(s,0)) - (K(s+1,1) - K(s,1)) \\
&\qquad + (R_\alpha(i,s+1) - R_\alpha(i,s)) - (Q_\alpha(i,s+1) - Q_\alpha(i,s)) \\
&\quad \geqq K(s+1,0) - K(s,0) - (K(s+1,1) - K(s,1)) + \alpha(\underline{M}_\infty - \bar{M}_\infty) \geqq 0 \quad \text{by 6.}
\end{aligned}$$

Therefore we can conclude that a_{LO} is better than a_{LC} for $(i,k=0,s+1)$. In a similar fashion, for $0 \leqq i \leqq I$, and $0 \leqq s \leqq S-1$,

$$C(i) + K(s+1,1) + E + G + Q_\alpha(0,s+1) \leqq C(i) + K(s+1,0) + R_\alpha(0,s+1)$$

implies

$$C(i) + K(s+2,1) + E + G + Q_\alpha(0,s+2) \leqq C(i) + K(s+2,0) + R_\alpha(0,s+2),$$

yielding that a_{RO} is better than a_{RC} for $(i,k=0,s+1)$ assuming a_{RO} is better than a_{RC} for $(i,k=0,s)$. □

Suppose that all the conditions in both Theorem 1 and Lemma 6 are satisfied. By Theorem 1, there is a stationary control limit policy with respect to operating machine which minimizes the total expected α-discounted cost. The i-s diagram of the optimal policy, as is seen in Fig. 3, is divided into upper and lower divisions for each k. The action of leaving an operating machine in operation is taken in each state in the upper region, and the action of repairing an operating machine is taken in each state in the lower region. The former has two alternatives a_{LO} and a_{LC}, while the latter has two alternatives a_{RO} and a_{RC}. We now focus on the possibility of subdividing each region having two alternatives. It is immediate from the first part of Lemma 6 that there exist critical numbers $s^L_{i,k}$ for each fixed k $(k=0,1)$ and i $(0 \leqq i \leqq I)$ such that for all (i,k,s) with $s < s^L_{i,k}$, a_{LC} is better than a_{LO}, and for all (i,k,s) with $s \geqq s^L_{i,k}$, a_{LO} is no worse than a_{LC}. This implies that the upper division can be divided into left and right subdivisions. a_{LC} is optimal in each state in the left subdivision, while a_{LO} is optimal in each state in the right subdivision. In a similar manner, we can show that there exist critical numbers $s^R_{i,k}$ for each fixed k and i such that for all (i,k,s) with $s < s^R_{i,k}$, a_{RC} is better than a_{RO}, and for all (i,k,s) with $s \geqq s^R_{i,k}$, a_{RO} is no worse than a_{RC}. Thus the lower division can be divided into two subdivisions, where a_{RC} is optimal in each state in the left subdivision, and a_{RO} is optimal in each state in the right subdivision. We call this type of policy a stationary *two-dimensional weak control limit policy*. One realization of a two-dimensional weak control limit policy, optimizing our problem, is shown in Fig. 4. The control limits found in this kind of policy are those on the action of repairing or leaving an operating machine, those on the action of a_{LC} or a_{LO}, and those on the action of a_{RC} or a_{RO}. Control limits on the action of opening or closing the repair shop might not exist. In this sense, this type of policy is weaker than a two-dimensional control limit policy.

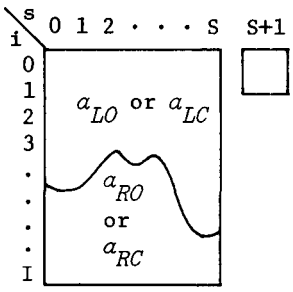

Figure 3. A typical optimal control limit policy with respect to operating machine

As in the previous case where the repair time is negligible, notice that the boundary of optimal regions of a_{RO} and a_{RC} is vertical.

As a conclusion of this section, we

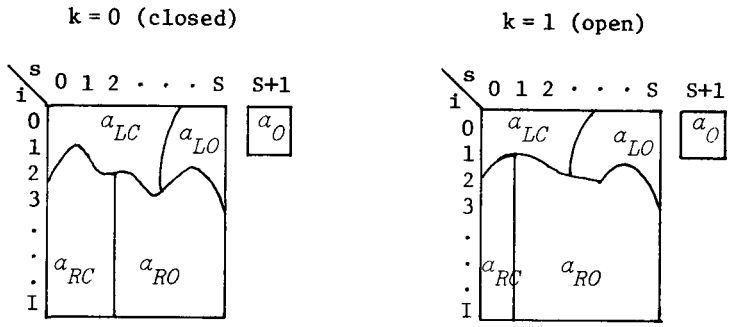

Figure 4. A typical optimal two-dimensional weak control limit policy

restate the above discussion as a theorem.

Theorem 4. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leqq i \leqq I$.
2. $A(i) - C(i)$ is nondecreasing in i for $0 \leqq i \leqq I$.
3. $K(s,k)$ is nondecreasing in s $(0 \leqq s \leqq S+1)$ for each $k = 0,1$.
4. $P \geqq min\{A(0),C(0)\}$.
5. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leqq i \leqq I-1$.
6. $K(s+1,0) - K(s,0) \geqq K(s+1,1) - K(s,1) + \alpha(\bar{M}_\infty - \underline{M}_\infty)$ for $0 \leqq s \leqq S$.

Then there exists a stationary two-dimensional weak control limit policy which minimizes the total expected α-discounted cost of the model.

Conditions 1, 2, 3 and 5 are the same as those in Theorem 3. 1 indicates that the material cost increases as the condition of the machine to be repaired gets worse. 2 says that the operating cost must increase more than the increase of the material cost for repairing a machine as its condition gets worse. 3 means that the holding cost increases as the number of machines in the repair system increases. 5 is called the IFR (increasing failure rate) property of a Markov chain since it says that the higher the state the greater the chance of further deterioration. 4 gives a lower bound on the penalty cost, which is usually very large. 6 is the only restrictive condition. It gives how much the increment of the holding cost when the gate is closed is bigger than the corresponding cost when the gate is open. It seems appropriate though that the former is more costly than the latter.

5. Computing Remarks and Future Topics

As each model treated here is a Markov decision model, the usual techniques such as policy improvement procedure and LP approach are applicable to compute an optimal policy. However if we know that an optimal policy is of a two-dimensional control limit form, better algorithms can be expected since this information should enable us to explore this structure, thereby decreasing significantly the number of policies that must be considered. One such a realization can be easily constructed where "good" policies are searched iteratively among stationary two-dimensional control limit policies whenever possible before switching to a usual policy improvement procedure.

Since the discrete time queueing control problem has not been fully studied, there are several extensions that can be made on our maintenance with control of queue models. Controlling the queue length by changing the repair service rate, controlling a multiple number of repair service stations by opening or closing them will be some topics for future research.

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