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AN APPROXIMATION FORMULA FOR THE MEAN WAITING TIME OF AN M/G/c QUEUE

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Abstract An approximation formula for the mean waiting time of an M/G/c queue is proposed. It estimates the mean waiting time from a moment of order $\alpha \leqq 2$, rather than the second moment, of the service distribution. Together with the Lee & Longton's formula [1] and the Page's formula [2], it was numerically tested on a variety of cases, and the test shows that the new formula is generally better than the previous two formulas, especially for queues with mixtures of Erlang distributions as the service distribution. A similar approximation formula for the variance of the waiting time is also proposed.

1. Introduction

Many approximation formulas for the mean waiting time of an $M/G/c$ queueing system have been proposed by Lee & Longton [1], Page [2] and other authors. Most of them estimate the mean waiting time $E_G(W)$ from the 2nd moment b_2 of the service distribution using the values of the mean waiting times $E_M(W)$ and $E_D(W)$ of the corresponding $M/M/c$ and $M/D/c$ queueing systems with the same mean service time and the same traffic intensity. For example, the Lee & Longton's formula is

$$\text{(1.1)} \qquad \hat{E}^L_G(W) = \frac{b_2}{2b_1^2} E_M(W) ,$$

and the Page's formula is

$$(1.2) \qquad \hat{E}^P_G(W) = \left(\frac{b_2}{b_1^2} - 1\right) E_M(W) + \left(2 - \frac{b_2}{b_1^2}\right) E_D(W) ,$$

where b_i is the ith moment of the service distribution.

The author tested the above two approximations on a variety of cases designated in Table 1 in Section 3. The result of the test, which is summarized in Table 2 in Section 3, shows that these approximations are fairly good for queueing systems with convolutions of Erlang distributions as the service distribution. But for some queueing systems with mixtures of Erlang distributions, they are not good. It seems that the use of the 2nd moment causes the unfitness. As will be discussed in the next section, the 2nd moment is not suitable for estimating the mean waiting time of a multi-channel queueing system, especially of a system with low traffic intensity. A moment of lower order than 2 will be rather adequate to estimate it.

In this paper we explore a new approximation formula which estimates the mean waiting time $E_G(W)$ from a moment b_α of order $\alpha \leqq 2$ of the service distribution. From several natural conditions stated in the next section, we can derive the following approximation formula.

$$(1.3) \qquad \hat{E}_G(W) = \left(\frac{b_\alpha}{b_1^\alpha}\right)^{\frac{1}{\alpha-1}} E_D(W) ,$$

where α is the unique positive number such that

$$(1.4) \qquad E_M(W) = (\Gamma(\alpha+1))^{\frac{1}{\alpha-1}} E_D(W) ,$$

where $\Gamma(z)$ is the gamma function. This formula was also tested in the cases designated in Table 1. The relative error has been less than 10% in every case in the test, and in most cases this approximation has been better than the previous two approximations, especially in cases of mixture type service distributions.

Similarly we can derive an approximation formula for the variance $V_G(W)$ of the waiting time. Let $V_M(W)$ and $V_D(W)$ be the variances of the waiting times of the corresponding $M/M/c$ and $M/D/c$ queueing systems with the same mean service time and the same traffic intensity, and β be the unique positive number such that

(1.5) $$V_M(W) - (E_M(W))^2 = (\Gamma(\beta+1))^{\frac{2}{\beta-1}} \{V_D(W) - (E_D(W))^2\} .$$

Then $V_G(W)$ is approximated by

(1.6) $$\hat{V}_G(W) = \left(\frac{b_\beta}{b_1{}^\beta} \right)^{\frac{2}{\beta-1}} \{V_D(W) - (E_D(W))^2\} + (\hat{E}_G(W))^2 .$$

The relative error of the approximation has been less than 21% in every case designated in Table 1.

Tables of α, β, $E_M(W)$, $V_M(W) - (E_M(W))^2$, $E_D(W)$ and $V_D(W) - (E_D(W))^2$ are presented in Appendix, together with some notes on calculations of $\hat{E}_G(W)$ and $\hat{V}_G(W)$.

2. A New Approximation Formula for the Mean Waiting Time

The purpose of this section is to show the derivation process of the new approximation formula (1.3) from several conditions. Let us consider a multi-channel queueing system $M/G/c$. In the system, customers arrive at a service facility with c channels in parallel via a Poisson process with rate λ . If all channels are busy, the customers form a single queue and are served in order of arrival. The service times are independent random variables subjecting to a distribution function $G(x)$. Let b_γ be the moment of order γ of the distribution G , and $\rho = \lambda b_1/c$ the traffic intensity. We denote by $E_G(W)$ the mean waiting time in the steady-state.

First we show a reason for considering an approximation formula which depends on b_α for some $\alpha \leqq 2$ rather than b_2 . Let L be the total number of customers being served or waiting in the queue. Since

(2.1) $$E_G(W) = \frac{1}{\lambda} E_G(L) = \frac{1}{\lambda} \sum_{n=1}^{\infty} nP\{L = c + n\} ,$$

if ρ is sufficiently small, $E_G(W)$ is approximated by $\frac{1}{\lambda} P\{L = c + 1\}$. As ρ tends to zero, $P\{L = c + 1\}$ is asymptotically equal to the conditional probability of the same event conditioned that the services of c customers being served have begun with their arrivals. Hence, when λ tends to zero,

$$\frac{1}{\lambda^{c+1}} P\{L = c+1\} = \int_0^\infty xK(x)\bar{G}(x)dx + o(\lambda) \quad , \tag{2.2}$$

where $\bar{G}(x) = 1 - G(x)$ and $K(x)$ is a function given by

$$K(x) = \int_x^\infty \bar{G}(x_2)dx_2 \int_{x_2}^\infty \bar{G}(x_3)dx_3 \cdots \int_{x_{c-1}}^\infty \bar{G}(x_c)dx_c \quad . \tag{2.3}$$

Hence, $E_G(W)$ is proportional to the integral in (2.2) for sufficiently small ρ . If $K(x)$ in the integrand is a constant function (this is the case if $c = 1$), then the integral is proportional to b_2 , and it becomes reasonable to estimate $E_G(W)$ from b_2 However, if $c > 1$, $K(x)$ is a non-increasing function vanishing at infinity. So, the masses of G at large x's less contribute to the integral, and it seems rather adequate to estimate $E_G(W)$ from a moment of lower order than 2 .

Next, associated with the above queueing system, let us consider another queueing system with c channels, a Poisson arrival process with rate $\lambda' = \lambda/p$, and a service distribution function

$$G'(x) = q + pG(x) \quad , \qquad x \geqq 0 \ , \tag{2.4}$$

where p and q are positive numbers such that $p + q = 1$. We denote by $E_{G'}(W)$ the mean waiting time of the new system and by b'_γ the moment of order γ of G' . Then $b'_\gamma = pb_\gamma$, and the traffic intensity of the new system is the same as that of the original system. This system is considered that customers requiring service times equal to zero are added to the original system in a random fashion. So both queueing systems have the same waiting time distribution, especially the same mean waiting time, i.e.,

$$E_{G'}(W) = E_G(W) \quad . \tag{2.5}$$

It is desirable that approximation formulas have the corresponding property to (2.5). The Lee & Longton's formula (1.1) satisfies this property, but the Page's formula (1.2) does not.

There is another property which should be satisfied by approximation formulas. If the time scale of a queueing system is changed, then the mean waiting time of the system changes proportionally to it. To say more precisely, let $E_{G^*}(W)$ be the mean waiting time of a queueing system formed from the queueing system under condisderation by changing the time scale so that the arrival rate $\lambda^* = \lambda/\theta$ and the service distribution function

$G^*(x) = G(x/\theta)$. Then $E_{G^*}(W) = \theta E_G(W)$. This property should be retained by approximation formulas. Both the Lee & Longton's formula and the Page's formula satisfy this property.

Now we shall consider a new approximation formula for $E_G(W)$ which depends on the service distribution only through b_1 and b_α for some $\alpha \leqq 2$, and which satisfies the two properties stated above. Let $\hat{E}_G(W) = A(b_1, b_\alpha)$ be the approximation formula. Then from the first property $\hat{E}_{G'}(W) = \hat{E}_G(W)$ corresponding to (2.5) ,

(2.6) $$A(pb_1, pb_\alpha) = A(b_1, b_\alpha) , \quad 0 < p \leqq 1 ,$$

and from the second property $\hat{E}_{G^*}(W) = \theta\hat{E}_G(W)$,

(2.7) $$A(\theta b_1, \theta^\alpha b_\alpha) = \theta A(b_1, b_\alpha) , \quad \theta > 0 .$$

It can be proved that if (2.6) and (2.7) hold for any possible values of b_1 , b_α , p and θ , then the function A must be of the form

(2.8) $$A(b_1, b_\alpha) = a\cdot\left(\frac{b_\alpha}{b_1}\right)^{\frac{1}{\alpha-1}} ,$$

where a is a proportional coefficient. To prove this, we note that from (2.6) and (2.7)

(2.9) $$A(1, 1) = A(p, p) = \frac{1}{\theta} A(\theta p, \theta^\alpha p)$$

for any p and θ such that $0 < p \leqq 1$ and $\theta > 0$. Substituting

(2.10) $$\theta = \left(\frac{b_\alpha}{b_1}\right)^{\frac{1}{\alpha-1}} \quad \text{and} \quad p = \left(\frac{b_1^{\alpha}}{b_\alpha}\right)^{\frac{1}{\alpha-1}}$$

into (2.9), we get (2.8) with $a = A(1, 1)$.

The function in (2.8) contains two unknown constants α and a . In order to determine the values of them, we will impose the condition that the approximate value coincides with the exact one if the service distribution is an exponential distribution, i.e., $G(x) = 1 - \exp(-x/b_1)$ for $x \geqq 0$, or if it is a deterministic distribution, i.e., $G(x) = 0$ for $x < b_1$ and $= 1$ for $x \geqq b_1$.

Consider the case where the service distribution is the deterministic distribution. In this case, we will write $E_D(W)$ and $\hat{E}_D(W)$ instead of

$E_G(W)$ and $\hat{E}_G(W)$. Then from the above condition we have $\hat{E}_D(W) = E_D(W)$. Since $b_\alpha = b_1^{\alpha}$, from (2.8) we have

$$(2.11)\qquad E_D(W) = \hat{E}_D(W) = A(b_1, b_\alpha) = ab_1 .$$

Hence the proportional coefficient a is given by

$$(2.12)\qquad a = \frac{1}{b_1} E_D(W) .$$

(Note that the right hand side of (2.12) does not depend on b_1 from the second property $E_{G*}(W) = \theta E_G(W)$ stated above.) Substituting (2.12) into (2.8), we have the right hand side of (1.3) in Section 1.

Next consider the case where the service distribution is the exponential distribution. In this case, we will write $E_M(W)$ and $\hat{E}_M(W)$ instead of $E_G(W)$ and $\hat{E}_G(W)$. From the above condition we have $\hat{E}_M(W) = E_M(W)$. Since $b_\alpha = b_1^{\alpha}\Gamma(\alpha+1)$,

$$(2.13)\qquad E_M(W) = \hat{E}_M(W) = A(b_1, b_\alpha) = ab_1(\Gamma(\alpha+1))^{\frac{1}{\alpha-1}} .$$

Combining (2.13) with (2.12) we obtain the equation (1.4). Since $\log \Gamma(z)$ is continuous and convex for $z \geqq 1$ and $\log \Gamma(2) = 0$, the function $f(\alpha) = (\Gamma(\alpha+1))^{\frac{1}{\alpha-1}}$ is continuous and monotone increasing for $\alpha \geqq 0$ if $f(1)$ is suitably defined. Since $f(0) = 1$, $f(2) = 2$ and

$$(2.14)\qquad 1 < E_M(W)/E_D(W) \leqq 2 ,$$

the equation (1.4) determines a unique positive $\alpha \leqq 2$.

Thus the approximation formula (1.3) has been derived from the following four conditions:

(i) it depends on the service distribution only through b_1 and b_α for some $\alpha \leqq 2$,

(ii) the approximate value is unchanged even if customers requiring service times equal to zero are added in a random fashion,

(iii) the approximate value changes proportionally as the time scale changes, and

(iv) $\hat{E}_M(W) = E_M(W)$ and $\hat{E}_D(W) = E_D(W)$, i.e., the approximate value coincides with the exact one if the service distribution is an exponential distribution or a deterministic distribution.

Note that the Lee & Longton's formula (1.1) satisfies (i) with $\alpha = 2$, (ii),

(iii) and a half of (iv), but it does not satisfy $\hat{E}_D(W) = E_D(W)$.
The Page's formula (1.2) satisfies (i) with $\alpha = 2$, (iii) and (iv), but it does not satisfy (ii).

The approximation formula (1.5) for the variance of the waiting time is derived by applying a similar argument to the quantity $V_G(W) - (E_G(W))^2$. The reason for considering the quantity rather than $V_G(W)$ is that the approximation formula derived from the quantity agrees with the Pollaczek's formula

$$V_G(W) = \frac{\lambda b_3}{3(1-p)} + (E_G(W))^2 \tag{2.15}$$

for a single channel queueing system, but the corresponding one derived from $V_G(W)$ does not.

3. Numerical Test for the Approximation Formulas

For the test of fitness of the approximation formulas, the author calculated the means and variances of waiting times for the cases described in Table 1. The calculation was achieved on NEAC 2000-700 at Tohoku University, using the algorithm proposed in [3].

Table 1. Calculated cases

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* Service distributions

a. Erlang distributions with phases from 2 to 7
(6 distributions)

b. Convolutions of two Erlang distributions with phases k and h such that

$k + h \leqq 4$

$b_2/b_1^2 = 1.3\ (0.1)\ 1.9$

(20 distributions)

c. Mixtures of two Erlang distributions with phases k and h such that

$1 \leqq k \leqq h \leqq 2$

The ratio of the means of the Erlang distributions $= 2^m,\ m = -3, -2, \ldots, 2, 3$

$b_2/b_1^2 = 1.6\ (0.1)\ 2.0\ (0.2)\ 3.0\ (1.0)\ 5.0$

c_1 : $k = h = 1$ (26 distributions)

c_2 : $k = 1,\ h = 2$ (67 distributions)

c_3 : $k = h = 2$ (34 distributions)

* Number of channels c = 1, 2, 3 and 4
* Traffic intensity ρ = 0.1 (0.1) 0.9

The approximation formulas (1.1), (1.2), (1.3) and (1.5) were tested for queues designated in Table 1. The maximum relative errors of the approximate values are tabulated in Table 2. The result of the test shows the following points:

Comparison between approximations $\hat{E}^L_G(W)$, $\hat{E}^P_G(W)$ and $\hat{E}_G(W)$

i) Generally, $\hat{E}_G(W)$ is more accurate than $\hat{E}^P_G(W)$, and $\hat{E}^P_G(W)$ is more accurate than $\hat{E}^L_G(W)$.
ii) For queues with service distributions in Group a (Erlang distributions) or in Group b (convolutions of Erlang distributions), $\hat{E}^P_G(W)$ and $\hat{E}_G(W)$ are satisfactorily accurate.
iii) For queues with service distributions in Group c (mixtures of Erlang distributions), sometimes $\hat{E}^L_G(W)$ and $\hat{E}^P_G(W)$ are not good, but $\hat{E}_G(W)$ is not so bad.
iv) For queues with service distributions in Group c such that $b_3 \leqq 2.0\ b_2{}^2/b_1$, i.e., roughly speaking, not long tailed distributions in Group c , $\hat{E}^P_G(W)$ and $\hat{E}^L_G(W)$ are not bad, but $\hat{E}_G(W)$ is better.

Relative errors of $\hat{E}_G(W)$ and $\hat{V}_G(W)$

i) The relative error of $\hat{E}_G(W)$ is within 10% in every case in the test. Also, the relative error of $\hat{V}_G(W)$ is within 21% in every case, and this indicates that the relative error of $\sqrt{\hat{V}_G(W)}$, which is an approximation of the standard deviation of the waiting time, is within 10% in every case.
ii) The result stated above ensures that $\hat{E}_G(W)$ and $\hat{V}_G(W)$ are fairly accurate for queues we ordinarily deal with. However this does not mean that they are always accurate. For a queue with a service distribution with $b_2 = \infty$, $E_G(W)$ is infinite, while $\hat{E}_G(W)$ may be finite.

Table 2. Maximum relative errors of approximate values

$$\text{maximum} \quad 100 \times \left| \frac{(\text{approximate value}) - (\text{exact value})}{(\text{exact value})} \right|$$

The maximization is taken over service distributions in the designated group and over the number of channels c from 1 to 4 .
a , b and c_i $(i = 1,2,3)$ in the first column represent groups of service distributions explained in Table 1, and $c_i{*}$ $(i = 1,2,3)$ represent the group of service distributions in c_i such that $b_3 \leqq 2.0b_2{}^2/b_1$.

Group		$\hat{E}^L_G(W)$	$\hat{E}^P_G(W)$	$\hat{E}_G(W)$	$\hat{V}_G(W)$
a	$\rho = .3$	17.0	.57	.40	1.77
	$= .6$	7.51	.80	.94	.70
	$= .9$	1.46	.32	.34	.13
b	$\rho = .3$	12.3	2.66	.58	2.45
	$= .6$	5.45	.73	.83	.55
	$= .9$	1.06	.23	.29	.11
c_1	$\rho = .3$	62.4	39.8	5.08	11.1
	$= .6$	30.6	24.2	7.47	14.5
	$= .9$	5.33	4.51	1.80	.91
c_2	$\rho = .3$	83.5	58.0	7.65	19.7
	$= .6$	39.0	32.3	9.61	20.6
	$= .9$	6.52	5.70	2.29	2.38
c_3	$\rho = .3$	50.3	36.4	5.89	14.3
	$= .6$	25.3	21.2	6.26	12.3
	$= .9$	4.47	3.93	1.47	1.11
$c_1{*}$	$\rho = .3$	19.8	6.27	2.10	7.77
	$= .6$	8.15	3.97	1.67	2.74
	$= .9$	1.50	.86	.57	.27
$c_2{*}$	$\rho = .3$	23.6	7.85	2.35	5.33
	$= .6$	9.56	4.26	2.01	2.25
	$= .9$	1.74	.96	.69	.56
$c_3{*}$	$\rho = .3$	7.08	8.24	1.84	6.85
	$= .6$	3.71	4.83	1.06	3.21
	$= .9$	.77	.94	.32	.81

Appendix. Some Notes on Calculations of $\hat{E}_G(W)$ and $\hat{V}_G(W)$

The values of α, β, $E_M(W)$, $V_M(W) - (E_M(W))^2$, $E_D(W)$ and $V_D(W) - (E_D(W))^2$ are tabulated in Tables 3, 4 and 5 below.

If sample data of the service distribution are available, then one can calculate $\hat{E}_G(W)$ by using Tables 3 and 5 and estimates

$$b_1 = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{and} \quad b_\alpha = \frac{1}{n} \sum_{i=1}^{n} x_i^\alpha .$$

If the service distribution is an Erlang distribution with phase k , then

$$\frac{b_\alpha}{b_1^\alpha} = \frac{1}{k^\alpha} \frac{1}{(k-1)!} \Gamma(k+\alpha) = \frac{1}{k^\alpha} (1+\alpha)(1+\frac{\alpha}{2}) \cdots$$
$$\cdots (1+\frac{\alpha}{k-1}) \Gamma(\alpha+1) .$$

So, using the relation (1.4), one can calculate $\hat{E}_G(W)$ from the formula

$$\hat{E}_G(W) = \left(\frac{1}{k^\alpha} (1+\alpha)(1+\frac{\alpha}{2}) \cdots (1+\frac{\alpha}{k-1}) \right)^{\frac{1}{\alpha-1}} E_M(W)$$

using Tables 3 and 4. This formula does not contain the gamma function explicitly.

If the service distribution is a convolution of several Erlang distributions or a mixture of such convolutions, then the distribution function of it can be written as

$$G(x) = \sum_i p_i E_{k_i} (x/\theta_i)$$

where $E_k(x)$ is the distrivution function of the Erlang distribution with phase k and mean equal to 1 , and p_i are (possibly negative) numbers such that $\sum p_i = 1$. Since

$$b_1 = \sum_i p_i \theta_i \quad \text{and}$$

$$b_\alpha = \sum_i p_i \left(\frac{\theta_i}{k_i} \right)^\alpha \frac{1}{(k_i - 1)!} \Gamma(k_i + \alpha) ,$$

we have

Table 3. The values of α and β

α and β are positive numbers satisfying

$$E_M(W) = (\Gamma(\alpha+1))^{\frac{1}{\alpha-1}} E_D(W)$$

and

$$V_M(W) - (E_M(W))^2 = (\Gamma(\beta+1))^{\frac{2}{\beta-1}} [V_D(W) - (E_D(W))^2]$$

respectively, where $E_M(W)$, $E_D(W)$, $V_M(W)$ and $V_D(W)$ are means and variances of the $M/M/c$ and $M/D/c$ queues with the same number of channels, the same traffic intensity, and the same mean service time.

c : the number of channels

ρ : the traffic intensity

The values of α

c	ρ = .1	.2	.3	.4	.5
1	2.0000	2.0000	2.0000	2.0000	2.0000
2	1.2068	1.4002	1.5479	1.6628	1.7534
3	.8423	1.0702	1.2742	1.4480	1.5924
4	.6389	.8565	1.0789	1.2854	1.4666

c	.6	.7	.8	.9
1	2.0000	2.0000	2.0000	2.0000
2	1.8256	1.8837	1.9307	1.9689
3	1.7109	1.8074	1.8858	1.9490
4	1.6199	1.7469	1.8503	1.9336

The values of β

c	ρ = .1	.2	.3	.4	.5
1	3.0000	3.0000	3.0000	3.0000	3.0000
2	1.7139	1.9384	2.1171	2.2617	2.3802
3	1.1731	1.4253	1.6596	1.8676	2.0481
4	.8836	1.1179	1.3640	1.6019	1.8202

c	.6	.7	.8	.9
1	3.0000	3.0000	3.0000	3.0000
2	2.4784	2.5603	2.6293	2.6876
3	2.2029	2.3348	2.4469	2.5420
4	2.0141	2.1829	2.3278	2.4512

Table 4. $E_M(W)$ and $V_M(W) - (E_M(W))^2$

$E_M(W)$ is the mean waiting time and $V_M(W)$ is the variance of the waiting time of the $M/M/c$ queue.

μ : the service rate, i.e., the reciprocal of the mean service time

c : the number of channels

ρ : the traffic intensity

The values of $\mu E_M(W)$

c	ρ = .1	.2	.3	.4	.5
1	.111111	.250000	.428571	.666667	1.00000
2	$.{}_{0}101010$	$.{}_{0}416667$	$.{}_{0}989011$	.190476	.333333
3	$.{}_{00}137174$	$.{}_{0}102740$	$.{}_{0}333471$	$.{}_{0}784314$	.157895
4	$.{}_{000}220677$	$.{}_{00}299401$	$.{}_{0}132321$	$.{}_{0}377916$	$.{}_{0}869565$

c	.6	.7	.8	.9
1	1.50000	2.33333	4.00000	9.00000
2	.562500	.960784	1.77778	4.26316
3	.295620	.547049	1.07865	2.72354
4	.179402	.357212	.745541	1.96938

The values of $\mu^2[V_M(W) - (E_M(W))^2]$

c	ρ = .1	.2	.3	.4	.5
1	.222222	.500000	.857143	1.33333	2.00000
2	$.{}_{0}110193$	$.{}_{0}486111$	.121724	.244898	.444444
3	$.{}_{00}101234$	$.{}_{00}835053$	$.{}_{0}295350$	$.{}_{0}748430$	.160665
4	$.{}_{000}122501$	$.{}_{00}185333$	$.{}_{00}910129$	$.{}_{0}286366$	$.{}_{0}718336$

c	.6	.7	.8	.9
1	3.00000	4.66667	8.00000	18.0000
2	.773438	1.35640	2.56790	6.28255
3	.317918	.617139	1.26853	3.32161
4	.159882	.340152	.752190	2.08998

Table 5. $E_D(W)$ and $V_D(W) - (E_D(W))^2$

$E_D(W)$ is the mean waiting time and $V_D(W)$ is the variance of the waiting time of the $M/D/c$ queue.

μ : the service rate, i.e., the reciprocal of the service time

c : the number of channels

ρ : the traffic intensity

The values of $\mu E_D(W)$

c	ρ = .1	.2	.3	.4	.5
1	$.{}_{0}555556$	.125000	.214286	.333333	.500000
2	$.{}_{00}620828$	$.{}_{0}242277$	$.{}_{0}552594$	.103311	.176741
3	$.{}_{000}947358$	$.{}_{00}658323$	$.{}_{0}200944$	$.{}_{0}450111$	$.{}_{0}872017$
4	$.{}_{000}164064$	$.{}_{00}205766$	$.{}_{00}845564$	$.{}_{0}226987$	$.{}_{0}496517$

c	.6	.7	.8	.9
1	.750000	1.16667	2.00000	4.50000
2	.293036	.493610	.903284	2.14692
3	.158409	.286247	.553900	1.37791
4	$.{}_{0}983759$	.189708	.386095	.999966

The values of $\mu^2[V_D(W) - (E_D(W))^2]$

c	ρ = .1	.2	.3	.4	.5
1	$.{}_{0}370370$	$.{}_{0}833333$	.142857	.222222	.333333
2	$.{}_{00}315930$	$.{}_{0}125048$	$.{}_{0}288624$	$.{}_{0}545179$	$.{}_{0}941175$
3	$.{}_{000}390189$	$.{}_{00}278456$	$.{}_{00}870264$	$.{}_{0}199065$	$.{}_{0}392936$
4	$.{}_{0000}567977$	$.{}_{000}738733$	$.{}_{00}313982$	$.{}_{00}869126$	$.{}_{0}195464$

c	.6	.7	.8	.9
1	.500000	.777778	1.33333	3.00000
2	.157322	.266970	.491863	1.17640
3	$.{}_{0}725915$	.133191	.261352	.658550
4	$.{}_{0}397126$	$.{}_{0}783496$	.162817	.429835

$$\hat{E}_G(W) = \left(\frac{\sum p_i \left(\frac{\theta_i}{k_i} \right)^{\alpha} (1+\alpha)(1+\frac{\alpha}{2}) \cdots (1+\frac{\alpha}{k_i-1})}{(\sum p_i\theta_i)^{\alpha}} \right)^{\frac{1}{\alpha-1}} E_M(W) \quad ,$$

which can be calculated using Tables 3 and 4.

These notes are also available for calculation of $\hat{V}_G(W)$ with trivial modifications.

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