ON LINEAR VECTOR MAXIMIZATION PROBLEMS

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Abstract   In this paper the solution sets of linear maximization problems with a vector criterion index are investigated. The properties of the solution sets are examined in terms of polar cones. A theorem is derived to show the necessary and sufficient conditions for the solution sets. An algorithm to find the solution sets is presented based on the theorem. An example is given to illustrate the theorem and the algorithm.

1.   Introduction

Several articles discussing vector valued optimization problems have been published, in which various solution points, non-inferior [12], efficient [2], Pareto-optimal [9], absolutely cooperative [9], optimal [12] and non-domination [10], [11] have been defined and investigated. The necessary and sufficient conditions for these solution points are given in [2], [3], [8], [9] and [12]. An algorithm to find a solution point is proposed in [5]. A non-inferior performance criterion is examined in a criterion index space [6]. These investigations are however, all based on scalarization of a vector valued criterion index and concern a solution point. Recently, a set of non-domination solutions has been investigated and methods to locate the set are preseted based on a concept of cone extreme points [10]. A method to find all solution points of linear problem is presented in [11] using a multicriteria simplex method.

One of the distinct features of the vector valued optimization problem, as is well known, is that the problem is solved by a set consisting of solution points. In order to maintain this characteristic it is significant [10] to

develop an algorithm to find the solution sets. In this paper we will be
concerned with solution sets of linear vector maximization problems (LVMP)
rather than a single solution point. After the formuration of the LVMP and the
definition of solution sets are mentioned, the necessary and sufficient condi-
tions for the solution sets will be derived by examining the intersection of a
polar cone and a convex cone. An algorithm to find the solution sets is
presented based on the theorem derived in this paper.

2.   Solution Sets of the Linear Vector Maximization Problems

     The LVMP considered in this paper is defined by

(1)   maximize A^Tx
                               $x \in B$

where A is an nxk matrix

(2) $A = [a_1, a_2, \ldots, a_k]$, $a_i \in R^n$

The constraint set B is given by

(3) $B = \{x| <b_i, x> - c^i \leq 0, i = 1, \ldots, s\}$,

where $<b_i, x>$ represents the scalar product between b_i and x. We use lower case
letters with or without subscripts as vectors and lower case letters with
superscripts as real numbers. Through this paper the ordering relations between
two vectors $a = b$, $a \geq b$, $a > b$, $a > b$ are used in the ordinary sense [4], [8].
Three kinds of solutions of the LVMP are known [2-3], [6], [8-10], [12], and
summarised as follows.

     Definition 1.   A point $x_o \in B$ is defined to be a non-inferior point if (4)
holds, a Pareto-optimal point if (5) holds, an optimal point if (6) holds; i.e.

(4)   there exists no other point in B such that $A^Tx > A^Tx_o$,

(5)   there exists no other point in B such that $A^Tx \geq A^Tx_o$,

(6) $A^Tx_o \geq A^Tx$,   for all $x \in B$.

Let Γ_n, Γ_p, and Γ_o, be the sets of non-inferior points, Preto-optimal points,
and optimal points of the LVMP, respectively.

     The following implications are clear by Definition 1.

(7) $\Gamma_n \supset \Gamma_p \supset \Gamma_o$

Although the set Γ_o may be the most desirable solution set, the solution sets
Γ_n and Γ_p are considered in this paper since we seldom encounter a problem when
Γ_o is nonempty.   In investigation of the solution sets a polar cone plays an

important role. The structure and the characteristics of the polar cone which are investigated in [7-8], [10] are briefly summarized.

Let $L(A)$ be a linear subspace spanned by the column vectors in A and let $L^{\perp}(A)$ be its orthogonal complement. A baiss for $L^{\perp}(A)$ is denoted by $\{e_1,\ldots,e_{n-r}\}$, $r \leq n$, where $L^{\perp}(A) = \{0\}$ for $n = r$. Let $C(A)$ be the cone which is generated by the column vectors in A, i.e.,

$$(8) \qquad C(A) = \{x \mid x = \sum_{i=1}^{k} \alpha^i a_i, \quad \alpha^i \geq 0\}.$$

The polar cone, the open polar cone, and semi-open polar cone of $C(A)$ are defined by

$$(9) \qquad P(A) = \{x \mid A^T x \geq 0\},$$

$$(10) \qquad P_o(A) = \{x \mid A^T x > 0\},$$

$$(11) \qquad P_s(A) = \{x \mid A^T x \geq 0\}.$$

It is proved in [8] that the polar cones given by a vector inequality are also represented in terms of generating vectors. For this purpose an edge vector w is introduced.

Definition 2. The vector $w \in P(A) \cap L(A)$ with unit norm is called an edge vector if w is not positively covered by any vectors in $P(A) \cap L(A) - C(w)$, where $C(w)$ is a one dimensional cone generated by w.

Proposition 1. Assume that rank $A = r$, then the vector w_i with unit norm is an edge vector of $P(A) \cap L(A)$ iff there exist subcollections $\{a_{i1},\ldots,a_{iq}\}$ of $\{a_1,\ldots,a_k\}$ such that

$$(12) \qquad \begin{array}{ll} \langle w_i, a_{ij} \rangle = 0, & j=1,\ldots q, \\ \langle w_i, a_t \rangle > 0, & t \notin \{i1,\ldots,iq\} \\ \langle w_i, e_j \rangle = 0, & j=1,\ldots,n-r, \\ \text{rank } [a_{i1},\ldots,a_{iq}] = r-1. \end{array}$$

Proposition 2. Given A by (2) and $C(A)$ by (8), then $P(A)$, $P_o(A)$, and $P_s(A)$ defined by (9), (10) and (11) respectively are represented by

$$(13) \qquad P(A) = \{x \mid x = \sum_{i=1}^{q} \alpha^i w_i + \sum_{i=1}^{n-r} \xi^i e_i, \ \alpha^i \geq 0, \ \xi^i \in R^1\},$$

$$(14) \qquad P_o(A) = \{x \mid x = \sum_{i=1}^{q} \alpha^i w_i + \sum_{i=1}^{n-r} \xi^i e_i, \ \alpha^i > 0, \ \xi^i \in R^1\},$$

$$\text{iff rank } A = \text{rank } [w_1,\ldots,w_q],$$

$$P_o(A) = \phi, \qquad \text{iff rank } A > \text{rank } [w_1,\ldots,w_q]$$

(15) $P_s(A) = \{x \mid x = \sum\limits_{i=1}^{q} \alpha^i w_i + \sum\limits_{i=1}^{n-r} \xi^i e_i, \; \alpha^i \geq 0,$

at least one $\alpha^i > 0, \; \xi^i \varepsilon \; R^1\}$,

where w_i, $i=1,\ldots,q$ is an edge vector of $P(A) \cap L(A)$

For the proof of the above propositions refer to [8]. Let $S(b_i)$ be an affine manifold defined by

(16) $S(b_i) = \{x \mid <b_i,x> - c^i = 0\}.$

Let F be a face [7], [8] of B and I(F) be the index set given by

(17) $I(F) = \{i \mid F \subset S(b_i), \; i = 1,\ldots,s\}.$

It is proved in [7] that F is represented by

(18) $F = B \cap (\bigcap\limits_{i \varepsilon I(F)} S(b_i))$

For the simplicity of representation assume that $I(F) = \{1,2,\ldots,m\}$. A closed convex cone is defined by

(19) $H^-(b_1,\ldots,b_m) = \{x \mid <b_i,x> \leq 0, \; i \; \varepsilon \; I(F)\}.$

It is obvious that $B \subset (H^-(b_1,\ldots,b_m) + x_s)$, where $x_s \varepsilon \; F$ and $H^-(b_1,\ldots,b_m) + x_s$ represents the transformation of the set $H^-(b_1,\ldots,b_m)$ by the vector x_s.

Let us define matrices Φ and Ψ and a vector θ by

(20) $\Phi = \begin{bmatrix} <w_1,b_1>,\ldots,<w_q,b_1> \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ <w_1,b_m>,\ldots,<w_q,b_m> \end{bmatrix}$

(21) $\Psi = \begin{bmatrix} <e_1,b_1>,\ldots,<e_{n-r},b_1> \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ <e_1,b_m>,\ldots,<e_{n-r},b_m> \end{bmatrix}$

(22) $\theta^T = [\theta^1,\ldots,\theta^m]$

Lemma 1. Given $P_o(A)$, $P_s(A)$ and $H^-(b_1,\ldots,b_m)$ by (14), (15) and (19) respectively, then (i), under the assumption such that $P_o(A) \neq \phi$,

(23) $H^-(b_1,\ldots,b_m) \cap P_o(A) = \phi,$

iff there exist nonnegative θ such that

(24) $\Phi^T\theta \geq 0,$
(25) $\Psi^T\theta = 0,$
and (ii)

(26) $H^-(b_1,\ldots,b_m) \cap P_s(A) = \phi,$

iff there exist nonnegative θ such that

(27) $$\phi^T \theta > 0,$$

(28) $$\psi^T \theta = 0.$$

Proof. First the "only if" part of (i) is shown. By Proposition 1 and $e_i \in L^\perp(A)$, $i = 1, \ldots, n-r$, the following relations are obtained,

$$A^T w_i \geq 0, \quad i = 1, \ldots, q,$$
$$A^T e_i = 0, \quad i = 1, \ldots, n-r.$$

It follows from (10) and (14) that

$$\sum_{i=1}^{q} \alpha^i A^T w_i > 0 \quad \text{for all } \alpha^i > 0, \quad i = 1, \ldots, q.$$

Hence, for every $j \in \{1, 2, \ldots, k\}$ there exists an integer $i \in \{1, 2, \ldots, q\}$ such that

(29) $$<a_j, w_i> \ > \ 0.$$

It is evident that $P_o(A) \cap H^-(b_1, \ldots, b_m) = \phi$, iff

(30) $$\{x \mid A^T x > 0, \quad -B_F^T x \geq 0\} = \phi,$$

where $B_F = [b_1, \ldots, b_m]$. Then Motzkin's alternative theorem [4] assures that

(31) $$\{y_1, y_2 \mid A \, y_1 - B_F y_2 = 0, \quad y_1 \geq 0, \quad y_2 \geq 0\} \neq \phi.$$

In other words, there exist y_1 and y_2 such that $A \, y_1 = B_F y_2$, $y_1 \geq 0$, $y_2 \geq 0$.
Therefore from (29) we obtain

$$<w_i, B_F y_2> \ = \ <w_i, A \, y_1> \ \geq \ 0, \ i = 1, \ldots, q,$$
$$\text{at least one } <w_i, A \, y_1> \ > \ 0,$$
$$<e_i, B_F y_2> \ = \ <e_i, A \, y_1> \ = \ 0, \ i = 1, \ldots, n-r,$$

which implies that (24) and (25) hold. The sufficiency of the theorem is evident by following convesely the above, since (30) and (31) are equivalent by the equivalency of Motzkin's theorem. This completes the proof of (i). The proof of (ii) is highly analogous to that of (i) and is not mentioned here.

Now we introduce two linear programming problems (LP). A solution of the LP will give us direct information about the solution set Γ_n, Γ_p. Assume that

(32) $$\Phi = [\phi_1, \phi_2, \ldots, \phi_q], \quad \phi_i \in R^m,$$

(33) $$\Psi = [\Psi_1, \Psi_2, \ldots, \Psi_{n-r}], \quad \Psi_i \in R^m.$$

Define $f^i(\theta)$, $g^i(\theta)$ and h by

$$f^i(\theta) = <\phi_i, \theta>, \quad i = 1, \ldots, s,$$

(34) $$g^i(\theta) = <\psi_i, \theta>, \quad i = 1, \ldots, n-r,$$

$$h = [1, \ldots, 1]^T.$$

The LP1 is given by

(35)
$$\max_{\theta} \sum_{i=1}^{s} f^i(\theta),$$

subject to

(36) $f^i(\theta) \geq 0,$ $i = 1,\ldots,s,$

(37) $\theta^i \geq 0,$ $i = 1,\ldots,m,$

(38) $1 - <h,\theta> \geq 0,$

(39) $g^i(\theta) = 0,$ $i = 1,\ldots,n-r$

The LP2 is given by

$$\max_{\theta,t} t$$

subject to (37), (38) and (39), and

(41) $f^i(\theta) \geq t,$ $i = 1,\ldots,s.$

Lemma 2.   (i) Assume that $P_o(A) \neq \phi$, then

(42) $H^-(b_1,\ldots,b_m) \cap P_o(A) = \phi,$

iff the LP1 has a solution such that

(43) $J(\theta_o) = \sum_{i=1}^{s} f^i(\theta_o) > 0.$

(ii) $H^-(b_1,\ldots,b_m) \cap P_s(A) = \phi,$

iff the LP2 has a solution such that

(44) $t^o > 0.$

In view of Lemma 1 the validity of this lemma is clear.

Theorem 1.   Let $P_o(A)$ and $P_s(A)$ be given by (14) and (15). Given a face F of B and the corresponding index set $I(F) = \{1,\ldots,m\}$. Then $F \subset \Gamma_n$, iff the LP1 has a solution which satisfies (43). $F \subset \Gamma_p$, iff the LP2 has a solution which satisfies (44).

Proof.   For the proof of this theorem, in view of Lemma 2, it is enough to show that the following two relations hold,

(45) $P_o(A) \cap H^-(b_1,\ldots,b_m) = \phi,$ iff $F \subset \Gamma_n,$

(46) $P_s(A) \cap H^-(b_1,\ldots,b_m) = \phi,$ iff $F \subset \Gamma_p.$

Assume that the relation in (45) holds, then
$$(P_o(A) + x_s) \cap (H^-(b_1,\ldots,b_m) + x_s) = \phi, \quad \text{for every } x_s \in F.$$
This shows that there exists no $x \in B$ such that $A^T x > A^T x_s$, since $B \subset H^-(b_1,\ldots,b_m) + x_s$. Thus $F \subset \Gamma_n$ is proved. Next assume that
$$P_o(A) \cap H^-(b_1,\ldots,b_m) \neq \phi,$$

then there exists x_s such that

(47) $\qquad\qquad\qquad A^T x_s > 0,$

(48) $\qquad\qquad\qquad <b_i, x_s> \leqq 0, \quad$ for all $i \in I(F)$.

Let x_k be a relative interior point of F, then

$$<b_i, x_k> = c^i, \quad i \in I(F)$$

(49)

$$<b_i, x_k> < c^i, \quad i \in \{\{1,\ldots,s\} - I(F)\}.$$

By combining (47), (48) and (49), we can choose a positive number ξ such that

$$<b_i, x_k + \xi x_s> \leqq c^i, \quad i = 1,\ldots,s,$$

$$A^T(x_k + \xi x_s) > A^T x_k,$$

which implies that $x_k \notin \Gamma_n$. This completes the proof of relation (45).
The proof of the other relation is almost the same as that of the first one and is not denoted here.

3. Algorithm to solve LVMP

In this section an algorithm, which is based on Theorem 1, is given to find a solution set Γ_n and Γ_p of the LVMP. For this algorithm it is necessary to obtain the representation of polar cones in terms of edge vectors as given in (14) and (15). In other words it is required that the edge vector w_i and basis vector e_i for a given matrix A are calculated. A calculation method for these vectors is given in [8] and is not repeated here. The outline of the algorithm to find $\Gamma_n(\Gamma_p)$ is as follows;

Step 1. If rank A = n, go to Step 2, otherwise (rank A = r < n) calculate a basis for $L^\perp(A)$.

Step 2. Calculate all edge vectors $\{w_1,\ldots,w_q\}$ of P(A) or $P(A) \cap L(A)$ (see [8] for the detail).

Step 3. If rank $[w_1,\ldots,w_q]$ < rank A, the $P_o(A) = \phi$ and $\Gamma_n = B$. In this case examine Γ_p only.

Step 4. $B* = \{b_i\}, i = 1,\ldots,s.$ Set m = 1.

Step 5. If $\bigcap_{j=1}^{m} S(b_{i_j}) \cap B = \phi$ for a m-tuples of distinct indices included in $1,\ldots,s$, take another m-tuples of them, otherwise use Theorem 1 to examine $\bigcap_{j=1}^{m} S(b_{i_j}) \cap B$ for a membership in $\Gamma_n(\Gamma_p)$.

Step 6. Suppose $i_k, k = 1,\ldots,s_m$ are indices such that

$\bigcap\limits_{k=1}^{m} S(b_{i_k}) \cap B \subset \Gamma_n(\Gamma_p)$. Delete b_{i_k}, $k = 1,\ldots,s_m$ from the set B^*,

m = m+1, and relabel subscripts. If $s - s_m \geq m$ and $m \leq n$ go to

Step 5, else algorithm terminates.

The main body of the algorithm is to solve the LP for applying Theorem 1 to a face. Through this algorithm, by solving the LP repeatedly, we can obtain the complete solution sets Γ_n and Γ_p. The elimination b_{i_k}, $k = 1,\ldots,s_m$, in Step 6 considerably reduces the number of faces of B to be examined. The validity of the elimination b_{i_k} is given in the following proposition.

Proposition 2. If a face F_1 of B with the index set $I(F_1)$ is a subset of the solution set Γ_n and/or Γ_p then any face with the index set $I(F_2)$ such that $I(F_2) \subset I(F_1)$ is also a subset of the solution sets.

This proposition is evident since $H^-(b_1,\ldots,b_m) \cap P_0(A) = \phi$ implies $H^-(b_1,\ldots,b_m,b_k) \cap P_0(A) = \phi$. It will be clear that the algorithm may be quite easily implemented when we have an interest in finding subsets of Γ_n and Γ_p which are large dimensional (n-1 or n-2 say), faces of B.

4. Example

For explanation of the algorithm in a concrete fashion, the following example is solved.

Example 1. Let us consider the following LVMP.

$$\text{maximize} \quad A^T x = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^T \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

subject to

$$[b_1 b_2 b_3 b_4 b_5]^T x = \begin{bmatrix} 1 & 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 & -1 \end{bmatrix}^T \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 6 \\ 10 \\ 0 \\ 0 \end{bmatrix}$$

To find solution sets, let us follow the algorithm.

Step 1. rank A = 2, $L^\perp(A) = \{0\}$

Step 2. The edge vectors of P(A) are obtained by.

$$W = [w_1 \quad w_2] = \begin{bmatrix} 1/\sqrt{2} & -2/\sqrt{13} \\ -1/\sqrt{2} & 3/\sqrt{13} \end{bmatrix}$$

Step 3. rank W = rank A = 2, and $P_o(A) \neq \phi$.

Step 4. $B* = \{b_1, b_2, b_3, b_4, b_5\}$, and $m = 1$.

Step 5. To examine if $F_1 = S(b_i) \cap B \subset \Gamma_n$ or not, solve the LP1 given by
$$\max_\theta \; f^1(\theta) + f^2(\theta).$$
subject to
$$f^j(\theta) \geq 0, \quad j = 1,2,$$
$$1 - \theta \geq 0, \quad \theta \geq 0,$$
where
$$f^j(\theta) = \langle w_j, b_i \rangle \theta, \quad j = 1,2.$$
By applying Theorem 1, we obtain
$$S(b_2) \cap B \subset \Gamma_n.$$

Step 6. $s_m = 1$, $s = 5 - 1 = 4$, $m = 2$. $B* = \{b_1, b_3, b_4, b_5\}$.

Step 5. To examine if $F_{14} = S(b_1) \cap S(b_4) \cap B \subset \Gamma_n$ or not, solve the LP1 given by
$$\max_\theta \; f^1(\theta) + f^2(\theta)$$
subject to
$$f^j(\theta) \geq 0, \quad j = 1,2,$$
$$1 - \langle h, \theta \rangle \geq 0, \quad \theta \geq 0,$$
where
$$f^j(\theta) = \langle w_j, b_1 \rangle \theta^1 + \langle w_j, b_4 \rangle \theta^2, \quad j = 1,2.$$
Examine faces F_{35} and F_{45} in the same way. By applying Theorem 1, we have no more solution subset.

Step 6. $s_m = 2$, $s = 2$, $m = 3 > n$, so that the algorithm terminates.
For the solution set Γ_p a slightly different procedure is applied. The solution sets obtained are
$$\Gamma_n = S(b_2) \cap B, \quad \text{and} \quad \Gamma_p = S(b_2) \cap S(b_3) \cap B.$$

These solution sets are illustrated in Fig. 1.

Example 2. Next we apply the algorithm to the following LVMP for $n = 4$.
$$\text{maximize } A^T x = \begin{bmatrix} -7 & 0 & -4 & -10 & -4 & 0 \\ 0 & 0 & -7 & 0 & -4 & -10 \\ -4 & 0 & 0 & 0 & -7 & 0 \\ -4 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}$$

subject to

$$[b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8]^T x = \begin{bmatrix} 1/5 & -5/8 & 0 & -1/50 & -1 & 0 & 0 & 0 \\ 1/25 & 1/4 & 1/10 & 1/20 & 0 & -1 & 0 & 0 \\ 2/25 & 1/8 & 1/5 & 5/8 & 0 & 0 & -1 & 0 \\ 4/25 & 2/25 & 1/8 & 1/8 & 0 & 0 & 0 & -1 \end{bmatrix}^T \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} \leqq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can obtain all solution sets as follows,

$$\Gamma_n = S(b_5) \cap B \cup S(b_6) \cap B \cup S(b_8) \cap B,$$
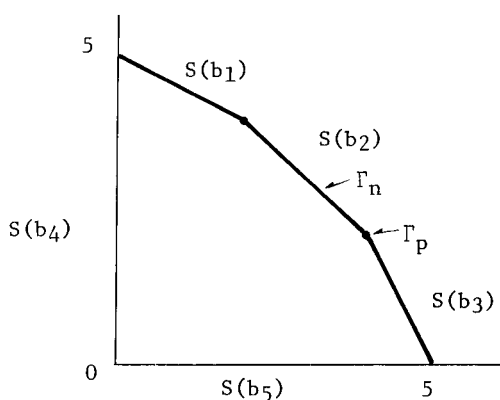$$\Gamma_p = S(b_5) \cap S(b_6) \cap S(b_7) \cap B.$$

Fig. 1.   The solution set Γ_n and Γ_p of Example 1.

5.   Concluding Remarks

This paper deals with the solution sets of the LVMP. The ideas contained in this article are displayed in Theorem 1, which can be used for the examination of a given face of a feasible set to decide whether it is a member of Γ_n and/or Γ_p. An algorithm to find the solution sets Γ_n and Γ_p is given. This algorithm is very useful when large dimensional faces of the feasible set are interested.

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