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TWO-PERSON ZERO-SUM GAMES WITH RANDOM PAYOFFS

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Abstract. This paper deals with two-person zero-sum rectangular games with random payoffs. It is assumed that each player knows the distribution functions of the random entries and that players must select their strategies before any observations of the random entries are made. In such a case, several models are considered and relations among the optimal values are obtained. A special case, in which these random entries are linear functions of a random variable is also treated and some properties of the optimal strategies are given. In the final section, illustrative examples are shown.

1. Preliminaries

In many of the practical situations which can be modeled as two-person zero-sum rectangular games, the elements of the payoff matrix may be known to the players as random variables with specified probability distributions. In this paper, we consider a two-person zero-sum rectangular game with an m by n random payoff matrix $A = (a_{ij})$. The random variable a_{ij} represents the payoff from player II to player I when player I plays row i and player II plays column j. We assume that each player knows the distribution of every random element in A and that the a_{ij} are independent of the mixed strategies selected by the players. We further assume that the players must select their strategies before any observations are taken on the a_{ij}. Thus, strategies are to be deterministic and are not to be explicit functions of the a_{ij}, although strategies will of course depend on the distributions of the a_{ij}. Under these circumstances, the question arises as to what is meant by playing the game in an optimal way. One possibility which suggests itself immediately is to replace a_{ij} by its expected value and then solve the resulting deterministic game. The model for player I is then rewritten as

$$\text{maximize}_{x\in X,\delta}\ \delta$$
$$\text{subject to}\ \ \min_{y\in Y} x^T \Lambda y \geqq \delta,$$

where

$$\Lambda = (\mathrm{E}(a_{ij})),$$

$$X = \{x^T = (x_1, x_2, \ldots , x_m) \mid \textstyle\sum_{i=1}^m x_i = 1,\ x_i \geqq 0 \text{ for } i = 1, 2, \ldots , m\}$$

and

$$Y = \{y = (y_1, y_2, \ldots , y_n)^T \mid \textstyle\sum_{j=1}^n y_j = 1,\ y_j \geqq 0 \text{ for } j = 1, 2, \ldots , n\}.$$

The corresponding problem for player II is

$$\text{minimize}_{y\in Y,\eta}\ \eta$$
$$\text{subject to}\ \ \max_{x\in X} x^T \Lambda y \leqq \eta.$$

Charnes et al. [3] considered chance-constrained games. The objective for player I is selecting a mixed strategy which maximizes the minimum value of the total payoff, δ, that he can attain with at least probability α, no matter what strategy player II may choose. The minimization is taken within the probability operator. In mathematical terms, player I wants to solve

(P1) $$\text{maximize}_{x\in X,\delta}\ \delta$$
$$\text{subject to}\ \ \mathrm{Prob}[\min_{y\in Y} x^T Ay \geqq \delta] \geqq \alpha,$$

where α $(0 < \alpha \leqq 1)$ is selected in advance by player I and unknown to player II. We denote the optimal value of δ in (P1) with a fixed probability level α by $\delta_1(\alpha)$. The corresponding problem for player II is

(P2) $$\text{minimize}_{y\in Y,\eta}\ \eta$$
$$\text{subject to}\ \ \mathrm{Prob}[\max_{x\in X} x^T Ay \leqq \eta] \geqq \beta,$$

where β $(0 < \beta \leqq 1)$ is pre-assigned by player II and unknown to player I. We denote the optimal value of η in (P2) with a fixed probability level β by $\eta_1(\beta)$.

Remarks

$$\mathrm{Prob}[\min_{y\in Y} x^T Ay \geqq \delta] \geqq \alpha$$

and

$$\mathrm{Prob}[\max_{x\in X} x^T Ay \leqq \eta] \geqq \beta$$

denote

$$\text{Prob}[x^T Ay \geqq \delta \quad \text{for all } y \in Y] \geqq \alpha$$

and

$$\text{Prob}[x^T Ay \leqq \eta \quad \text{for all } x \in X] \geqq \beta,$$

respectively. In what follows we use the former expression.

Recently, Blau [1] considered payoff-maximization problems and probability-maximization problems. For player I, the payoff-maximization problem by Blau is formulated mathematically as

$$\text{(P3)} \quad \text{maximize}_{x \in X, \delta} \ \delta$$
$$\text{subject to} \ \min_{y \in Y} \text{Prob}[x^T Ay \geqq \delta] \geqq \alpha,$$

where α ($0 < \alpha \leqq 1$) is a pre-assigned probability level selected by player I and unknown to his opponent. The interpretation of (P3) is that player I seeks a strategy that gives him the greatest payoff level, while, at the same time, guaranteeing that the probability of his total payoff exceeding the payoff level is always bounded below by α, no matter what strategy his opponent may use. We denote the optimal value of δ in (P3) with a fixed probability level α by $\delta_2(\alpha)$. Correspondingly, player II seeks a strategy that gives him the least payoff level, while, at the same time, guaranteeing that the probability of his total payoff not exceeding the payoff level is always bounded below by β, no matter what strategy his opponent may use, i.e., the problem for player II is written as follows:

$$\text{(P4)} \quad \text{minimize}_{y \in Y, \eta} \ \eta$$
$$\text{subject to} \ \min_{x \in X} \text{Prob}[x^T Ay \leqq \eta] \geqq \beta,$$

where β ($0 < \beta \leqq 1$) is chosen in advance by player II and unknown to player I. We denote the optimal value of η in (P4) with a fixed probability level β by $\eta_2(\beta)$. In the probability-maximization problems by Blau, player I chooses a payoff level δ and wishes to determine the maximum probability of his total payoff being bounded below by this level, independent of any strategy that his opponent may select. In mathematical terms, player I specifies δ and solves

$$\text{(P5)} \quad \text{maximize}_{x \in X, \alpha} \ \alpha$$
$$\text{subject to} \ \min_{y \in Y} \text{Prob}[x^T Ay \geqq \delta] \geqq \alpha.$$

Similarly, player II selects η, which is unknown to player I, and solves

(P6) $\quad \text{maximize}_{y\varepsilon Y,\beta}\ \beta$

$$\text{subject to}\quad \min_{x\varepsilon X} \text{Prob}[x^T Ay \leqq \eta] \geqq \beta.$$

Let $\alpha_2(\delta)$ be the optimal probability level α in (P5) with a fixed payoff level δ and let $\beta_2(\eta)$ be the optimal probability level β in (P6) with a fixed payoff level η.

Now, we establish other probability-maximization models. Suppose that player I chooses a payoff level δ which is unknown to player II and wishes to select a strategy which maximizes the probability of his minimum payoff being bounded below by δ no matter what strategy player II may use. The minimization is taken within the probability operator, and hence, player I is preparing against the possibility that his opponent will choose the most damaging strategy for whatever realization of a_{ij} may obtain. Thus, player I solves

(P7) $\quad \text{maximize}_{x\varepsilon X,\alpha}\ \alpha$

$$\text{subject to}\quad \text{Prob}[\min_{y\varepsilon Y} x^T Ay \geqq \delta] \geqq \alpha.$$

Correspondingly, player II specifies η which is unknown to his opponent and solves

(P8) $\quad \text{maximize}_{y\varepsilon Y,\beta}\ \beta$

$$\text{subject to}\quad \text{Prob}[\max_{x\varepsilon X} x^T Ay \leqq \eta] \geqq \beta.$$

We denote the optimal value of α in (P7) with a fixed payoff level δ by $\alpha_1(\delta)$ and the optimal value of β in (P8) with a fixed payoff level η by $\beta_1(\eta)$.

In the above models, it is not necessarily assumed that the random variables a_{ij} are mutually independent, although independence of the a_{ij} is assumed in the papers by Blau [1] and Charnes et al [3].

2. Relations among Models

In this section, we give several relations among the models in the preceding section.

Lemma 1. For any probability levels α and β,

$$\delta_1(\alpha) \leqq \delta_2(\alpha) \qquad \text{and} \qquad \eta_1(\beta) \geqq \eta_2(\beta).$$

Proof: Let x^* be an optimal strategy for (P1) with a probability level α. We have

$$\text{Prob}[x^{*T}Ay \geqq \delta_1(\alpha)] \geqq \text{Prob}[\min_{y\varepsilon Y} x^{*T}Ay \geqq \delta_1(\alpha)] \geqq \alpha$$

for all $y \in Y$,

and hence,

(1) $$\min_{y \in Y} \mathrm{Prob}[x^{*T}Ay \geqq \delta_1(\alpha)] \geqq \alpha.$$

It follows, from (1), that x^* and $\delta_1(\alpha)$ are feasible for (P3) with the probability level α, and so, $\delta_1(\alpha) \leqq \delta_2(\alpha)$. The proof of the second inequality is similar. This terminates our proof.

Lemma 2. For any payoff levels δ and η,

$$\alpha_1(\delta) \leqq \alpha_2(\delta) \qquad \text{and} \qquad \beta_1(\eta) \leqq \beta_2(\eta).$$

Proof: Let x^* be an optimal strategy for (P7) with a payoff level δ. Then we have

$$\mathrm{Prob}[x^{*T}Ay \geqq \delta] \geqq \mathrm{Prob}[\min_{y \in Y} x^{*T}Ay \geqq \delta] \geqq \alpha_1(\delta) \qquad \text{for all } y \in Y,$$

and hence,

(2) $$\min_{y \in Y} \mathrm{Prob}[x^{*T}Ay \geqq \delta] \geqq \alpha_1(\delta).$$

It follows, from (2), that x^* and $\alpha_1(\delta)$ are feasible for (P5) with the payoff level δ, and so, $\alpha_1(\delta) \leqq \alpha_2(\delta)$. The proof of the second inequality is similar. This terminates our proof.

Theorem 1. If $\alpha + \beta > 1$, then $\delta_2(\alpha) \leqq \eta_2(\beta)$.

Proof: Let x^* be an optimal strategy for (P3) with a probability level α and let y^* be an optimal strategy for (P4) with a probability level β. Then we have

$$\mathrm{Prob}[x^{*T}Ay \geqq \delta_2(\alpha)] \geqq \alpha \qquad \text{for all } y \in Y,$$

and thus,

(3) $$\mathrm{Prob}[x^{*T}Ay^* \geqq \delta_2(\alpha)] \geqq \alpha.$$

Similarly, we get

(4) $$\mathrm{Prob}[x^{*T}Ay^* \leqq \eta_2(\beta)] \geqq \beta.$$

If $\alpha + \beta > 1$, then (4) yields

$$\mathrm{Prob}[x^{*T}Ay^* \leqq \eta_2(\beta)] > 1 - \alpha,$$

and hence,

(5) $$\mathrm{Prob}[x^{*T}Ay^* > \eta_2(\beta)] < \alpha.$$

From (3) and (5), we obtain $\delta_2(\alpha) \leqq \eta_2(\beta)$. This terminates our proof.

The same type of proof as above establishes

Corollary 1. Let the distribution functions of the a_{ij} be strictly increasing over $(-\infty, \infty)$. If $\alpha + \beta \geqq 1$, then $\delta_2(\alpha) \leqq \eta_2(\beta)$.

Corollary 2. If $\alpha + \beta > 1$, then $\delta_1(\alpha) \leqq \eta_1(\beta)$.

Proof: This is a direct consequence of Lemma 1 and Theorem 1.

Blau [1] showed $\delta_2(\alpha) \leqq \eta_2(\beta)$ when $\alpha \geqq 0.5$ and $\beta \geqq 0.5$ under the following assumptions:

1. The elements of $A = (a_{ij})$ are mutually independent and belong to a symmetric stable distribution with the common characteristic exponent τ such that $1 < \tau \leqq 2$.
2. Each a_{ij} has the common scale parameter $\theta > 0$.

(Note that if a_{ij} has a symmetric stable distribution, then the distribution function is strictly increasing over $(-\infty, \infty)$.) Theorem 1 and Corollaries 1 and 2 hold without such assumptions. In fact, the random variables a_{ij} may be dependent.

3. Linear Payoff Functions

In what follows, we treat a two-person zero-sum rectangular games with an m by n payoff matrix $A = (b_{ij}Z + c_{ij})$, where b_{ij} and c_{ij} are constants and Z is a random variable with a known distribution function. Such a case might occur many times in practical situation. Suppose, for example, that the payoff from player II to player I distributes normally with mean μ_{ij} and variance σ_{ij}^2 when player I plays row i and player II plays column j. Then the situation reduces to our model by putting Z to be the standard nornal distribution and $b_{ij} = \sigma_{ij}$ and $c_{ij} = \mu_{ij}$. Furthermore, suppose that p (the probability that an incoming plane is a friend) in I.F.F. game and p^* (the rate of effectiveness) in advertising game (see Chapter 4 in Karlin [4]) are not constants but random variables with a known distribution function. Then the games can be treated as our model.

We denote problems (P1), (P2), ... , (P8) with such linear payoffs by (P1'), (P2'), ... , (P8'), respectively, and we denote the optimal values $\delta_i(\alpha)$, $\eta_i(\beta)$, $\alpha_i(\delta)$ and $\beta_i(\eta)$ by $\delta_i^*(\alpha)$, $\eta_i^*(\beta)$, $\alpha_i^*(\delta)$ and $\beta_i^*(\eta)$, respectively. Note that (P1') and (P3') with $0.5 < \alpha \leqq 1$ and (P2') and (P4') with $0.5 < \beta \leqq 1$ should appeal to some conservative players and we assume them henceforth.

We shall use the following notation:

$B = (b_{ij}) = (B_1, B_2, \dots, B_n)$,

$C = (c_{ij}) = (C_1, C_2, \dots, C_n)$,

$X_1 = \{x \in X \mid x^T B_j \geqq 0 \quad \text{for all } j\}$,

$X_2 = \{x \in X \mid x^T B_j \leqq 0 \quad \text{for all } j\}$,

$X_3 = X - X_1 - X_2$,

$Y_1(x) = \{y \in Y \mid x^T By \geqq 0\}$,

$Y_2(x) = \{y \in Y \mid x^T By \leqq 0\}$,

$Y_3(x) = \{y \in Y \mid x^T By = 0\}$,

$Y^*(x) = \{y \in Y \mid x^T By > 0\}$,

$J_1(x) = \{j \mid x^T B_j > 0\}$,

$J_2(x) = \{j \mid x^T B_j < 0\}$,

$J_3(x) = \{j \mid x^T B_j = 0\}$,

$F_1^{-1}(\alpha) = \sup\{\omega \mid \alpha \geqq \text{Prob}[Z < \omega]\}$

and

$F_2^{-1}(\alpha) = \inf\{\omega \mid \alpha \leqq \text{Prob}[Z \leqq \omega]\}$.

3.1. Optimal strategies for (P3') and (P4')

Let $G_1(s, t)$ be a two-person zero-sum rectangular game with the m by $2n$ payoff matrix

$$\begin{pmatrix} F_1^{-1}(s)b_{11}+c_{11} & \cdots & F_1^{-1}(s)b_{1n}+c_{1n} & F_2^{-1}(t)b_{11}+c_{11} & \cdots & F_2^{-1}(t)b_{1n}+c_{1n} \\ F_1^{-1}(s)b_{21}+c_{21} & \cdots & F_1^{-1}(s)b_{2n}+c_{2n} & F_2^{-1}(t)b_{21}+c_{21} & \cdots & F_2^{-1}(t)b_{2n}+c_{2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ F_1^{-1}(s)b_{m1}+c_{m1} & \cdots & F_1^{-1}(s)b_{mn}+c_{mn} & F_2^{-1}(t)b_{m1}+c_{m1} & \cdots & F_2^{-1}(t)b_{mn}+c_{mn} \end{pmatrix}$$

and let $v_1(s, t)$ be the value of the game. Similarly. let $G_2(s, t)$ be a two-person zero-sum rectangular game with the $2m$ by n payoff matrix

$$\begin{pmatrix} F_1^{-1}(s)b_{11}+c_{11} & F_1^{-1}(s)b_{12}+c_{12} & \cdots & F_1^{-1}(s)b_{1n}+c_{1n} \\ \vdots & \vdots & & \vdots \\ F_1^{-1}(s)b_{m1}+c_{m1} & F_1^{-1}(s)b_{m2}+c_{m2} & \cdots & F_1^{-1}(s)b_{mn}+c_{mn} \\ F_2^{-1}(t)b_{11}+c_{11} & F_2^{-1}(t)b_{12}+c_{12} & \cdots & F_2^{-1}(t)b_{1n}+c_{1n} \\ \vdots & \vdots & & \vdots \\ F_2^{-1}(t)b_{m1}+c_{m1} & F_2^{-1}(t)b_{m2}+c_{m2} & \cdots & F_2^{-1}(t)b_{mn}+c_{mn} \end{pmatrix}$$

and let $v_2(s, t)$ be the value of the game.

Since $\mathrm{Prob}[x^T(BZ + C)y \geqq \delta] \geqq \alpha$ is identical with

$$x^T\{F_1^{-1}(1 - \alpha)B + C\}y \geqq \delta \qquad \text{if } x^TBy \geqq 0$$

and

$$x^T\{F_2^{-1}(\alpha)B + C\}y \geqq \delta \qquad \text{if } x^TBy \leqq 0,$$

the stochastic problem (P3') is reduced to the following deterministic problem (PI):

$$\text{(PI)} \quad \text{maximize}_{x\in X,\delta}\ \delta$$
$$\text{subject to } x^T\{F_1^{-1}(1 - \alpha)B + C\}y \geqq \delta \qquad \text{for all } y \in Y_1(x),$$
$$x^T\{F_2^{-1}(\alpha)B + C\}y \geqq \delta \qquad \text{for all } y \in Y_2(x).$$

Now, let us consider a problem (PII):

$$\text{(PII)} \quad \text{maximize}_{x\in X,\delta}\ \delta$$
$$\text{subject to } x^T\{F_1^{-1}(1 - \alpha)B + C\}y \geqq \delta \qquad \text{for all } y \in Y,$$
$$x^T\{F_2^{-1}(\alpha)B + C\}y \geqq \delta \qquad \text{for all } y \in Y.$$

As is well known, the optimal value of δ in (PII) is equal to $v_1(1-\alpha, \alpha)$ and an optimal vector x for (PII) is a player I's optimal strategy for $G_1(1-\alpha, \alpha)$ and the reverse is also true.

The following theorem gives a technique to obtain $\delta_2^*(\alpha)$ and an optimal strategy for (P3') with a probability level α (> 0.5).

Theorem 2. If $0.5 < \alpha$, then $\delta_2^*(\alpha) = v_1(1-\alpha, \alpha)$ and player I's optimal strategies for the rectangular game $G_1(1-\alpha, \alpha)$ are optimal strategies for (P3') with the probability level α.

Proof: Let x^* be an optimal vector for (PI). We have

$$x^{*T}\{F_1^{-1}(1-\alpha)B + C\}y \geqq \delta_2^*(\alpha) \qquad \text{for all } y \in Y_1(x^*)$$

and

$$x^{*T}\{F_2^{-1}(\alpha)B + C\}y \geqq \delta_2^*(\alpha) \qquad \text{for all } y \in Y_2(x^*).$$

Since $0.5 < \alpha$,

$$\mathrm{Prob}[Z < F_1^{-1}(1-\alpha)] \leqq 1 - \alpha < \alpha \leqq \mathrm{Prob}[Z \leqq F_2^{-1}(\alpha)],$$

and thus,

$$F_1^{-1}(1-\alpha) \leqq F_2^{-1}(\alpha).$$

Therefore, we get

$$x^{*T}\{F_2^{-1}(\alpha)B + C\}y \geqq x^{*T}\{F_1^{-1}(1-\alpha)B + C\}y \geqq \delta_2^*(\alpha) \qquad \text{for all } y \in Y_1(x^*)$$

and

$$x^{*T}\{F_1^{-1}(1-\alpha)B + C\}y \geqq x^{*T}\{F_2^{-1}(\alpha)B + C\}y \geqq \delta_2^*(\alpha) \qquad \text{for all } y \in Y_2(x^*).$$

Thus,

$$x^{*T}\{F_1^{-1}(1-\alpha)B + C\}y \geqq \delta_2^*(\alpha) \qquad \text{for all } y \in Y \tag{6}$$

and

$$x^{*T}\{F_2^{-1}(\alpha)B + C\}y \geqq \delta_2^*(\alpha) \qquad \text{for all } y \in Y. \tag{7}$$

It follows, from (6) and (7), that x^* and $\delta_2^*(\alpha)$ are feasible for (PII), and hence, $\delta_2^*(\alpha) \leqq v_1(1-\alpha, \alpha)$. On the other hand, it is obvious that the optimal value of δ for (PII) is less than or equal to $\delta_2^*(\alpha)$, i.e., $\delta_2^*(\alpha) \geqq v_1(1-\alpha, \alpha)$. Therefore, we get $\delta_2^*(\alpha) = v_1(1-\alpha, \alpha)$, and so, player I's optimal strategies for $G_1(1-\alpha, \alpha)$ are optimal strategies for (P3'). This terminates our proof.

When $\alpha > 0.5$, Theorem 2 implies that $\delta_2^*(\alpha)$ depends on the distribution function only through $F_1^{-1}(1-\alpha)$ and $F_2^{-1}(\alpha)$ and not through the other properties of the distribution. Thus, for two random variables Z and Z' with the same values of $F_1^{-1}(1-\alpha)$ and $F_2^{-1}(\alpha)$, the optimal values of δ in (P3') with the probability level α are identical. The following theorem can be proved by the similar method as above.

Theorem 2'. If $0.5 < \beta$, then $\eta_2^*(\beta) = v_2(1-\beta, \beta)$ and player II's optimal strategies for the rectangular game $G_2(1-\beta, \beta)$ are optimal strategies for (P4') with the probability level β.

Corollary 3. Let $F_1^{-1}(0.5) = F_2^{-1}(0.5)$. If $0.5 \leqq \alpha$ and $0.5 \leqq \beta$, then

$$\delta_2^*(\alpha) \leqq v_1(0.5,\ 0.5) = v_2(0.5,\ 0.5) \leqq \eta_2^*(\beta).$$

Proof: If $F_1^{-1}(0.5) = F_2^{-1}(0.5)$, then we can show that $v_1^*(0.5,\ 0.5) = \delta_2^*(0.5)$ and $v_2^*(0.5,\ 0.5) = \eta_2^*(0.5)$ by the similar method as in the proof of Theorem 2. The corollary follows directly from $\delta_2^*(\alpha)$ and $\eta_2^*(\beta)$ being non-increasing and non-decreasing functions of α and β, respectively, and from $v_1(0.5,\ 0.5) = v_2(0.5,\ 0.5)$.

3.2. Optimal strategies for (P5') and (P6')

In the remaining parts of this section, we assume that the distribution function of Z is absolutely continuous. We denote by F^{-1} the inverse function of the distribution function of Z. Since the results that can be developed for player II are often obvious analogues of those for player I, these analogues will not be stated when they are apparent.

Theorem 3. If $\delta \leqq v_1(0.5,\ 0.5)$, then

$$\alpha_2^*(\delta) = \max\{\alpha \mid v_1(1-\alpha,\ \alpha) \geqq \delta\}.$$

Proof: Let α be a probability level with $v_1(1-\alpha,\ \alpha) \geqq \delta$. Then there is an $x \in X$ such that

$$x^T\{F^{-1}(1-\alpha)B + C\}y \geqq \delta \qquad \text{for all } y \in Y$$

and

$$x^T\{F^{-1}(\alpha)B + C\}y \geqq \delta \qquad \text{for all } y \in Y.$$

Hence,

$$\mathrm{Prob}[x^T(BZ + C)y \geqq \delta] \geqq \alpha \qquad \text{for all } y \in Y,$$

and thus, $\alpha_2^*(\delta) \geqq \alpha$. Therefore, we get

$$\alpha_2^*(\delta) \geqq \max\{\alpha \mid v_1(1-\alpha,\ \alpha) \geqq \delta\}.$$

To prove the converse inequality, let an x^* ($\in X$) be an optimal strategy for (P5') with the payoff level δ. From $\delta \leqq v_1(0.5,\ 0.5)$, it follows that $\alpha_2^*(\delta) \geqq 0.5$, and hence, we get

$$x^{*T}\{F^{-1}(1-\alpha_2^*(\delta))B + C\}y \geqq \delta \qquad \text{for all } y \in Y$$

and

$$x^{*T}\{F^{-1}(\alpha_2^*(\delta))B + C\}y \geqq \delta \qquad \text{for all } y \in Y.$$

Thus, $v_1(1-\alpha_2^*(\delta),\ \alpha_2^*(\delta)) \geqq \delta$. Therefore, we obtain the desired result.

Since $F^{-1}(\alpha)$ is continuous, $\delta_2^*(\alpha) = v_1(1-\alpha,\ \alpha)$ is a continuous decreasing function of α. Theorem 3 implies that α^* is the optimal probability level in (P5') with a payoff level δ if and only if α^* is the maximum root of the equation $v_1(1-\alpha, \alpha) = \delta$. Hence, in order to solve (P5'), it suffices to give a technique for finding the maximum root of a continuous decreasing function. As there are many available techniques, we do not specify any details.

3.3. Optimal strategies for (P1') and (P2')

Theorem 4. If there is an optimal strategy x^* for (P3') with a probability level α such that $x^* \in X_1 \cup X_2$, then $\delta_1^*(\alpha) = \delta_2^*(\alpha)$ and the x^* is also an optimal strategy for (P1') with the probability level α.

Proof: Let x^* be an optimal strategy for (P3') with a probability level α. Then we have

$$\text{(8)} \qquad \text{Prob}[x^{*T}(BZ + C)y \geqq \delta_2^*(\alpha)] \geqq \alpha \qquad \text{for all } y \in Y.$$

If $x^* \in X_1$, then we get

$$\text{Prob}[Z \geqq \{\delta_2^*(\alpha) - x^{*T}Cy\} \,/\, x^{*T}By] \geqq \alpha \qquad \text{for all } y \in Y^*(x^*),$$

and hence,

$$\text{Prob}[Z \geqq \max_{y \in Y^*(x^*)} \{\delta_2^*(\alpha) - x^{*T}Cy\} \,/\, x^{*T}By] \geqq \alpha.$$

Therefore,

$$\text{(9)} \qquad \text{Prob}[x^{*T}(BZ + C)y \geqq \delta_2^*(\alpha) \quad \text{for all } y \in Y^*(x^*)] \geqq \alpha.$$

From (8), we further obtain

$$\text{(10)} \qquad x^{*T}Cy \geqq \delta_2^*(\alpha) \qquad \text{for all } y \in Y_3(x^*).$$

Now, (9) and (10) yield

$$\text{Prob}[\min_{y \in Y} x^{*T}(BZ + C)y \geqq \delta_2^*(\alpha)] \geqq \alpha,$$

and hence, $\delta_2^*(\alpha) \leqq \delta_1^*(\alpha)$. Since $\delta_2^*(\alpha) \geqq \delta_1^*(\alpha)$ from Lemma 1, we have $\delta_2^*(\alpha) = \delta_1^*(\alpha)$, and so, x^* is an optimal strategy for (P1') with the probability level α. If $x^* \in X_2$, then the similar argument yields the same conclusion. This terminates our proof.

Theorem 5. For any probability levels α and β,

$$\delta_1^*(\alpha) = \max_{0 \leqq \gamma \leqq 1-\alpha} v_1(\gamma, \alpha + \gamma)$$

and

$$\eta_1^*(\beta) = \min_{0 \leqq \gamma \leqq 1-\beta} v_2(\gamma, \beta + \gamma).$$

Proof: We first note that, for any x such that $\min_{j\varepsilon J_3(x)} x^T C_{\cdot j} \geqq \delta$,

$$\text{Prob}[\min_{y\varepsilon Y} x^T(BZ + C)y \geqq \delta] \geqq \alpha$$

is equivalent to

$$\text{Prob}[\max_{j\varepsilon J_1(x)} (\delta - x^T C_{\cdot j}) / x^T B_{\cdot j} \leqq Z] \geqq \alpha,$$

$$\text{Prob}[Z \leqq \min_{j\varepsilon J_2(x)} (\delta - x^T C_{\cdot j}) / x^T B_{\cdot j}] \geqq \alpha$$

and

$$\text{Prob}[\max_{j\varepsilon J_1(x)} (\delta - x^T C_{\cdot j}) / x^T B_{\cdot j} \leqq Z \leqq \min_{j\varepsilon J_2(x)} (\delta - x^T C_{\cdot j}) / x^T B_{\cdot j}] \geqq \alpha,$$

if $x \varepsilon X_1$, $x \varepsilon X_2$ and $x \varepsilon X_3$, respectively.

Let x^* be a player I's optimal strategy for a rectangular game $G_1(\gamma, \alpha + \gamma)$, where $0 \leqq \gamma \leqq 1 - \alpha$. We have

$$x^{*T}\{F^{-1}(\gamma)B_{\cdot j} + C_{\cdot j}\} \geqq v_1(\gamma, \alpha + \gamma) \qquad \text{for all } j$$

and

$$x^{*T}\{F^{-1}(\alpha + \gamma)B_{\cdot j} + C_{\cdot j}\} \geqq v_1(\gamma, \alpha + \gamma) \qquad \text{for all } j.$$

Hence, if $x^* \varepsilon X_3$, then

$$[v_1(\gamma, \alpha + \gamma) - x^{*T} C_{\cdot j}] / x^{*T} B_{\cdot j} \leqq F^{-1}(\gamma) \qquad \text{for all } j \varepsilon J_1(x^*),$$

$$[v_1(\gamma, \alpha + \gamma) - x^{*T} C_{\cdot j}] / x^{*T} B_{\cdot j} \geqq F^{-1}(\alpha + \gamma) \qquad \text{for all } j \varepsilon J_2(x^*)$$

and

$$x^{*T} C_{\cdot j} \geqq v_1(\gamma, \alpha + \gamma) \qquad \text{for all } j \varepsilon J_3(x^*).$$

Thus, we get

$$\text{(11)} \qquad \text{Prob}[\min_{y\varepsilon Y} x^{*T}(BZ + C)y \geqq v_1(\gamma, \alpha + \gamma)] \geqq \alpha.$$

Similarly, we obtain (11) even if $x^* \varepsilon X_1 \cup X_2$, and so,

$$\delta_1^*(\alpha) \geqq v_1(\gamma, \alpha + \gamma) \qquad \text{for all } \gamma \text{ such that } 0 \leqq \gamma \leqq 1 - \alpha.$$

Therefore, we have

$$\delta_1^*(\alpha) \geqq \max_{0 \leqq \gamma \leqq 1-\alpha} v_1(\gamma, \alpha + \gamma).$$

To prove the converse inequality, let an x' satisfy

$$\text{Prob}[\min_{y\varepsilon Y} x'^T(BZ + C)y \geqq \delta_1^*(\alpha)] \geqq \alpha.$$

If $x' \varepsilon X_3$, then

$$\min_{j\varepsilon J_3(x')} x'^T C_{\cdot j} \geqq \delta_1^*(\alpha)$$

and there is a γ^* $(0 \leqq \gamma^* \leqq 1 - \alpha)$ which satisfies

$$\text{Prob}[\max_{j \varepsilon J_1(x')} \{\delta_1^*(\alpha) - x'^T C_j\} / x'^T B_j \leqq Z] = 1 - \gamma^*$$

and

$$\text{Prob}[\min_{j \varepsilon J_2(x')} \{\delta_1^*(\alpha) - x'^T C_j\} / x'^T B_j \geqq Z] \geqq \alpha + \gamma^*.$$

Thus, we get

$$x'^T\{F^{-1}(\gamma^*)B_j + C_j\} \geqq \delta_1^*(\alpha) \qquad \text{for all } j \varepsilon J_1(x')$$

and

$$x'^T\{F^{-1}(\alpha + \gamma^*)B_j + C_j\} \geqq \delta_1^*(\alpha) \qquad \text{for all } j \varepsilon J_2(x').$$

Whereas, if $j \varepsilon J_1(x')$, then

$$x'^T\{F^{-1}(\alpha + \gamma^*)B_j + C_j\} \geqq x'^T\{F^{-1}(\gamma^*)B_j + C_j\}$$

and if $j \varepsilon J_2(x')$, then

$$x'^T\{F^{-1}(\gamma^*)B_j + C_j\} \geqq x'^T\{F^{-1}(\alpha + \gamma^*)B_j + C_j\}.$$

Hence, we have

$$x'^T\{F^{-1}(\alpha + \gamma^*)B_j + C_j\} \geqq \delta_1^*(\alpha) \qquad \text{for all } j$$

and

$$x'^T\{F^{-1}(\gamma^*)B_j + C_j\} \geqq \delta_1^*(\alpha) \qquad \text{for all } j.$$

Therefore,

$$v_1(\gamma^*, \alpha + \gamma^*) \geqq \delta_1^*(\alpha) \qquad \text{for a } \gamma^* \text{ such that } 0 \leqq \gamma^* \leqq 1 - \alpha.$$

We can get the same conclusion even if $x' \varepsilon X_1 \cup X_2$. Accordingly, we obtain

$$\delta_1^*(\alpha) = \max_{0 \leqq \gamma \leqq 1-\alpha} v_1(\gamma, \alpha + \gamma).$$

The second equality of the theorem is proved by the similar method. This terminates our proof.

The following corollary is a direct consequence of the theorem.

Corollary 4. For any probability levels α and β,

$$\eta_1^*(\beta) - \delta_1^*(\alpha) \leqq v_2\{(1-\beta)/2, (1+\beta)/2\} - v_1\{(1-\alpha)/2, (1+\alpha)/2\}.$$

3.4. Optimal strategies for (P7') and (P8')

Theorem 6. If there is an optimal strategy x^* for (P5') with a payoff level δ such that $x^* \varepsilon X_1 \cup X_2$, then $\alpha_1^*(\delta) = \alpha_2^*(\delta)$ and the x^* is also an optimal strategy for (P7') with the payoff level δ.

Proof: Let $x^* \varepsilon X_1$ be an optimal strategy for (P5') with a payoff level δ. Since

(12) $\quad \text{Prob}[x^{*T}(BZ + C)y \geqq \delta] \geqq \alpha_2^*(\delta) \qquad$ for all $y \in Y$,

we get

$$\text{Prob}[Z \geqq (\delta - x^{*T}Cy) \;/\; x^{*T}By] \geqq \alpha_2^*(\delta) \qquad \text{for all } y \in Y^*(x^*),$$

and hence,

(13) $\quad \text{Prob}[x^{*T}(BZ + C)y \geqq \delta \quad \text{for all } y \in Y^*(x^*)]$

$$= \text{Prob}[Z \geqq \max_{y \in Y^*(x^*)} (\delta - x^{*T}Cy) \;/\; x^{*T}By] \geqq \alpha_2^*(\delta).$$

From (12), we further obtain

(14) $\quad x^{*T}Cy \geqq \delta \qquad$ for all $y \in Y_3(x^*)$.

Now, (13) and (14) yield

$$\text{Prob}[\min_{y \in Y} x^{*T}(BZ + C)y \geqq \delta] \geqq \alpha_2^*(\delta),$$

and so, $\alpha_2^*(\delta) \leqq \alpha_1^*(\delta)$. Since $\alpha_2^*(\delta) \geqq \alpha_1^*(\delta)$ from Lemma 2, we get $\alpha_2^*(\delta) = \alpha_1^*(\delta)$. Thus, x^* is an optimal strategy for (P7') with the payoff level δ. If $x^* \in X_2$, then the similar argument yields the same conclusion. This terminates our proof.

The following theorem is proved by the similar method as in Theorem 5.

Theorem 7. For any payoff levels δ and η,

$$\alpha_1^*(\delta) = \max\{u - v \mid v_1(u, v) \geqq \delta\}$$

and

$$\beta_1^*(\eta) = \max\{u - v \mid v_2(u, v) \leqq \eta\}.$$

4. Examples

In this section, we give brief examples which illustrate some of the results in the preceding sections.

Example 1. Suppose that (b_{ij}) and (c_{ij}) are given as follows:

$$(b_{ij}) = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} \qquad (c_{ij}) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

Let Z be an uniformly distributed random variable over (0, 1). If $\alpha = 0.7$, then (P3') is equivalent to the deterministic game

$$\begin{pmatrix} 0.6 & 0.7 & 1.4 & 0.3 \\ 1.4 & 0.9 & 0.6 & 2.1 \end{pmatrix}.$$

The value of the game is 0.84 and $x = (0.3, 0.7)$ is a player I's optimal strategy for the game. Hence, $\delta_2^*(0.7) = 0.84$ and $x = (0.3, 0.7)$ is an optimal strategy for (P3') with $\alpha = 0.7$. Similarly, (P4') with $\beta = 0.7$ is equivalent to the deterministic game

$$\begin{pmatrix} 0.6 & 0.7 \\ 1.4 & 0.9 \\ 1.4 & 0.3 \\ 0.6 & 2.1 \end{pmatrix}.$$

Therefore, $\eta_2^*(0.7) = 1.2$ and $y^* = (y_1^*, y_2^*) = (0.6, 0.4)$ is an optimal strategy for (P4') with $\beta = 0.7$. Since

$$b_{i1}y_1^* + b_{i2}y_2^* \geqq 0 \qquad \text{for } i = 1 \text{ and } 2,$$

y^* is also an optimal strategy for (P2') with $\beta = 0.7$ and $\eta_1^*(0.7) = 1.2$. From Theorem 5, it follows that

$$\delta_1^*(0.7) = \max_{0 \leqq \gamma \leqq 0.3} \text{val} \begin{pmatrix} 2\gamma & 1-\gamma & 1.4+2\gamma & 0.3-\gamma \\ 2-2\gamma & 3\gamma & 0.6-2\gamma & 2.1+3\gamma \end{pmatrix}.$$

The value of the game

$$\begin{pmatrix} 2\gamma & 1 - \gamma & 1.4 + 2\gamma & 0.3 - \gamma \\ 2 - 2\gamma & 3\gamma & 0.6 - 2\gamma & 2.1 + 3\gamma \end{pmatrix}$$

is

$$(4\gamma^2 + 4\gamma - 2) / (8\gamma - 3) \qquad \text{for } 0 \leqq \gamma \leqq 0.25$$

and

$$(4\gamma^2 + 6.8\gamma - 0.6) / (8\gamma - 0.2) \qquad \text{for } 0.25 \leqq \gamma \leqq 0.3,$$

which is increasing over (0, 0.3). Hence, $\delta_1^*(0.7) = 9/11$ and $x^* = (9/22, 13/22)$ is an optimal strategy for (P1') with $\alpha = 0.7$. Thus, we have

$$\delta_1^*(0.7) < \delta_2^*(0.7) < \eta_2^*(0.7) = \eta_1^*(0.7).$$

Example 2. Suppose that (b_{ij}), (c_{ij}) and Z are the same as in Example 1. Let us solve (P5') with $\delta = 0.8$. From $\delta_2^*(0.7) = 0.84$, it follows that $0.7 \leqq \alpha_2^*(0.8)$. Since (P3') with $\alpha = 0.8$ is equivalent to the deterministic game

$$\begin{pmatrix} 0.4 & 0.8 & 1.6 & 0.2 \\ 1.6 & 0.6 & 0.4 & 2.4 \end{pmatrix},$$

$\delta_2^*(0.8) = 26/35$, and so, $0.7 \leqq \alpha_2^*(0.8) \leqq 0.8$. Problem (P3') with $\alpha = 0.74$ reduces to the game

$$\begin{bmatrix} 0.52 & 0.74 & 1.48 & 0.26 \\ 1.48 & 0.78 & 0.52 & 2.22 \end{bmatrix}.$$

Hence, $\delta_2^*(0.74) = 0.7696$ so that $0.7 \leqq \alpha_2^*(0.8) \leqq 0.74$. If we let $\alpha = 0.723$, then (P3') reduces to the game

$$\begin{bmatrix} 0.554 & 0.723 & 1.446 & 0.277 \\ 1.446 & 0.831 & 0.554 & 2.169 \end{bmatrix},$$

and hence, $\delta_2^*(0.723) = 0.8011$. Therefore, $0.723 \leqq \alpha_2^*(0.8) \leqq 0.74$. Thus, (P3') with $\alpha = 0.7236$ is equivalent to the game

$$\begin{bmatrix} 0.5528 & 0.7236 & 1.4472 & 0.2764 \\ 1.4472 & 0.8292 & 0.5528 & 2.1708 \end{bmatrix}.$$

The value of the game is 0.8000 and $x' = (0.2764, 0.7236)$ is a player I's optimal strategy for the game. Hence, $\alpha_2^*(0.8) = 0.7236$ and x' is an optimal strategy for (P5') with $\delta = 0.8$. Now, let us solve (P6') with $\eta = 1.2$. Since $\eta_2^*(0.7) = 1.2$, we have $\beta_2^*(1.2) \geqq 0.7$. Problem (P4') with $\beta = 1.0$ is equivalent to the game

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

The value of the game is 1.2 and player II's optimal strategy for the game is $y^* = (y_1^*, y_2^*) = (0.6, 0.4)$, and so, $\beta_2^*(1.2) = 1.0$ and y^* is an optimal strategy for (P6') with $\eta = 1.2$. Since

$$b_{i1}y_1^* + b_{i2}y_2^* \geqq 0 \qquad \text{for } i = 1 \text{ and } 2,$$

$\beta_1^*(1.2) = 1.0$ and $y^* = (0.6, 0.4)$ is also an optimal strategy for (P8') with the payoff level $\eta = 1.2$. Finally,

$$\mathrm{val}\begin{bmatrix} 2-2u & u & 2-2v & v \\ 2u & 3-3u & 2v & 3-3v \end{bmatrix}$$

$$= \begin{cases} (6 - 12u + 4u^2) \,/\, (5 - 8u), \\ \qquad \text{if} \quad 0 \leqq u \leqq 0.5, \ u \geqq v \quad \text{or} \quad 0.75 \leqq u \leqq 1, \ u \geqq v, \\ (6 - 6u - 6v + 4uv) \,/\, (5 - 4u - 4v), \\ \qquad \text{if} \quad 0.5 \leqq u \leqq 0.75, \ 0 \leqq v \leqq 0.5, \ 3u + 2v \leqq 3, \\ 3 - 3u, \qquad \text{if} \quad v \leqq u \leqq 0.75, \ 3u + 2v \geqq 3, \\ 2v, \qquad \text{if} \quad 0.5 \leqq v \leqq u, \ 3u + 2v \leqq 3. \end{cases}$$

Hence, from Theorem 7, $\alpha_1^*(0.8) = 5/7$ and $x'' = (0.4, 0.6)$ is an optimal strategy for (P7') with $\delta = 0.8$. Thus, we get $\alpha_1^*(0.8) < \alpha_2^*(0.8)$ and $\beta_1^*(1.2) = \beta_2^*(1.2)$.

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References

1. Blau, R. A.: Random-Payoff Two-Person Zero-sum Games. *Operations Research*, Vol. 22 (1974), pp. 1243-1251.
2. Cassiby, R. G., Field, C. A., and Kirby, M. J. L.: Solution of a satisficing Model for Random Payoff Games. *Management Science*, Vol. 19 (1972), pp. 266-271.
3. Charnes, A., Kirby, M. J. L., and Raike, W. M.: Zero-Zero Chance-Constrained Games, *Theory of Probability and Its Applications*, Vol. 13 (1968), pp. 628-646.
4. Karlin, S.: *Mathematical Methods and Theory in Games, Programming and Economics, Vol. I*, Addison-Wesley, Reading, 1959.
5. McKinsey, J. C. C.: *Introduction to the Theory of Games*, McGrow-Hill, New York, 1952.

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