

## A PRIMAL APPROACH TO THE INDEPENDENT ASSIGNMENT PROBLEM

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(Received April 9, 1976; Revised July 13, 1976)

### Abstract

We show a theorem which characterizes optimal independent assignments. Based on the theorem, we propose an algorithm for finding an optimal independent assignment, which is of a primal type in that we start from a maximum independent matching and that we get maximum independent matchings having smaller total weights than the old ones as the computation proceeds.

### 1. Introduction

The independent assignment problem has recently been formulated and solved by M. Iri and N. Tomizawa [8]. Given a bipartite graph with matroidal structures on both of the two sets of end-vertices, the independent assignment problem is to find a maximum independent matching [13] having the smallest total weight, where a nonnegative weight is given to each arc. It is a natural extension of the ordinary assignment problem. The Iri-Tomizawa algorithm for finding an optimal independent assignment gives us the smallest-total-weight independent matchings of cardinalities 1, 2, ... and, finally, of the largest cardinality as the computation proceeds.

E. L. Lawler [10] has also considered a related problem called the weighted matroid intersection problem, i.e., the problem of finding a common independent set, of two matroids, having the largest total weight, where a weight is given to each element of the set on which the two matroids are defined. Two algorithms are presented: one is based on the linear-programming formulation and its primal-dual-type method, similar to the approach adopted in [4] for the minimum-cost flow problem; and the other is similar to the Iri-Tomizawa algorithm [8].

We shall consider the independent assignment problem and propose an algorithm of a primal type in the sense that we begin with a maximum independent matching and that we get maximum independent matchings having smaller total weights than the old ones as the computation proceeds. Theorem 5.1 in Section 5 characterizes optimal independent assignments and gives a basis for obtaining the algorithm. The algorithm presented here is a matroidal counterpart of the primal-type algorithm for the ordinary assignment problem due to M. Klein [9]. The approach adopted in the present paper will give further light on the independent assignment problem.

## 2. Definitions and Preliminary Lemmas

In this section we shall give several lemmas which will be used in the following sections. We assume a familiarity with fundamental properties of a matroid as described in [14,15].

Let  $M(V, F)$  be a matroid defined on a finite set  $V$  with a nonempty family  $F$  of subsets of  $V$ , where  $F$  satisfies

(i) if  $I \in F$  and  $I' \subseteq I$ , then  $I' \in F$

and

(ii) if  $I, I' \in F$  and  $|I| > |I'|$ , then there exists an element  $v$ , in  $I - I'$ , such that  $I' \cup \{v\} \in F$ .

An element of  $F$  is called an independent set and an element of  $2^V - F$  a dependent set, where  $2^V$  is the family of all subsets of  $V$ . A minimal dependent set is called a circuit. The closure function  $cl : 2^V \rightarrow 2^V$  is defined in terms of circuits as follows. For any subset  $U$  of  $V$ ,

$cl(U) \equiv U \cup \{v | v \notin U, \text{ there exists a circuit containing } v \text{ on } U \cup \{v\}\}.$

We shall make use of the following lemmas. The proofs are immediate from the above definitions and will be omitted.

Lemma 2.1 : For any subsets  $U_1$  and  $U_2$  of  $V$  such that  $U_1 \subseteq U_2$ ,

$$cl(U_1) \subseteq cl(U_2),$$

and for any subset  $U$  of  $V$ ,

$$cl(cl(U)) = cl(U).$$

Lemma 2.2 : For any independent set  $I$  and any element  $v \in cl(I) - I$ , let  $U_0$  be given by

$$U_0 = \{u \mid u \in I, v \notin \text{cl}(I - \{u\})\}.$$

Then we have

$$v \in \text{cl}(U_0).$$

Lemma 2.3 : If  $I_1$  and  $I_2$  are independent sets and if

$$\text{cl}(I_1) \supseteq I_2,$$

then

$$|I_1| \geq |I_2|.$$

Lemma 2.4 : If  $v \notin \text{cl}(I)$  for an element  $v$  of  $V$  and an independent set  $I$ , then  $\{v\} \cup I$  is also an independent set.

### 3. Formulation of the Problem

Consider a finite bipartite graph  $G(V_1, V_2; A)$  with vertex sets  $V_1$  and  $V_2$  and an arc set  $A (\subseteq V_1 \times V_2; \text{arcs are assumed to have initial vertices in } V_1 \text{ and terminal vertices in } V_2)$ . A real weight function  $w$  is defined on the arc set  $A$ ; and matroids  $M_1(V_1, F_1)$  and  $M_2(V_2, F_2)$  are also defined on the end-vertex sets  $V_1$  and  $V_2$ , respectively, where  $F_1$  and  $F_2$  are families of independent sets on  $V_1$  and  $V_2$ , respectively. For an arc set  $B (\subseteq A)$ , let us denote by  $\partial_i B$  the set of the end vertices of  $B$  in  $V_i$  for  $i = 1, 2$ . An independent matching [13] is an arc set  $B (\subseteq A)$  such that

$$|B| = |\partial_1 B| = |\partial_2 B|$$

and

$$\partial_1 B \in F_1, \quad \partial_2 B \in F_2.$$

A maximum independent matching is an independent matching of the largest cardinality.

The independent assignment problem to be considered in the present paper is to find a maximum independent matching  $B$  which has the smallest total weight:

$$\sum_{a \in B} w(a).$$

A solution of the problem will be called an optimal independent assignment on the given bipartite graph  $G(V_1, V_2; A)$  with regard to the weight function

w and to the matroids  $M'_i(V_i, F_i)$  ( $i=1,2$ ).

The independent assignment problem was first treated by M. Iri and N. Tomizawa [8] who proposed an algorithm which is a matroidal counterpart of the ordinary assignment algorithms described in [1,7].

#### 4. Auxiliary Graph associated with an Independent Matching

We shall define an auxiliary graph associated with an independent matching as follows.

Let  $B$  be an independent matching on the bipartite graph  $G(V_1, V_2; A)$  with regard to the matroids  $M_i(V_i, F_i)$  ( $i=1,2$ ). The auxiliary graph  $\bar{G}_B(\bar{V}, \bar{A})$  associated with the independent matching  $B$  is a directed graph with vertex set  $\bar{V}$  and arc set  $\bar{A}$ . Here,  $\bar{V}$  is given by

$$(4.1) \quad \bar{V} = V_1 \cup V_2 \cup \{s, t\},$$

where  $s$  and  $t$  are two added vertices. The arc set  $\bar{A}$  is composed of six disjoint parts:

$$(4.2) \quad A_0 = A - B,$$

$$(4.3) \quad B^* = \text{the set of the arcs obtained by reversing the direction of the arcs of } B,$$

$$(4.4) \quad A_1 = \{(u, v) \mid u \in \partial_1 B, v \in cl_1(\partial_1 B) - \partial_1 B, v \notin cl_1(\partial_1 B - \{u\})\},$$

$$(4.5) \quad A_2 = \{(u, v) \mid v \in \partial_2 B, u \in cl_2(\partial_2 B) - \partial_2 B, u \notin cl_2(\partial_2 B - \{v\})\},$$

$$(4.6) \quad S_1 = \{(s, v) \mid v \in V_1 - cl_1(\partial_1 B)\} \cup \{(v, s) \mid v \in \partial_1 B\},$$

and

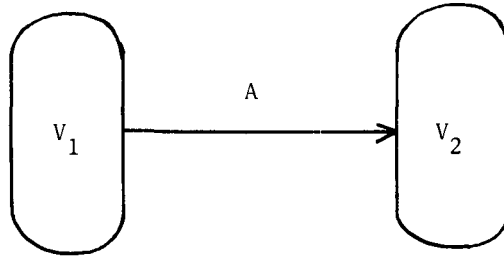
$$(4.7) \quad S_2 = \{(v, t) \mid v \in V_2 - cl_2(\partial_2 B)\} \cup \{(t, v) \mid v \in \partial_2 B\},$$

where  $cl_i$  is the closure function associated with the matroid  $M_i(V_i, F_i)$  for  $i = 1, 2$ . Moreover, we define the weight function  $\bar{w}$  on the arc set  $\bar{A}$  of the auxiliary graph  $\bar{G}_B(\bar{V}, \bar{A})$  as follows:

$$(4.8) \quad \begin{aligned} \bar{w}(a) &= w(a) && \text{if } a \in A_0, \\ &= -w(a') && \text{if } a \in B^*, \text{ where } a' \text{ is the arc in } A \\ &&& \text{corresponding to } a, \\ &= 0 && \text{otherwise.} \end{aligned}$$

The auxiliary graph  $\bar{G}_B(\bar{V}, \bar{A})$  thus defined is slightly different from but essentially the same as the one defined in [8,11] and is useful for

(1)  $G(V_1, V_2; A)$



(2)  $\bar{G}_B(\bar{V}, \bar{A})$

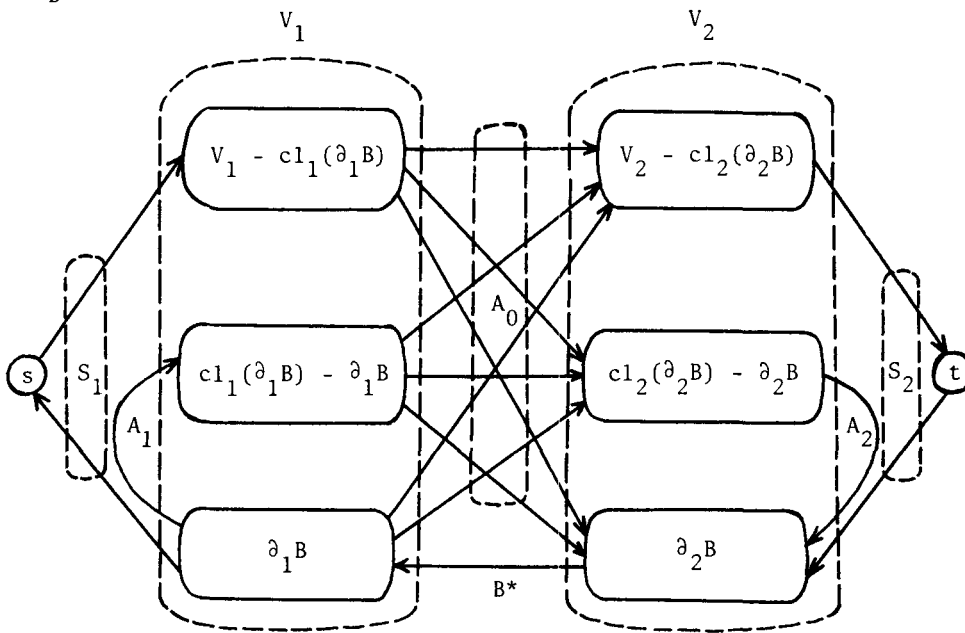


Fig. 1. Illustrations of the original bipartite graph  $G(V_1, V_2; A)$  and the auxiliary graph  $\bar{G}_B(\bar{V}, \bar{A})$  associated with an independent matching  $B$ . The each arrow stands for a (possibly empty) set of arcs in its direction.

characterizing the optimal independent assignments and for obtaining a primal-type algorithm.

### 5. A Fundamental Theorem

In this section we shall show a fundamental theorem which characterizes the optimal independent assignments and gives a basis for a primal-type algorithm.

First, we shall give, without proofs, the lemmas due to N. Tomizawa and M. Iri [11]. In the following lemmas,  $B$  is the independent matching appearing in Section 4.

Lemma 5.1 : Let  $\{(u_i, v_i) | i=1, 2, \dots, p\}$  be a subset of the arc set  $A_1$  defined by (4.4), where  $p$  is a positive integer. If there is no arc such that

$$(5.1) \quad (u_i, v_j), \quad i < j, \quad i, j = 1, 2, \dots, p$$

in  $A_1$ , then

$$(5.2)_1 \quad I_1 \equiv (\partial_1 B - \{u_1, u_2, \dots, u_p\}) \cup \{v_1, v_2, \dots, v_p\} \in F_1$$

and

$$(5.2)_2 \quad cl_1(I_1) = cl_1(\partial_1 B).$$

Lemma 5.2 : Let  $\{(u_i, v_i) | i=1, 2, \dots, p\}$  be a subset of the arc set  $A_2$  defined by (4.5), where  $p$  is a positive integer. If there is no arc such that

$$(5.3) \quad (u_i, v_j), \quad i < j, \quad i, j = 1, 2, \dots, p$$

in  $A_2$ , then

$$(5.4)_1 \quad I_2 \equiv (\partial_2 B - \{v_1, v_2, \dots, v_p\}) \cup \{u_1, u_2, \dots, u_p\} \in F_2$$

and

$$(5.4)_2 \quad cl_2(I_2) = cl_2(\partial_2 B).$$

It should be noted that (5.2) (resp. (5.4)) is valid if the condition of Lemma 5.1 (resp. Lemma 5.2) holds by appropriately numbering the arcs.

We call an independent matching of cardinality  $r$  an  $r$ -independent matching. The length of a directed path  $P$  is defined as the sum of the weights of the arcs lying on  $P$ , and, similarly, the length of a directed cycle. A directed cycle having a negative length is called a negative

directed cycle. A directed cycle  $L$  is called elementary if  $L$  contains no directed cycle having a smaller number of arcs than  $L$ .

Now, we show a fundamental theorem.

Theorem 5.1 : Suppose there exists an  $r$ -independent matching on  $G(V_1, V_2; A)$  for a positive integer  $r$ . An  $r$ -independent matching  $B$  has the smallest total weight among all  $r$ -independent matchings if and only if there is no negative directed cycle on the auxiliary graph  $\bar{G}_B(\bar{V}, \bar{A})$  associated with  $B$ .

(Proof)

(I) "if" part : Suppose that there is no negative directed cycle on the auxiliary graph  $\bar{G}_B(\bar{V}, \bar{A})$ . Let  $\bar{B}$  be an arbitrary  $r$ -independent matching on  $G(V_1, V_2; A)$ . In the following we shall restrict ourselves to the sub-bipartite graph with the arc set  $B_0 \equiv B \cup \bar{B}$ . We shall denote by  $G(B_0)$  the sub-bipartite graph and by  $\bar{G}_B(B_0)$  the auxiliary graph associated with the independent matching  $B$  on  $G(B_0)$ , where the matroids  $M_i(V_i, F_i)$  ( $i=1, 2$ ) should be restricted on  $\partial_i B \cup \partial_i \bar{B}$ , respectively: for each  $i = 1, 2$ , the restriction of  $M_i(V_i, F_i)$  on  $\partial_i B \cup \partial_i \bar{B}$  is the matroid  $\bar{M}_i(\bar{V}_i, \bar{F}_i)$  defined by

$$\bar{V}_i = \partial_i B \cup \partial_i \bar{B}, \quad \bar{F}_i = \{I \mid I \subseteq \partial_i B \cup \partial_i \bar{B}, I \in F_i\}.$$

The end-vertex sets  $\partial_i B \cup \partial_i \bar{B}$  ( $i=1, 2$ ) are composed of the following disjoint sets:

$$(5.5) \quad \left. \begin{aligned} V_{i1} &= \partial_i B_0 - \overline{cl}_i(\partial_i B), \\ V_{i2} &= \partial_i B \cap \partial_i \bar{B}, \\ V_{i3} &= \overline{cl}_i(\partial_i B) - \partial_i B, \\ V_{i4} &= \partial_i B - \partial_i B \cap \partial_i \bar{B} \end{aligned} \right\} \quad i = 1, 2,$$

where  $\overline{cl}_i$  is the closure function associated with  $\bar{M}_i(\bar{V}_i, \bar{F}_i)$  ( $i=1, 2$ ).

Let us define: for an arc  $a$ ,  $\partial^+ a$  and  $\partial^- a$  be, respectively, the initial end-vertex and the terminal end-vertex of  $a$ ; for a vertex  $v$ ,

$$(5.6) \quad \delta^+ v \equiv \{a \mid \partial^+ a = v\}, \quad \delta^- v \equiv \{a \mid \partial^- a = v\};$$

and for an arc set  $A$  and a vertex set  $V$ ,

$$(5.7) \quad \partial^+ A \equiv \bigcup_{a \in A} \partial^+ a, \quad \delta^+ V \equiv \bigcup_{v \in V} \delta^+ v.$$

Since, for any vertex set  $C_1$  (resp.  $C_2$ ) such that

$$(5.8) \quad V_{13} \supseteq C_1 \quad (\text{resp. } V_{23} \supseteq C_2),$$

we have

$$(5.9) \quad V_{12} \cup C_1 \in F_1 \quad (\text{resp. } V_{22} \cup C_2 \in F_2)$$

and

$$(5.10) \quad V_{12} \cup (\partial^+ \delta^- C_1) \in F_1 \quad (\text{resp. } V_{22} \cup (\partial^- \delta^+ C_2) \in F_2)$$

and since from Lemmas 2.1 and 2.2 and the definitions (4.2) and (4.3)

$$(5.11) \quad \begin{aligned} \overline{c1}_1(V_{12} \cup (\partial^+ \delta^- C_1)) &\supseteq V_{12} \cup C_1 \\ (\text{resp. } \overline{c1}_2(V_{22} \cup (\partial^- \delta^+ C_2))) &\supseteq V_{22} \cup C_2, \end{aligned}$$

then it follows from Lemma 2.3 that

$$(5.12) \quad |(\partial^+ \delta^- C_1) \cap V_{14}| \geq |C_1| \quad (\text{resp. } |(\partial^- \delta^+ C_2) \cap V_{24}| \geq |C_2|),$$

where  $\partial^\pm, \delta^\pm$  are to be understood as the incidence functions with regard to the auxiliary graph  $\overline{G}_B(B_0)$ .

Let us denote by  $\overline{A}_1$  (resp.  $\overline{A}_2$ ) the arc set from  $V_{14}$  to  $V_{13}$  (resp. from  $V_{23}$  to  $V_{24}$ ) on  $\overline{G}_B(B_0)$ . Note that (5.8) and (5.12) correspond to a sufficient condition for the existence of a complete matching on the bipartite graph with a vertex set  $V_{14} \cup V_{13}$  (resp.  $V_{23} \cup V_{24}$ ) and an arc set  $\overline{A}_1$  (resp.  $\overline{A}_2$ ) [5]. Therefore, we can choose an arc set  $A_1^\circ \subseteq \overline{A}_1$  (resp.  $A_2^\circ \subseteq \overline{A}_2$ ) such that

$$(5.13) \quad \begin{aligned} \partial^- A_1^\circ &= V_{13}, \quad |A_1^\circ| = |\partial^+ A_1^\circ| = |\partial^- A_1^\circ| \\ (\text{resp. } \partial^+ A_2^\circ &= V_{23}, \quad |A_2^\circ| = |\partial^+ A_2^\circ| = |\partial^- A_2^\circ|). \end{aligned}$$

Let us remove from  $\overline{G}_B(B_0)$  the arcs of  $(\overline{A}_1 - A_1^\circ) \cup (\overline{A}_2 - A_2^\circ)$  and the arcs given by

$$(v, s), \quad v \in \partial^+ A_1^\circ \quad \text{and} \quad (t, v), \quad v \in \partial^- A_2^\circ.$$

Moreover, remove from it the arcs corresponding to  $B \cap \overline{B}$  as well as their end-vertices and the arcs given by

$$(v, s), \quad v \in \partial_1 B \cap \partial_1 \overline{B} \quad \text{and} \quad (t, v), \quad v \in \partial_2 B \cap \partial_2 \overline{B}.$$

We denote by  $\hat{G}$  the resultant graph. It is to be noted that  $\hat{G}$  is a subgraph of the auxiliary graph  $\overline{G}_B(\overline{V}, \overline{A})$  of  $G(V_1, V_2; A)$  with regard to  $B$ . We shall thus apply the weight function  $\overline{w}$  of (4.8) defined on  $\overline{G}_B(\overline{V}, \overline{A})$  to  $\hat{G}$ .

From the way of constructing  $\hat{G}$ , we see that for any vertex  $v$  on  $\hat{G}$  its positive and negative degrees are equal to each other, i.e.,

$$(5.14) \quad |\delta^+ v| = |\delta^- v|,$$

where  $\delta^\pm$  are to be understood as the incidence functions defined on  $\hat{G}$ ; that is to say, each connected component of  $\hat{G}$  is a directed Euler graph. Therefore, the graph  $\hat{G}$  can be covered by directed cycles having no common arcs. Let these cycles be given by

$$(5.15) \quad C_i, \quad i \in I.$$

Consequently, we have

$$(5.16) \quad \begin{aligned} & (\text{the total weight of } \bar{B}) - (\text{the total weight of } B) \\ &= \sum_{i \in I} (\text{the length of } C_i), \end{aligned}$$

where the length of  $C_i$  is given with regard to the weight function  $\bar{w}$  of (4.8), while the weights of  $\bar{B}$  and  $B$  are with regard to the original weight function  $w$ .

Since the directed cycles  $C_i$  ( $i \in I$ ) on  $\hat{G}$  are also the directed cycles on the auxiliary graph  $\bar{G}_B(\bar{V}, \bar{A})$ , the lengths of  $C_i$  ( $i \in I$ ) must be nonnegative. From (5.16), we thus have

$$(\text{the total weight of } \bar{B}) \geq (\text{the total weight of } B).$$

(II) "only if" part : Suppose that  $B$  is an  $r$ -independent matching having the smallest total weight among all  $r$ -independent matchings on  $G(V_1, V_2; A)$ . We shall show that, if there is a negative directed cycle on the auxiliary graph  $\bar{G}_B(\bar{V}, \bar{A})$  associated with  $B$ , then it leads to contradiction.

Assume there is a negative directed cycle on  $\bar{G}_B(\bar{V}, \bar{A})$ . Let  $L$  be a negative directed cycle having the smallest number of arcs. Define the arc sets  $\tilde{A}_1$  and  $\tilde{A}_2$  by

$$(5.17) \quad \tilde{A}_1 \equiv A_1 \cap \tilde{L}, \quad \tilde{A}_2 \equiv A_2 \cap \tilde{L},$$

where  $\tilde{L}$  is the set of the arcs on  $L$ , and  $A_1$  and  $A_2$  are those defined by (4.4) and (4.5). Furthermore, we express the arc sets  $\tilde{A}_1$  and  $\tilde{A}_2$  as

$$(5.18) \quad \begin{aligned} \tilde{A}_1 &= \{(a_i^1, a_i^2) \mid i=1, 2, \dots, k_1\}, \\ \tilde{A}_2 &= \{(b_i^1, b_i^2) \mid i=1, 2, \dots, k_2\}, \end{aligned}$$

where for  $j = 1, 2$ ,  $k_j$  is equal to  $|\tilde{A}_j|$ .

We define a graph  $\hat{G}(\tilde{A}_1)$  (resp.  $\hat{G}(\tilde{A}_2)$ ) with a "vertex" set  $\tilde{A}_1$  (resp.  $\tilde{A}_2$ ) as follows:

there exists an "arc" from  $(a_p^1, a_p^2)$  to  $(a_q^1, a_q^2)$  (resp. from  $(b_r^1, b_r^2)$  to  $(b_s^1, b_s^2)$ ) on  $\mathcal{G}(\mathcal{A}_1)$  (resp.  $\mathcal{G}(\mathcal{A}_2)$ ) if and only if there exists an arc  $(a_p^1, a_q^2)$  (resp.  $(b_r^1, b_s^2)$ ) on the auxiliary graph  $\bar{G}_B(\bar{V}, \bar{A})$ , where  $p, q = 1, 2, \dots, k_1$ ;  $p \neq q$  (resp.  $r, s = 1, 2, \dots, k_2$ ;  $r \neq s$ ).

Suppose there exists a directed cycle on  $\mathcal{G}(\mathcal{A}_1)$ . Let the directed cycle be given by

$$(5.19) \quad ((\bar{a}_1^1, \bar{a}_1^2), (\bar{a}_2^1, \bar{a}_2^2), \dots, (\bar{a}_\ell^1, \bar{a}_\ell^2), (\bar{a}_1^1, \bar{a}_1^2)), \quad \ell \leq k_1.$$

By the definition of  $\mathcal{G}(\mathcal{A}_1)$ , there are arcs:

$$(5.20) \quad (\bar{a}_i^1, \bar{a}_{i+1}^2), \quad i = 1, 2, \dots, \ell$$

in  $A_1$ , where  $\bar{a}_{\ell+1}^2 \equiv \bar{a}_1^2$ . For each  $i = 1, 2, \dots, \ell$ , let us define a directed cycle  $\tilde{\mathcal{C}}_i$  as the one obtained from  $L$  by removing the arcs belonging to the directed path on  $L$  from vertex  $\bar{a}_i^1$  to vertex  $\bar{a}_{i+1}^2$  and by adding to it the arc  $(\bar{a}_i^1, \bar{a}_{i+1}^2)$ . Let us also define:

$$(5.21) \quad \begin{aligned} w_i &\equiv \text{the length of the directed path on } L \text{ from } \bar{a}_i^1 \text{ to } \bar{a}_{i+1}^2, \\ \tilde{w}_i &\equiv W - w_i, \quad i = 1, 2, \dots, \ell, \end{aligned}$$

where  $W$  is the length of  $L$ , and  $\tilde{w}_i$  is equal to the length of  $\tilde{\mathcal{C}}_i$  since the length (or weight) of the arc  $(\bar{a}_i^1, \bar{a}_{i+1}^2)$  is zero. From the definition (5.21), we get

$$(5.22) \quad \sum_{i=1}^{\ell} \tilde{w}_i = \ell W - \sum_{i=1}^{\ell} w_i = \ell W - \ell' W,$$

where  $\ell'$  is a positive integer less than  $\ell$ ; this is because the length of each arc  $(\bar{a}_i^1, \bar{a}_i^2)$  ( $i=1, 2, \dots, \ell$ ) is zero so that the length of the directed path on  $L$  from  $\bar{a}_i^1$  to  $\bar{a}_{i+1}^2$  is equal to the length of the directed path on  $L$  from  $\bar{a}_i^1$  to  $\bar{a}_{i+1}^2$  for each  $i = 1, 2, \dots, \ell$ . Therefore, since  $W$ , the length of  $L$ , is negative and since  $\ell > \ell'$ , we have from (5.22)

$$(5.23) \quad \tilde{w}_{i_0} < 0$$

for some  $i_0 \in \{1, 2, \dots, \ell\}$ . This means that there exists a negative directed cycle  $\tilde{\mathcal{C}}_{i_0}$  with a smaller number of arcs than  $L$ , which contradicts the smallest cardinality of  $L$ . Consequently, there is no directed cycle on  $\mathcal{G}(\mathcal{A}_1)$ .

Similarly we can show that there is no directed cycle on  $\mathcal{G}(\mathcal{A}_2)$ .

Since there is no directed cycle either on  $\mathcal{G}(\mathcal{A}_1)$  or on  $\mathcal{G}(\mathcal{A}_2)$ , we

can see that by appropriately renumbering the elements of (5.18) the assumptions of Lemmas 5.1 and 5.2 hold; and we thus have

$$(5.24) \quad E_1 \equiv (\partial_1 B - \{a_1^1, a_2^1, \dots, a_{k_1}^1\}) \cup \{a_1^2, a_2^2, \dots, a_{k_1}^2\} \in F_1,$$

$$E_2 \equiv (\partial_2 B - \{b_1^2, b_2^2, \dots, b_{k_2}^2\}) \cup \{b_1^1, b_2^1, \dots, b_{k_2}^1\} \in F_2$$

and

$$(5.25) \quad cl_1(E_1) = cl_1(\partial_1 B), \quad cl_2(E_2) = cl_2(\partial_2 B).$$

Let us transform  $B$  into  $B'$  by adding those arcs of  $A$  (from  $V_1$  to  $V_2$ ) which belong to  $L$  and by removing from it those arcs (from  $V_1$  to  $V_2$ ) which correspond to the arcs from  $V_2$  to  $V_1$  belonging to  $L$ . Here,  $L$  is an elementary directed cycle, since  $L$  is a negative directed cycle having the smallest number of arcs. Therefore, for each  $i = 1, 2$ ,  $\partial_i B'$  is obtained by adding at most one vertex in  $V_i - cl_i(\partial_i B)$  to  $E_i$  of (5.24) and by removing from it at most one vertex in  $\partial_i B$ . It follows from (5.24), (5.25) and Lemma 2.4 that  $B'$  is also an  $r$ -independent matching and, since the length of  $L$  is negative,  $B'$  has a smaller total weight than  $B$ . This contradicts the assumption that  $B$  has the smallest total weight among all  $r$ -independent matchings. Q.E.D.

## 6. A Primal-Type Algorithm for Finding an Optimal Independent Assignment

In this section we shall propose a primal-type algorithm for finding an optimal independent assignment based on Theorem 5.1. The algorithm presented here is to first find a maximum independent matching by applying Tomizawa and Iri's method [11]. Given a maximum independent matching, we then transform the maximum independent matching into the one having a smaller total weight, and repeat the process until we reach an optimal solution. The way of the transformation and the optimality criterion (or the stopping rule) have been already suggested by Theorem 5.1 and its proof.

### Algorithm for finding an optimal independent assignment

- 1° Find a maximum independent matching  $B$  by applying the Tomizawa-Iri algorithm [11]. Go to 2°.
- 2° Construct the auxiliary graph  $\bar{G}_B(\bar{V}, \bar{A})$  associated with  $B$  as described in Section 4. Go to 3°.
- 3° If there is no negative directed cycle on  $\bar{G}_B(\bar{V}, \bar{A})$ , then  $B$  is an optimal

solution and the algorithm terminates; and if there is at least one negative directed cycle on  $\bar{G}_B(\bar{V}, \bar{A})$ , find a negative directed cycle  $L$  of the smallest cardinality and then go to 4°.

- 4° Let  $L_1$  be the set of the arcs (from  $V_1$  to  $V_2$ ) which belong to  $L$  and let  $L_2$  be the set of the arcs (from  $V_1$  to  $V_2$ ) obtained by reversing the direction of the arcs from  $V_2$  to  $V_1$  which belong to  $L$ .

Then set

$$B \leftarrow (B - L_2) \cup L_1$$

and go to 2°.

Remark 1 : The validity of the above algorithm is clear from Theorem 5.1 and its proof. We get a maximum independent matching having a smaller total weight than the old ones every time we go through steps 2°, 3° and 4° of the algorithm.

Remark 2 : In step 3° of the algorithm, by use of a power method we can examine whether there exists a negative directed cycle on  $\bar{G}_B(\bar{V}, \bar{A})$  or not and find a negative directed cycle of the smallest cardinality if it exists. However, it is not necessary to find a negative directed cycle of the smallest cardinality. The above algorithm works as well if in step 3° we find an elementary negative directed cycle  $L$  such that there is no negative directed cycle composed of a directed path on  $L$  (from vertex  $a$  to vertex  $b$ , say) and an arc  $(b, a)$  on the auxiliary graph  $\bar{G}_B(\bar{V}, \bar{A})$  but not on  $L$ . A negative directed cycle of this kind can be effectively found with a slight modification by the method described in [2,6] based on the Warshall-Floyd shortest-path algorithm [12,3].

Remark 3 : Since the total weight of the maximum independent matching is strictly decreased in step 4° and the number of maximum independent matchings is finite, the algorithm terminates in a finite number of steps. It is, however, difficult to obtain a good estimate of the computational complexity; this is also the case for primal-type methods for the ordinary assignment problem (cf. [6,9]).

Remark 4 : We describe briefly the Tomizawa-Iri maximum-independent-matching algorithm [11] for the purpose of completing the above algorithm. The Tomizawa-Iri algorithm starts from an arbitrary independent matching  $B$  (for example, the trivial independent matching  $B = \emptyset$ ).

- (1) Construct the auxiliary graph  $\bar{G}_B(\bar{V}, \bar{A})$  associated with  $B$  as described

- in Section 4. Go to (2).
- (2) If there is at least one directed path from vertex  $s$  to vertex  $t$  on  $\bar{G}_B(\bar{V}, \bar{A})$ , then find a directed path  $L$ , from vertex  $s$  to vertex  $t$ , of the smallest cardinality on  $\bar{G}_B(\bar{V}, \bar{A})$ , and go to (3); and if there is no directed path from vertex  $s$  to vertex  $t$  on  $\bar{G}_B(\bar{V}, \bar{A})$ , then  $B$  is a maximum independent matching.
- (3) (The same as 4° of the above algorithm except that "go to 2°" should be replaced by "go to (1)").

The cardinality of the independent matching increases by 1 in step (3). Note that step (2) can be easily carried out since a directed path of the smallest cardinality is to be found. For large-scale problems we may heuristically find an independent matching of large cardinality, which can be employed as the initial independent matching instead of the trivial one  $\emptyset$ .

Remark 5 : By the Iri-Tomizawa optimal-independent-assignment algorithm of primal-dual type [8], the cardinality of an independent matching increases by 1 every time we find a shortest path on an auxiliary graph containing "arcs of negative length". Accordingly, in the case where the cardinality of the maximum independent matching is large (or estimated to be large), the primal-type algorithm proposed in the present paper may be more recommendable than the Iri-Tomizawa algorithm because of the following facts: (i) the maximum independent matching itself can be easily obtained (see Remark 4) and (ii) by the primal-type algorithm we get maximum independent matchings having smaller total weights than the old ones as the computation proceeds, which is desirable if there is a possibility of truncating the computation due to the computational cost.

#### Acknowledgments

The author is deeply indebted to Professor Masao IRI of the University of Tokyo for his useful suggestions and valuable discussions on the matter. The author is supported by the Sakkokai Foundation. This paper was supported in part by the Grant in Aid for Scientific Research of the Ministry of Education, Science and Culture of Japan under Grant: Cooperative Research (A) 135017 (1976).

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