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A DICHOTOMOUS SEARCH WITH TRAVEL COST

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Abstract

An object to be searched is represented by a point in an interval of length n. The searching operation will be started from the left endpoint of the interval. A point at the distance x from the left endpoint will be chosen as the first searching point. It will be assumed that the search at the point finds out whether the object lies to the left or to the right of the point and that a travel cost required to move distance x is ax ($a \geq 0$) and a search cost at the point is b (≥ 0). The problem is that of determining a sequence of searching points so as to minimize the maximum cost required to diminish the existing interval of length n up to unit length, and that of getting the maximum cost required by using this sequence. Furthermore, several properties of minimax policy and considerations of the assumptions are described.

1. Introduction

In this paper, it is considered that the cost required to locate a target consists of search cost and travel cost, and minimax policy and the corresponding maximum cost are derived.

In papers on dichotomous search [1], [2], [3], [4] and [5], the travel cost has not been considered. However, among many problems encountered actually, there is a case where the travel cost is not negligible. It is the experiment which seek *recrystallization temperature* in physics:

Each metal has a proper recrystallization temperature, *i.e.*, the lowest temperature required to recrystallize perfectly within an hour. A metal softens remarkably when its temperature exceeds the recrystallization temperature. Therefore, when we wish to use some metal as a material, it is necessary to determine the recrystallization temperature of the metal.

Here, we shall describe a process of determining the recrystallization temperature by taking a pure iron as an example. It is said that pure iron's recrystallization temperature is about 450°C. However, it varies considerably depending on the composition and the degree of deformation. We must determine it for every test pieces respectively. It is assumed that we know in advance the recrystallization temperature of the test piece to be between 400°C and 600°C. In the first place, we tentatively raise the temperature of the furnace up to 520°C. Then, we keep the test piece in the furnace on that level of the temperature for about an hour. We must take an X-ray photograph in order to know whether or not the heat-treated test piece has been recrystallized. This operation requires about half an hour. It becomes clear that the recrystallization temperature is below 520°C if we find by the X-ray pattern that the test piece has been recrystallized. We take about $0.01x$ hours until we vary the temperature of the furnace by x°C and stabilize it. Then, we take about an hour until we vary the temperature of the furnace, for example, up to 420°C and stabilize it. If we find out that the recrystallization temperature is above 420°C, we raise again the temperature of the furnace, for example, up to 480°C. By repeating those similar trial procedures, it is possible to determine the recrystallization temperature of the test piece, for example, with an accuracy equal to 5°C. In this experiment, the problem is that of finding the trial procedures so as to minimize the total time required to locate the recrystallization temperature within the interval of 5°C. An abstract model for the above-mentioned example is as follows:

There is a stationary point target in an interval of length n. Let us consider a searching operation which enables us to determine whether the target lies to the left or to the right of a chosen point within the interval. We name this chosen point a searching point. Here, ax is the travel cost required for a searching point to move distance x. b is the search cost required to search the target at that searching point. By repeating such a target searching, we can diminish endlessly the length of an existing interval, *i.e.*, an interval in which the point target lies. However, in fact, it is enough to diminish the interval up to a unit length. Here, at least, the following problems arise:

(I) *To determine a sequence of searching points so as to minimize the expected* cost required to diminish the existence interval of length n up to unit length, and to get the expected cost $f(n)$ required by using this sequence.

(II) To determine a sequence of searching points so as to minimize the maximum cost required to diminish the existing interval of length n up to unit length, and to get the maximum cost $h(n)$ required by using this sequence.

In this paper, we shall treat the case (II). The cost $h(n)$ required by starting a search procedure from the left endpoint of the existing interval equals to that of the right endpoint. By such a symmetric property, we easily formulate the problem and can solve it.

In the case (I), it seems that we can analyze the problem in the case where a prior probability density of the target is uniform in the existing interval. We shall study this problem in the future.

2. Formulation and Derivation of Solutions

It is assumed without loss of generality that we start the searching operation from the left endpoint of the interval of length n. We shall choose a point at the distance x from the left endpoint as the first searching point. Here, the travel cost up to that point is ax $(a \geq 0)$ and the search cost at that point is b (≥ 0). After searching completed at that point, if we find out that the target lies to the left of that point, $h(x)$ is the maximum cost required by way of a minimax policy in the subsequent searching. If we find out that the target lies to the right of that point, $h(n-x)$ is the maximum cost required by way of a minimax policy in the subsequent searching. Using the principle of optimality enunciated by Bellman, we obtain the functional equation governing $h(n)$.

$$(2.1) \quad h(n) = \begin{cases} 0 & \text{for } 1 \geq n \geq 0 \\ \min\limits_{0 \leq x \leq n} [ax + b + \max\{h(x),\ h(n-x)\}] & \text{for } n > 1 \end{cases}$$

where a, b, x, n are all non-negative real numbers.

We take

$$(2.2) \quad H_n(x) = ax + b + \max\{h(x),\ h(n-x)\}$$

and we take S_n as a set of x that minimizes $H_n(x)$ for a given n (> 1). Then, equation (2.1) is rewritten as follows:

$$(2.3) \quad h(n) = \min_{0 \leq x \leq n} H_n(x) = H_n(x;\ x \in S_n) < H_n(x;\ x \notin S_n) \quad \text{for } n > 1$$

Here, putting $A_i = [2^i,\ 2^{i+1}]$, we obtain a unique integer i satisfying $m \in A_i$ for an arbitrary positive number m. Using $g(m)$ for that i, we obtain

$$(2.4) \quad 2^{g(m)} < m \leq 2^{g(m)+1}$$

The desired solution of equation (2.1) is obtained by the following theorem. (The proof is given in appendix.)

Theorem I

$$(2.5)\quad h(n) = \begin{cases} 0 & \text{for } 1 \ge n > 0 \\ (n-1)a + \{g(n)+1\}b & \text{for } n > 1 \end{cases}$$

When $n > 2$,

$$(2.6)\quad \begin{aligned} S_n &= \{x;\ n - 2^{g(n)} \le x \le n/2\} && a, b > 0 \\ &= \{x;\ 0 \le x \le n/2\} && a > 0,\ b = 0 \\ &= \{x;\ n - 2^{g(n)} \le x \le 2^{g(n)}\} && a = 0,\ b > 0 \end{aligned}$$

When $2 \ge n > 1$,

$$(2.7)\quad \begin{aligned} S_n &= \{x;\ x = n - 1\} && a, b > 0 \\ &= \{x;\ 0 \le x \le n-1\} && a > 0,\ b = 0 \\ &= \{x;\ n-1 \le x \le 1\} && a = 0,\ b > 0 \end{aligned}$$

From this theorem, we have the following trivial but interesting properties in regard to S_n and $h(n)$.

Corollary 1. S_n is independent of a, b with an exception that a structure of S_n changes depending on values of a, b positive or zero, respectively.

Corollary 2. We shall use $S_n(a, b)$ for S_n, because S_n is a function of a and b. For arbitrary $n > 1$, we have $S_n(a, b) = S_n(a, 0) \cap S_n(0, b)$.

Corollary 3. $h(n)$ is the sum of the travel cost $(n-1)a$ and the search cost $\{g(n)+1\}b$.

It is interesting to note that all Corollaries 1, 2 and 3 suggest the independent relation between travel and search. It seems that the linear property of the cost function $ax + b$ and the use of minimax criterion are reasons why such a property holds. Considering Corollary 3, the meaning of the travel cost $(n-1)a$ is clear. Also, the search cost $\{g(n)+1\}b$ is a minimax cost of a special case that $k = 1$ in the reference [4]. In this paper, this corresponds to the case $a = 0$. Thus, we easily obtain the following Corollary.

Corollary 4. The graph of the function $h(n)$ is the broken line inscribed in the graph of $(n-1)a + b\log_2 n$ with vertices at the points n which are powers of 2.

Corollary 5. $h(n)$ is a concave function for $n \ge 1$.

Taking $a = b = 1$ as an example, the graph of $h(n)$ is shown in Fig. 2.

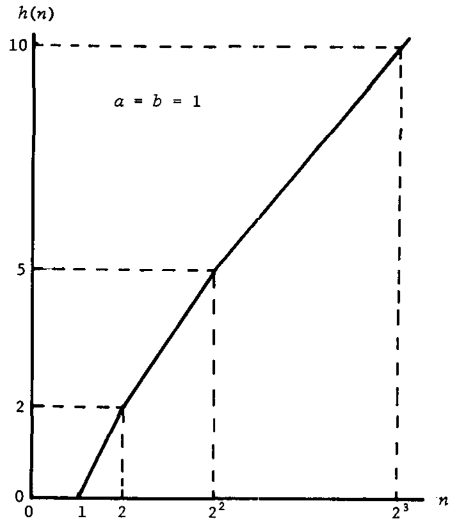

Fig. 2. Optimal Value Function, $h(n)$.

Now, we shall describe a minimax search procedure for three different cases.

(1) a and $b > 0$

From Theorem I, it is easy to obtain a minimax search procedure. If $2 \geq n > 1$, from equation (2.7), searching is completed in a single search when choosing $x = n - 1$ as a search point. When $n > 2$, for given arbitrary n we choose arbitrary element of S_n defined by equation (2.6) as the first searching point. Here, the length of the interval diminished by the searching is considered to be a new value of n. Therefore, we choose arbitrary element of S_n as the second searching point. By repeating the similar searching procedure, it happens that $2 \geq n > 1$ (or $1 \geq n > 0$).
In this case, we may follow the searching procedure when $2 \geq n > 1$ as described above.

The searching procedure mentioned above is nothing but a faithful practice of Theorem I. For any $n > 2$, any elements of S_n given by equation (2.6) are equivalent in the sense of minimax policy. Therefore, for simplicity of searching procedure, we see that a searching procedure of choosing the middle point of an existing interval as a searching point is concise and optimal.

(2) $a = 0$ and $b > 0$

This is a special case that $k = 1$ in reference [4]. The searching procedure are obtained by considerations analogous to those given in (1).

(3) $a > 0$ and $b = 0$

In this case, we see from (2.6) of Theorem I that the first searching point may be any point at the left side of a middle point in an existing interval. However, such a searching procedure is meant for only the worst case. In the special case when $b=0$, as we can intuitively conjecture, the following can be said with respect to optimal policy in a broader sense:

The total cost required for a policy that continuously repeats searches of infinite times from the left endpoint in the existing interval is less than any cost required for any other policy irrespective of a target location.

3. A Numerical Example

We shall discuss the example described in introduction. In this example, if we take 5°C as a unit, we see that $n=40$, $a=0.05$ and $b=1.5$. Using (2.4) yields $g(40)=5$. Therefore, from Theorem I, we have $h(40) = 39\times 0.05 + 6\times 1.5 = 10.95 \fallingdotseq 11$. Accordingly, if we start the experiment at 400°C, by repeating the experiment according to the minimax procedures, we can determine the recrystallization temperature of the test piece with an accuracy equal to 5°C. In the worst case, the accuracy is obtained by X-ray photometry which is repeated six times within 11 hours. When such an experiment is made, a besetting error is to overestimate the travel cost (the time required to vary the temperature of the furnace) and make the overcautious experiment. For example, suppose that some experimenter takes the X-ray photograph every 10°C. Then, in the worst case, he must take the X-ray photograph 20 times and require the experiment time of about 32 hours. Therefore, we see that this procedure is very inefficient beside the minimax procedure. It is instructive to become an ironical result that in this inefficient experiment he requires very much search cost by taking care of the travel cost.

4. Considerations of Assumptions

In this section, we shall describe several considerations of assumptions. In the search model treated in this paper, it is assumed that the target is stationary. However, this assumption is not essential. It is essential that whether the target to be searched lies to the left or to the right of the searching point, and not essential that whether or not it is stationary in the existing interval reduced. This is clear from (2.1). Furthermore, it is not essential that the target is a point. If the target is not a point, we must consider a case where the target is on the searching point. In this case, the search should be stopped at this time point. Accordingly, taking the length of the target as a unit, it follows that

$$(4.8) \quad h(n) = \begin{cases} 0 & \text{for } 1 \geq n \geq 0 \\ \min\limits_{0 \leq x \leq n} \max\{ax+b+h(x),\ ax+b+h(n-x),\ ax+b\} & \text{for } n > 1 \end{cases}$$

Since $h(x)$ and $h(n-x)$ are nonnegative respectively, we see that equation (4.8) is equivalent to equation (2.1). Therefore, Theorem I also holds for a mobile target of unit length. The properties mentioned above also hold for the case where the travel cost is zero.

Appendix Proof of Theorem I.

The union set $\bigcup_{i=0}^{\infty} A_i$ of the half open intervals A_i includes all real numbers which is greater than one. Let us give an arbitrary integer $i \geq 1$. If we show that equations (2.5) and (2.6) satisfy (2.3) for any $n \in A_i$ and that (2.5) and (2.7) satisfy (2.3) for any $n \in A_0$, the theorem is proved for all real number $n \geq 1$.

Let us consider a (n, x) plane and introduce the following set:

$R_i = \{(n, x);\ n \in A_i,\ 0 \leq x \leq n\}$

$C_i = \{(n, x);\ n \in A_i,\ n - 2^{g(n)} \leq x \leq n/2\}$

$U_i = \{(n, x);\ n \in A_i,\ n/2 < x \leq n\}$

$L_i = \{(n, x);\ n \in A_i,\ 0 \leq x < n - 2^{g(n)}\}$

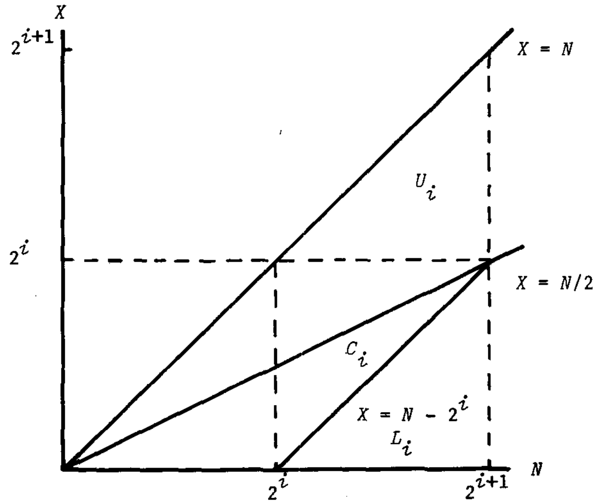

Fig. 1. The Relation of Each Regions

Here, it is clear that $R_i = C_i + U_i + L_i$, where + notation implies the direct sum of sets. The relation of each regions is shown in Fig. 1. Next, let us obtain values which $H_n(x)$ takes in each region.

(I) When $a, b > 0$

(i) The case $(n, x) \in U_i$

It is easily seen from (2.5) that $h(n)$ is strictly increasing with $n \geq 1$. From definition of U_i, we have $ax+b+h(x) > ax+b+h(n-x)$. Then, it follows that

(A.1) $H_n(x) = ax+b+h(x)$

It is clear that this is strictly increasing with x. From (2.4), we have $g(n/2) = g(n) - 1$. Therefore, for arbitrary x such that $n/2 < x \leq n$, it follows that $H_n(x) > H_n(n/2) = an/2 + b + (n/2-1)a + (g(n/2) + 1)b = (n-1)a + \{g(n) + 1\}b = h(n)$.

(ii) The case $(n, x) \in C_i$

It follows from definition of C_i and (2.4) that

(A.2) $2^{g(n)-1} < n/2 \leq n-x \leq 2^{g(n)}$.

Then, we obtain

(A.3) $g(n-x) = g(n) - 1$.

Also, since $x \leq n/2$, we have $h(x) \leq h(n-x)$. Therefore, we have

(A.4) $H_n(x) = ax + b + h(n-x)$.

(a) When $n > 2$

Equation (A.2) yields $1 < n/2 \leq n-x$. From this and (2.5), we see that $h(n-x) = (n-x-1)a + (g(n-x)+1)b$. Furthermore, using (A.3) and (A.4), for arbitrary x such that $n - 2^{g(n)} \leq x \leq n/2$, we have

$H_n(x) = ax + b + h(n-x) = (n-1)a + (g(n-x)+2)b = (n-1)a + (g(n)+1)b = h(n)$.

(b) When $2 \geq n > 1$

In this case, using (2.4) yields

(A.5) $g(n) = 0$.

It follows from (A.2) and (A.5) that $n/2 \leq n-x \leq 1$. Therefore, it is easily seen from (2.5) that $h(n-x) = 0$. From this and (A.4), we have

(A.6) $H_n(x) = ax + b$.

Because $H_n(x)$ is strictly increasing with x, for arbitrary x such that $n-1 \leq x \leq n/2$, we have $H_n(x) \geq H_n(n-1) = a(n-1)+b = a(n-1) + \{g(n)+1\}b = h(n)$ where the sign of equality holds only when $x = n-1$.

(iii) The case $(n, x) \in L_i$

From the definition of L_i and (2.4), we obtain $2^{g(n)} < n-x \leq n \leq 2^{g(n)+1}$. Using this and (2.4) yields

(A.7) $g(n) = g(n-x)$.

It is clear from $x < n - 2^{g(n)} \leq n/2$ that $h(x) < h(n-x)$. Therefore, we see that $H_n(x) = ax + b + h(n-x)$. Further, we see from $n-x > 2^{g(n)} \geq 1$ that

$h(n-x) = (n-x-1)a + (g(n-x)+1)b$. From this and (A.7), for arbitrary x such that $0 \le x < n - 2^{g(n)}$, we obtain

$$\text{(A.8)} \quad \begin{aligned} H_n(x) &= ax + b + h(n-x) \\ &= (n-1)a + b + \{g(n-x)+1\}b \\ &= (n-1)a + \{g(n)+2\}b > h(n). \end{aligned}$$

From (i), (ii) and (iii), when $n > 2$, we see that

$$x \in S_n \Longrightarrow H_n(x) = h(n)$$

$$x \notin S_n \Longrightarrow H_n(x) > h(n)$$

That is, $H_n(x)$ takes the minimum value $h(n)$ only when $n - 2^{g(n)} \le x \le n/2$. When $2 \ge n > 1$, $H_n(x)$ takes the minimum value $h(n)$ only when $x = n-1$.

(II) When $a > 0$, $b = 0$

The proof is identical to that of (I) except the case $(n, x) \in L_i$. In the case $(n, x) \in L_i$, we obtain from (A.8), $H_n(x) = (n-1)a = h(n)$. Therefore, when $n > 2$, $H_n(x)$ takes the minimum value $h(n)$ only when $0 \le x \le n/2$. When $2 \ge n > 1$, $H_n(x)$ takes the minimum value $h(n)$ only when $0 \le x \le n-1$.

(III) When $a = 0$, $b > 0$

(i) The case $(n, x) \in U_i$

Using (2.5) and (A.1) yields $H_n(x) = b + h(x) = \{g(x)+2\}b$. We see from definition of U_i that $n/2 < x \le n$. From (2.4), we have $g(x) = g(n) - 1$ for x such that $n/2 < x \le 2^{g(n)}$. Therefore, it follows that $H_n(x) = \{g(n)+1\}b = h(n)$. Also, we have $g(x) = g(n)$ for x such that $2^{g(n)} < x \le n$. Therefore, it follows that $H_n(x) = \{g(n)+2\}b > h(n)$.

(ii) The case $(n, x) \in C_i$

(a) When $n > 2$

In the same way as (I), we have $H_n(x) = h(n)$.

(b) When $2 \ge n > 1$

From (A.5) and (A.6), $H_n(x) = b = \{g(n)+1\}b = h(n)$.

From (a) and (b), we have $H_n(x) = h(n)$ for arbitrary x such that $n - 2^{g(n)} \le x \le n/2$.

(iii) The case $(n, x) \in L_i$

Using (A.8) yields $H_n(x) = \{g(n)+2\}b > h(n)$. Summarizing (i), (ii) and (iii) described above, it follows for any $n > 0$ that $H_n(x)$ takes the minimum value $h(n)$ only when $n - 2^{g(n)} \le x \le 2^{g(n)}$. Let us note that $g(n) = 0$ for $2 \ge n > 1$. This completes the proof.

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