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AN APPLICATION OF FUZZY GRAPHS TO THE PROBLEM CONCERNING GROUP STRUCTURE

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Abstract

A fuzzy graph is utilized to characterize the role played by an individual member in such a group that a class of group members having relationship with any given member has no sharply defined boundary. The concepts of weakening and strengthening points of an ordinary graph presented by Ross and Harary are generalized to those of a fuzzy graph.

1. Introduction

The theory of graphs is one of the most important tools in the study of the group structure. For instance, Ross and Harary [4] utilized the graph to characterize such a role of an individual member in a group that: A strengthening member of the group is one whose presence causes the graph corresponding to the group to be more highly connected than that obtained when he is absent, while a weakening member is one whose presence causes the graph to belong to a weaker category of connectedness. Besides this, the graph has been widely utilized to study the problems concerning redundancies, liaison persons, cliques, structural balance and so forth.

In many cases, however, the mere presence or absence of a relation is not adequate to represent a given group structure. As was pointed out in [1], there may be different strengths of the relations between individuals. There may even be situations in which it is fuzzy rather than well-defined whether or not an arbitrary individual has relationship with a given member, that is, a class of group members being in relationship with any given member does not have a sharply defined boundary. In such cases, the ordinary graph may not fully represent the group structure. Instead, the fuzzy graph seems to be a more relevant mathematical model.

The purpose of the present paper is to present an application of the fuzzy graph to the group structure. We shall confine our attention to extending the concepts of weakening and strengthening points of an ordinary graph given by Ross and Harary to those of a fuzzy graph.

In the next section, we shall briefly review the concepts presented by Ross and Harary [4]. In Section 3, we discuss the connectedness of the fuzzy graph. In the final section, the concepts of weakening and strengthening points of a fuzzy graph are introduced and their fundamental properties are investigated.

2. Weakening and Strengthening Points of a Directed Graph

To begin with, we will briefly review various kinds of connectedness of directed graphs, or more briefly digraphs [2].

A finite digraph G is *strongly connected*, or *strong*, if every two points are mutually reachable; G is *unilaterally connected*, or *unilateral*, if for any two points at least one is reachable from the other. We say that G is *weakly connected*, or *weak*, if every two points are joined by a semipath. Finally, G is *disconnected* if it is not even weak. For completeness, we note that a digraph G consisting of exactly one point is strong, for since it does not contain two distinct points, the definition is vacuously satisfied. Let U_3, U_2, U_1, and U_0 be collections of all strong digraphs, all unilateral digraphs, all weak digraphs, and all disconnected digraphs, respectively. Obviously we have

(2.1) $$U_3 \subset U_2 \subset U_1 .$$

In order to divide all digraphs into mutually exclusive connectedness categories, let

(2.2) $$C_3 = U_3, \quad C_2 = U_2 \setminus U_3, \quad C_1 = U_1 \setminus U_2, \text{ and } C_0 = U_0 .$$

Then, each digraph belongs to exactly one of the above categories C_i, i=0,1,2,3.

Ross and Harary [4] characterized weakening and strengthening members of a group using these disjoint connectedness categories. Let b be any point of a digraph G, and let G_b be the subgraph obtained from G by the removal of b. A point b is said to be of the type (i,j) if G is in C_i, while G_b is in C_j. The (i,j) point b is called a strengthening point if $i > j$; it is a neutral point if $i = j$; and it is a weakening point if $i < j$. The main results in [4] are the following.

Theorem 1. In any group whatsoever, there are at most two weakening members.

Theorem 2. There are no (1,3) members in any group, and all other (i,j) members can occur.

3. Connectedness of the Fuzzy Graph

As was stated in the introduction, one may be concerned with a group where a class of group members being in relationship with any given member is one with an unsharp boundary in which the transition from membership to nonmembership is gradual rather than abrupt. A fuzzy graph may be utilized to represent such a group.

Definition 1 [3]. Let X be a finite set of points $x_1, x_2, \ldots, x_n$, and let Γ be the function which associates each point of X, say x_i, with a fuzzy set Γ_{x_i} in X whose membership function is $\mu_{\Gamma_{x_i}}$. Then, $FG=(X,\Gamma)$ is called a fuzzy graph.

In this definition, if each $\mu_{\Gamma_{x_i}}$, $i=1,2,\ldots,n$, takes only two values 0 and 1, FG reduces to an ordinary graph. A more detailed discussion of fuzzy graphs is found in [3].

In order to evaluate the effect of the removal of a point on the connectedness of its fuzzy graph, we introduce

Definition 2. A fuzzy subgraph of $FG=(X,\Gamma)$ is defined to be a fuzzy graph of the form (Y,Γ'), where Y is a (non-fuzzy) subset of X and the function Γ' is defined as

(3.1) $$\Gamma'_{x_i} = \Gamma_{x_i} \cap Y \quad \text{for any } x_i \in Y.$$

Definition 3. For a fuzzy set A in X, with membership function μ_A, two fuzzy sets ΓA and $\Gamma^{-1}A$ in X are defined by

$$(3.2)\qquad \mu_{\Gamma A}(x_i) = \max_{x_j \in X} \mathrm{Min}\{\mu_A(x_j),\ \mu_{\Gamma_{x_j}}(x_i)\} \quad \text{for all } x_i \in X,$$

and

$$(3.3)\qquad \mu_{\Gamma^{-1}A}(x_i) = \max_{x_j \in X} \mathrm{Min}\{\mu_A(x_j),\ \mu_{\Gamma_{x_i}}(x_j)\} \quad \text{for all } x_i \in X,$$

respectively.

We have the following

Proposition 1. Let A and B be two fuzzy sets in X, with μ_A and μ_B denoting their respective membership functions. Then,

(a) $\Gamma A \subset \Gamma B$ if $A \subset B$,

(b) $\Gamma^{-1}A \subset \Gamma^{-1}B$ if $A \subset B$,

(c) $\Gamma(A \cap B) \subset \Gamma A \cap \Gamma B$,

(d) $\Gamma^{-1}(A \cap B) \subset \Gamma^{-1}A \cap \Gamma^{-1}B$,

(e) $\Gamma(A \cup B) = \Gamma A \cup \Gamma B$,

(f) $\Gamma^{-1}(A \cup B) = \Gamma^{-1}A \cup \Gamma^{-1}B$.

Proof. Properties (a) and (b) are obvious from Definition 3. Properties (c) and (d) directly follow from (a) and (b), respectively.

(e)
$$\begin{aligned}\mu_{\Gamma(A\cup B)}(x_i) &= \max_{x_j \in X} \mathrm{Min}[\ \mathrm{Max}\{\mu_A(x_j),\ \mu_B(x_j)\},\ \mu_{\Gamma_{x_j}}(x_i)\ ] \\ &= \mathrm{Max}[\ \max_{x_j \in X} \mathrm{Min}\{\mu_A(x_j), \mu_{\Gamma_{x_j}}(x_i)\},\ \max_{x_j \in X} \mathrm{Min}\{\mu_B(x_j), \mu_{\Gamma_{x_j}}(x_i)\}] \\ &= \mathrm{Max}[\ \mu_{\Gamma A}(x_i),\ \mu_{\Gamma B}(x_i)\ ] \\ &= \mu_{\Gamma A \cup \Gamma B}(x_i).\end{aligned}$$

The property (f) is shown in the same way as (e).

Definition 4. For a fuzzy graph $FG=(X,\Gamma)$, the transitive closure of Γ, denoted by $\hat{\Gamma}$, is defined by

$$(3.4)\qquad \hat{\Gamma}_{x_i} = \{x_i\} \cup \Gamma_{x_i} \cup \Gamma^2_{x_i} \cup \cdots \cup \Gamma^{n-1}_{x_i} \quad \text{for } x_i \in X,$$

where $\Gamma^m_{x_i} = \Gamma(\Gamma^{m-1}_{x_i})$, $m=2,3,\ldots,n-1$. In the same way, the inverse transitive closure $\hat{\Gamma}^{-1}$ is

$$(3.5)\qquad \hat{\Gamma}^{-1}_{x_i} = \{x_i\} \cup \Gamma^{-1}_{x_i} \cup \Gamma^{-2}_{x_i} \cup \cdots \cup \Gamma^{-n+1}_{x_i} \quad \text{for } x_i \in X,$$

where $\Gamma^{-1}_{x_i} = \Gamma^{-1}\{x_i\}$ and $\Gamma^{-m}_{x_i} = \Gamma^{-1}(\Gamma^{-m+1}_{x_i})$, $m=2,3,\ldots,n-1$.

We can easily see that

$$(3.6)\qquad \mu_{\hat{\Gamma}_{x_i}}(x_j) = \mu_{\hat{\Gamma}^{-1}_{x_j}}(x_i) \quad \text{for any } x_i,\ x_j \in X.$$

The grades of membership $\mu_{\hat{\Gamma}_{x_i}}(x_j)$ and $\mu_{\hat{\Gamma}^{-1}_{x_i}}(x_j)$ may be interpreted as the degree of the existence of a directed path from x_i to x_j and that from x_j to x_i, respectively. Let us define

$$(3.7)\qquad \Delta_{x_i} = \Gamma_{x_i} \cup \Gamma^{-1}_{x_i} \quad \text{for } x_i \in X,$$

and

$$(3.8)\qquad \hat{\Delta}_{x_i} = \{x_i\} \cup \Delta_{x_i} \cup \Delta^2_{x_i} \cup \cdots \cup \Delta^{n-1}_{x_i} \quad \text{for } x_i \in X,$$

where $\Delta^m_{x_i} = \Delta(\Delta^{m-1}_{x_i})$, $m=2,3,\ldots,n-1$.

The value of the membership function $\mu_{\hat{\Delta}_{x_i}}(x_j)$ may be interpreted as the degree for two points x_i and x_j to be joined by a semipath.

With the above preparation, we reach the following

Definition 5. The grades of membership of a fuzzy graph $FG=(X,\Gamma)$ in U_3, U_2, U_1, and U_0 are defined by

$$(3.9)\qquad \begin{cases} \mu_{U_3}(FG) = \operatorname*{Min}_{i,j} \mu_{\hat{\Gamma}_{x_i}}(x_j), \\ \mu_{U_2}(FG) = \operatorname*{Min}_{i,j} \operatorname{Max}\{\mu_{\hat{\Gamma}_{x_i}}(x_j),\ \mu_{\hat{\Gamma}_{x_j}}(x_i)\}, \\ \mu_{U_1}(FG) = \operatorname*{Min}_{i,j} \mu_{\hat{\Delta}_{x_i}}(x_j), \\ \mu_{U_0}(FG) = 1 - \operatorname*{Min}_{i,j} \mu_{\hat{\Delta}_{x_i}}(x_j), \end{cases}$$

respectively.

It follows that

$$(3.10)\qquad \mu_{U_3}(FG) \leqq \mu_{U_2}(FG) \leqq \mu_{U_1}(FG) \quad \text{for any } FG=(X,\Gamma).$$

Specifically, we can see that for any digraph G in C_i,

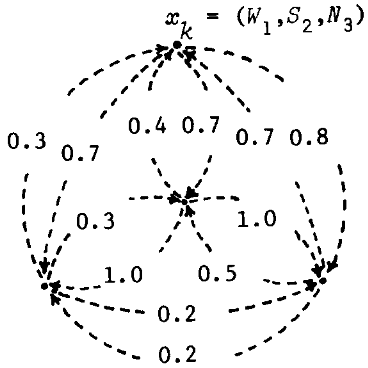
$$\mu_{U_j}(G) = 0 \quad \text{for } 3 \geqq j > i;\quad \mu_{U_j}(G) = 1 \quad \text{for } i \geqq j \geqq 1.$$

4. Weakening and Strengthening Points of a Fuzzy Graph

In this section we define weakening and strengthening points of a fuzzy graph as a natural extension of those of an ordinary digraph. We then investigate their fundamental properties.

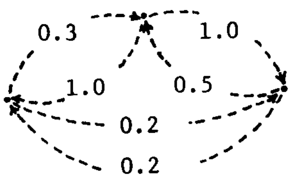
Definition 6. For a fuzzy graph $FG=(X,\Gamma)$, let FG_k be the fuzzy subgraph $(X\setminus\{x_k\},\Gamma')$ obtained from FG by the removal of a point x_k. Then, the point x_k is a weakening point for U_i (a W_i point, for short) if $\mu_{U_i}(FG) < \mu_{U_i}(FG_k)$; it is a neutral point for U_i (an N_i point) if $\mu_{U_i}(FG) = \mu_{U_i}(FG_k)$; and it is a strengthening point for U_i (an S_i point) if $\mu_{U_i}(FG) > \mu_{U_i}(FG_k)$, where $i=1,2,3$.

For instance, a point x_k, as shown in Figure 1, is a weakening point for U_1 because the grade of membership in U_1 of the fuzzy subgraph FG_k is greater than that of FG. In the similar way it is also an S_2 point and an N_3 point, so we say x_k is a point of the type (W_1,S_2,N_3).



A fuzzy graph FG.

$(\mu_{U_1}(FG),\mu_{U_2}(FG),\mu_{U_3}(FG))$
$= (\ 0.8,\ 0.7,\ 0.3\ ).$



A fuzzy subgraph FG_k.

$(\mu_{U_1}(FG_k),\mu_{U_2}(FG_k),\mu_{U_3}(FG_k))$
$= (\ 1.0,\ 0.5,\ 0.3\ ).$

Figure 1.

In what follows, for brevity of notation, let

(4.1) $\quad p_{ij} = \mu_{\hat{\Gamma}_{x_i}}(x_j),\quad i,j=1,2,\ldots,n,$

(4.2) $\quad q_{ij} = \mu_{\hat{\Delta}_{x_i}}(x_j),\quad i,j=1,2,\ldots,n,$

and

(4.3) $\quad r_{ij} = \mu_{\hat{\Gamma}'_{x_i}}(x_j),\quad i,j\neq k;\ i,j=1,2,\ldots,n,$

where $\hat{\Gamma}$ and $\hat{\Delta}$ are respectively as defined in (3.4) and (3.8), and $\hat{\Gamma}'$ is the transitive closure of Γ'.

Let P and Q denote respectively $n \times n$ matrices with elements p_{ij} and q_{ij} and let R be an $n \times n$ matrix, whose elements in the k-th row and in the k-th column are zeros and each (i,j) element is r_{ij}, where $i,j\neq k$; $i,j=1,2,\ldots,n$.

The next lemma serves to characterize weakening points for each connectedness category.

Lemma 1. (i) A point x_k is a W_3 point if and only if the elements of P which are equal to $\mu_{U_3}(FG)$ are all in the k-th row or in the k-th column of P.
(ii) A point x_k is a W_2 point if and only if any (i,j) elements of P such that $\mathrm{Max}\{p_{ij}, p_{ji}\} = \mu_{U_2}(FG)$ are in the k-th row and in the k-th column of P.
(iii) A point x_k is a W_1 point if and only if all the elements of Q which are equal to $\mu_{U_1}(FG)$ are in the k-th row and in the k-th column of Q.

Proof. (i) Let x_k be a W_3 point. Suppose that there exists an element, say an (l,m) element, $l,m\neq k$, which is equal to $\mu_{U_3}(FG)$. Since

$$r_{ij} \leqq p_{ij}\ , \quad i,j\neq k;\ i,j=1,2,\ldots,n,$$

we have

$$\mu_{U_3}(FG_k) = \underset{i,j\neq k}{\mathrm{Min}}\ \{\ r_{ij}\ \} \leqq p_{lm} = \mu_{U_3}(FG),$$

which contradicts the assumption that x_k is a W_3 point. Therefore, every element of P which is equal to $\mu_{U_3}(FG)$ is in the k-th row or in the k-th column of P.

Conversely, assume that the elements of P which are equal to $\mu_{U_3}(FG)$ are all in the k-th row or in the k-th column of P. First, notice that if an element which is equal to $\mu_{U_3}(FG)$ is in the k-th row (column) of P, then every non-diagonal element in the k-th row (column) of P is equal to $\mu_{U_3}(FG)$. Hence

$$\mathrm{Min}\{p_{ik}, p_{kj}\} = \mu_{U_3}(FG) < p_{ij}\ , \quad i,j\neq k;\ i,j=1,2,\ldots,n,$$

which yields

$$r_{ij} = p_{ij} > \mu_{U_3}(FG), \quad i,j\neq k;\ i,j=1,2,\ldots,n.$$

Therefore

$$\mu_{U_3}(FG_k) = \underset{i,j\neq k}{\mathrm{Min}}\ \{\ r_{ij}\ \} > \mu_{U_3}(FG),$$

so that x_k is a W_3 point, which completes the proof of (i).

The proofs of (ii) and (iii) are similar to that of (i).

The following theorem is an immediate consequence of Lemma 1.

Theorem 3. There exist at most two W_i points in any fuzzy graph, where i=1,2,3. Further, any fuzzy graph with n ($n \geqq 3$) points has at most one W_1

(W_3) point.

Lemma 2. For any fuzzy graph $FG=(X,\Gamma)$, there exists a path $\{x_{i_1}, x_{i_2},\ldots, x_{i_s}\}$ ($s \geqq n$) such that:

(1) every point of X appears in the path;

(2) $\mu_{\Gamma_{x_{i_l}}}(x_{i_{l+1}}) \geqq \mu_{U_2}(FG)$, $l=1,2,\ldots,s-1$.

Proof. Let us construct an ordinary digraph $G=(X,\Gamma'')$ from FG as follows:

$$\mu_{\Gamma''_{x_i}}(x_j)\left\{\begin{matrix}1\\0\end{matrix}\right\} \text{ according as } p_{ij}\left\{\begin{matrix}\geqq\\<\end{matrix}\right\}\mu_{U_2}(FG),\quad i,j=1,2,\ldots,n.$$

Since $\mathrm{Max}\{p_{ij}, p_{ji}\} \geqq \mu_{U_2}(FG)$, G includes a tournament as a partial graph of G. Since every tournament has a Hamiltonian path, G has a Hamiltonian path. On the other hand, we can easily see from Definition 4 that if $p_{ij} \geqq \mu_{U_2}(FG)$, then there exists at least a path $\{x_i, x_u,\ldots, x_v, x_j\}$ such that

$$\mu_{\Gamma_{x_i}}(x_u) \geqq \mu_{U_2}(FG),$$
$$\vdots$$
$$\mu_{\Gamma_{x_v}}(x_j) \geqq \mu_{U_2}(FG).$$

Thus, we obtain the desired result.

The following theorem shows that in any fuzzy graph with n ($n \geqq 2$) points, it is impossible for all points to be strengthening ones for U_2 (U_1).

Theorem 4. In any fuzzy graph FG with n ($n \geqq 2$) points, there exist at least two points which are either weakening or neutral ones for U_2 (U_1).

Proof. Let a path $\{x_{i_1}, x_{i_2},\ldots, x_{i_s}\}$ satisfy (1) and (2) of Lemma 2. Without loss of generality, we can assume that the initial and final points x_{i_1} and x_{i_s} appear exactly once in the path. For, if the initial point (the final point) appears more than once in the path, we can delete the first point (the last point) of the path, so that the remaining path also meets the requirements (1) and (2).

Now, according to the above assumption, a path $\{x_{i_2}, x_{i_3},\ldots, x_{i_s}\}$ and a path $\{x_{i_1}, x_{i_2},\ldots, x_{i_{s-1}}\}$ contain respectively all points in $X\setminus\{x_{i_1}\}$ and all points in $X\setminus\{x_{i_s}\}$. Therefore

$$\mu_{U_2}(FG_{i_1}) \geqq \mu_{U_2}(FG),$$

and

$\mu_{U_2}(FG_{i_s}) \geqq \mu_{U_2}(FG)$.

Thus, each of x_{i_1} and x_{i_s} is either a W_2 or N_2 point, which completes the proof for U_2.

The proof for U_1 is similar.

Corollary 1. Any fuzzy graph with n ($n \geqq 3$) points has at least one N_1 point.

Proof. It is an immediate consequence of Theorems 3 and 4.

Theorem 5. If a fuzzy graph FG with n ($n \geqq 3$) points has two W_2 points, then

$$\mu_{U_2}(FG) < \mu_{U_1}(FG).$$

Proof. Let x_k and x_l be W_2 points, *i.e.*,

$$\mu_{U_2}(FG_k) > \mu_{U_2}(FG), \tag{4.4}$$

and

$$\mu_{U_2}(FG_l) > \mu_{U_2}(FG). \tag{4.5}$$

Suppose that

$$\mu_{U_2}(FG) = \mu_{U_1}(FG). \tag{4.6}$$

From (3.10) and (4.4) through (4.6), we find that x_k and x_l must be W_1 points, which contradicts Theorem 3. This completes the proof.

Theorem 6. Any W_3 point is either a W_2 one or an N_2 one.

Proof. Let x_k be a W_3 point. From the proof of Lemma 1, we get

$$r_{ij} = p_{ij}, \quad i,j \neq k; \ i,j=1,2,\ldots,n.$$

Therefore,

$$\mu_{U_2}(FG_k) \geqq \mu_{U_2}(FG),$$

which ends the proof.

The following theorem directly follows from Definition 6 and (3.10).

Theorem 7. If $\mu_{U_i}(FG) = \mu_{U_j}(FG)$ for some $i < j$, then an S_i point is also an S_j point.

Theorem 8. If $\mu_{U_i}(FG) = \mu_{U_j}(FG)$ for some $i > j$, then a W_i point is also a W_l point, where $1 \leqq l \leqq i$.

Proof. Let $i = 3$ and $j = 1$, *i.e.*,

(4.7) $$\mu_{U_3}(FG) = \mu_{U_1}(FG).$$

Let x_k be a W_3 point. From (3.10) and (4.7) we obtain

$$\mu_{U_2}(FG_k) > \mu_{U_2}(FG),$$

and

$$\mu_{U_1}(FG_k) > \mu_{U_1}(FG).$$

Thus, x_k is a W_l point, where $1 \leq l \leq 3$. Next, assume that x_k is a W_2 point and that $\mu_{U_2}(FG) = \mu_{U_1}(FG)$. It follows that

$$\mu_{U_1}(FG_k) > \mu_{U_1}(FG).$$

Thus, x_k is a W_l point, where $1 \leq l \leq 2$. Finally, we shall prove that if x_k is a W_3 point and $\mu_{U_3}(FG) = \mu_{U_2}(FG)$ then it is a W_l point, where $1 \leq l \leq 3$. Since it is obvious that x_k is a W_2 point, it suffices to show that x_k is a W_1 point. Using Lemma 1, it follows that both in the k-th row and in the k-th column of P there exists an element which is equal to $\mu_{U_3}(FG)$. Hence we get from the proof of Lemma 1

$$p_{kj} = p_{jk} = \mu_{U_3}(FG), \quad j \neq k;\ j=1,2,\ldots,n.$$

Thus we have

$$\mu_{\Gamma_{x_k} \cup \Gamma_{x_k}^{-1}}(x_j) \leq \mu_{U_3}(FG), \quad j \neq k;\ j=1,2,\ldots,n,$$

which yields

$$q_{kj} = q_{jk} \leq \mu_{U_3}(FG), \quad j \neq k;\ j=1,2,\ldots,n.$$

Therefore we get

$$\mu_{U_1}(FG) = \mu_{U_3}(FG),$$

so that

$$\mu_{U_1}(FG_k) > \mu_{U_1}(FG).$$

This completes the proof.

Theorem 9. Let x_k be a W_i point. If $\mu_{U_i}(FG_k) = \mu_{U_j}(FG_k)$ for some $i < j$, Then x_k is also a W_l point, where $1 \leq l \leq j$.

Proof. The proof of this theorem is similar to that of Theorem 8.

In closing, we shall show how results of Ross and Harary can be obtained from our results as the special cases. First, note that, in the case of the ordinary digraph G, $\mu_{U_i}(G_k) > \mu_{U_i}(G)$ if and only if $\mu_{U_i}(G_k) = 1$ and $\mu_{U_i}(G) = 0$, that is, $G_k \in U_i$ and $G \notin U_i$. With the understanding that a weakening point for U_0 is one whose presence makes its fuzzy graph more highly disconnected than it would be without the point, the W_0 point is defined to be the W_1 point. We can easily see from Theorem 3 that any digraph has at most two weakening points. And, from Theorem 8, we can find that there are no (1,3) points in any digraph.

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