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JOINT DISTRIBUTION OF UPTIME AND DOWNTIME FOR SOME REPAIRABLE SYSTEMS

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ABSTRACT

An uptime and a downtime for some redundant repairable systems are not independent each other. The joint distributions of a single uptime and a single downtime for such systems are derived by using renewal equations. In particular, the downtime distributions are easily obtained by the marginal distributions of the joint ones. Explicit forms of the joint distributions are obtained for a few systems.

1. INTRODUCTION

Consider a repairable system repeating up and down alternately. Such systems are a one-unit system, a two-unit redundant system, and so on. Stochastic behavior of uptime and downtime for such systems was analyzed independently by many authors: The distributions of the first uptime, i.e., the first-passage time distributions to system down for redundant repairable systems were discussed by Gaver [6, 7], Downton [5], Gnedenko, et al. [8], Srinivasan [13], Osaki [12], and others. The mean uptime ratio is well-known as the interval availability [9]. On the other hand, Barlow and Hunter [1] discussed the total downtime distribution for a simple one-unit system using the so-called 'sojourn time problem' given by Takács [14]. In this case, the uptime and the downtime are indenpendent each other. However, the uptime and the downtime for some redundant repairable systems are not independent each other. In such models, it is not easy to derive the downtime distribution directly because the time instant at which the system down occurs is not a regeneration point.

In this paper we are interested in the joint distributions of the first uptime and the next following downtime which are dependent each other, when all units are new at time 0. We derive such joint distributions using renewal equations. In particular, the downtime distributions are easily obtained by the marginal distributions of the joint ones.

2. TWO-UNIT STANDBY REDUNDANT SYSTEM

Consider a two-unit standby redundant system of two identical units [6, 11, 12, 13]. Assume that the cumulative distribution function (cdf) of the time to failure for a unit is $F(t)$ and the cdf of the repair time for a failed unit is $G(t)$. A failed unit is repaired immediately upon failure and then is put back in standby immediately upon repair completion. A repaired unit is as good as new. A unit in standby neither deteriorates nor fails in the standby interval. Assume that each switchover is perfect and each switchover time is instantaneous. It is finally assumed that repair is made for a single unit, then there will be a queue for repair if the two units fail simultaneously. The repair discipline is 'first come first served.'

In this model, the uptime and the downtime are dependent each other because the repair time distribution is arbitrary. That is, the downtime depends on the history of how long the repair time elapses. Let T and X denote the random variables (r.v.'s) of the single uptime and the single downtime, respectively. It is noted that the system is 'up' when one of the two units is at least up, and the system is 'down' when both the two units are down simultaneously. We define the state i of the process ($i = 1, 2$) as the number of units operating or being in standby. We assume that both units are new at time 0. Define the joint distribution $\Pi_i(t, x) \equiv \Pr\{T \leq t \text{ and } X > x \mid \text{the process starts in state } i \text{ at time } 0\}$ ($i = 1, 2$). Let T_k and X_k ($k = 1, 2, \ldots$) are the r.v.'s of the k^{th} uptime and the k^{th} downtime, respectively. A realization of the behavior is shown in Fig. 1. It is evident that $\Pi_2(t, x) = \Pr\{T_1 \leq t \text{ and } X_1 > x\}$ and $\Pi_1(t, x) = \Pr\{T_k \leq t \text{ and } X_k > x\}$ ($k = 2, 3, \ldots$). Thus, both $\Pi_i(t, x)$ ($i = 1, 2$) specify the behavior of the process completely since the time instant at which the system recovery occurs is a regeneration point. Then, we have the following renewal equations:

(1) $$\Pi_2(t, x) = \int_0^t \Pi_1(t-u, x)dF(u),$$

(2) $$\Pi_1(t, x) = \int_0^t \overline{G}(x+u)dF(u) + \int_0^t \Pi_1(t-u, x)G(u)dF(u),$$

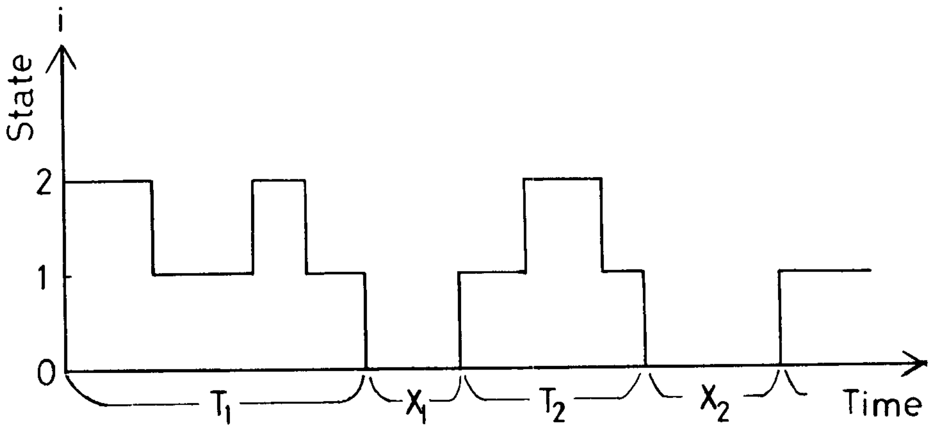

Fig. 1. A realization of the behavior of the process, where the r.v.'s T_k and X_k are the k^{th} uptime and the k^{th} downtime, respectively.

where, in general, $\bar{G}(t) \equiv 1 - G(t)$. The first term of the right-hand side of (2) is the probability that an operating unit fails during (u, u+du] ($0 < u < t$) and repair of the failed unit is not completed up to time $x + u$. The second term is the probability that after repair completion of a failed unit, the operating unit fails during (u, u+du], and then the process obeys $\Pi_1(t-u, x)$.

Taking the Laplace-Stieltjes (LS) transforms on both sides of (1) and (2) with respect to t, we have

(3) $$\pi_2(s, x) = f(s)\pi_1(s, x),$$

(4) $$\pi_1(s, x) = \int_0^\infty e^{-st}\, \bar{G}(x+t)dF(t) \Big/ \left[1 - \int_0^\infty e^{-st}\, G(t)dF(t)\right],$$

where $f(s)$ is the LS transform of $F(t)$, and $\pi_2(s, 0)$ denotes the LS transform of the first-passage time distribution to system down and has been derived by Gnedenko, et al. [8] and Srinivasan [13]. The survival probability of the downtime, $\Pi_1(x) \equiv \Pr\{X > x\}$, is given by the marginal distribution of $\Pi_1(t, x)$, i.e.,

(5) $$\Pi_1(x) = \pi_1(0, x) = \int_0^\infty \bar{G}(x+t)dF(t) \Big/ \int_0^\infty \bar{G}(t)dF(t),$$

and the mean downtime is

(6) $$\int_0^\infty \Pi_1(x)dx = \int_0^\infty \bar{G}(t)F(t)dt \Big/ \int_0^\infty \bar{G}(t)dF(t).$$

In particular, if $G(t) = 1 - \exp(-\mu t)$, the uptime and the downtime are independent each other, and it is evident that $\pi_1(s, x) = \pi_1(s, 0)\pi_1(0, x)$.

3. TWO-UNIT PARALLELED REDUNDANT SYSTEM

Consider a two-unit paralleled redundant system with a single repairman, where the exponential failure is assumed (see Gaver [6] and Osaki [12]). The failure rates in state k (k = 1, 2) are λ_k, respectively, which are independent of time t (i.e., the exponential assumption). That is, the probability that either one of the two units fails during $(t, t+\Delta t)$, given that the two units are operating at time t, is $\lambda_2\Delta t + o(\Delta t)$, and the probability that one unit fails during $(t, t+\Delta t)$, given that one unit is operating at time t, is $\lambda_1\Delta t + o(\Delta t)$. The arbitrariness of assuming λ_k implies several interesting models as special cases.

In a similar fashion of (1) and (2), we have

$$\Pi_2(t, x) = \int_0^t \Pi_1(t-u, x)\lambda_2 e^{-\lambda_2 u} du, \tag{7}$$

$$\Pi_1(t, x) = \int_0^t \bar{G}(x+u)\lambda_1 e^{-\lambda_1 u} du + \int_0^t \Pi_2(t-u, x)e^{-\lambda_1 u} dG(u). \tag{8}$$

Thus,

$$\pi_2(s, x) = \{\lambda_2/(s + \lambda_2)\}\, \pi_1(s, x), \tag{9}$$

$$\pi_1(s, x) = \int_0^\infty \lambda_1 e^{-(s+\lambda_1)t}\, \bar{G}(x+t)dt \Big/ [1 - \{\lambda_2/(s + \lambda_2)\}\, g(s+\lambda_1)], \tag{10}$$

where g(s) is the LS transform of G(t). Note that $\pi_2(s, 0)$ is coincident with (19) of Gaver [6]. The survival probability of the downtime is

$$\Pi_1(x) = \int_0^\infty \lambda_1 e^{-\lambda_1 t}\, \bar{G}(x+t)dt \Big/ [1 - g(\lambda_1)], \tag{11}$$

and the mean downtime is

$$\int_0^\infty \Pi_1(x)dx = \int_0^\infty \bar{G}(t)(1 - e^{-\lambda_1 t})dt \Big/ [1 - g(\lambda_1)]. \tag{12}$$

The model discussed above includes several interesting redundant repairable models as special cases. We show four cases which are well-known and applicalbe in practical fields.

(i) $\lambda_1 = \lambda_2 \equiv \lambda$

In this case, the model corresponds to a two-unit standby redundant system discussed in Section 2. If the failure time is exponential, i.e., $F(t) = 1 - \exp(-\lambda t)$, equations (3) and (4) are equal to (9) and (10), respectively.

(ii) $\lambda_2 = 2\lambda$ and $\lambda_1 = \lambda$

The model correponds to a two-unit paralleled redundant system. In this case, $\pi_2(s, 0)$ is coincident with (1) of Gaver [6].

(iii) $\lambda_2 = \lambda + \lambda'$ and $\lambda_1 = \lambda$

The model corresponds to a two-unit standby redundant system with standby failure. That is, each unit has the failure rate λ when it is operating and λ' when it is in standby. Gnedenko, et al. [8, p. 135] has obtained the LS transform and the mean time to first system down for such a system.

(iv) $\lambda_2 = n\lambda$ and $\lambda_1 = (n - 1)\lambda$ $(n \geq 2)$

The model is a 2-out-of-n system $(n \geq 2)$. It is composed of n parallel units with a single repairman, and the system down occurs if 2 out of n units fail simultaneously. Downton [5] has obtained the time distribution to the first system down for a general m-out-of-n system $(m \leq n)$.

4. TWO-UNIT REDUNDANT SYSTEM WITH ALLOWED DOWNTIME

In addition to the two-unit standby and paralleled redundant systems discussed in Section 2 and Section 3, we assume that the system fails when both units have been down for more than A which is not always constant (Calabro [4] called such the time A the 'allowed downtime'). Kodama, et al. [10] discussed a one-unit system considering the allowed downtime. As an example of such a two-unit system, we give fuel charge/discharge system for a nuclear reactor discussed by Buzacott [3]. In the model, the reactor shuts down spontaneously when both fuel charge machines have been failed for more than a specified time, say 28 days.

Assume that the r.v. A has the cdf $H(t)$. Then, for the two-unit standby system, we easily have

$$\Pi_2(t, x) = \int_0^t \Pi_1(t-u, x)dF(u), \tag{13}$$

$$\Pi_1(t, x) = \int_0^t \overline{G}(x+u) \int_0^u dF(v)dH(u-v) + \int_0^t \Pi_1(t-u, x)G(u)dF(u) + \int_0^t \Pi_1(t-u, x) \int_0^u \overline{H}(u-v)dF(v)dG(u). \tag{14}$$

Thus,

$$\pi_1(s, x) = \frac{\int_0^\infty e^{-st}\, \overline{G}(x+t) \int_0^t dF(v)dH(t-v)}{1 - \int_0^\infty e^{-st}\, G(t)dF(t) - \int_0^\infty e^{-st} \int_0^t \overline{H}(t-v)dF(v)dG(t)} . \tag{15}$$

Suppose that H is the degenerate distribution placing unit mass at a, i.e.,

$H(t) = 0$ for $t < a$ and 1 for $t \geq a$. Then, the survival probability of the downtime is

$$(16)\qquad \Pi_1(x) = \frac{\int_0^\infty \bar{G}(x+t+a)dF(t)}{\int_0^\infty \bar{G}(t+a)dF(t)},$$

and the mean downtime is

$$(17)\qquad \int_0^\infty \Pi_1(x)dx = \frac{\int_0^\infty \bar{G}(t+a)F(t)dt}{\int_0^\infty \bar{G}(t+a)dF(t)},$$

which agree with (5) and (6) if $a = 0$, respectively. Further, the LS transform of the first-passage time distribution to system down is

$$(18)\qquad \pi_2(s,\ 0) = \frac{f(s)\int_0^\infty e^{-s(t+a)}\bar{G}(t+a)dF(t)}{1-\int_0^\infty e^{-st}G(t)dF(t)-\int_0^\infty e^{-st}F(t)dG(t)+\int_0^\infty e^{-s(t+a)}F(t)dG(t+a)}.$$

For the two-unit paralleled redundant system,

$$(19)\qquad \Pi_1(t,\ x) = \int_0^t \bar{G}(x+u)\int_0^u \lambda_1 e^{-\lambda_1 v}dvdH(u-v) + \int_0^t \Pi_2(t-u,\ x)e^{-\lambda_1 v}dG(u)$$
$$+ \int_0^t \Pi_1(t-u,\ x)\int_0^u \bar{H}(u-v)\ \lambda_1 e^{-\lambda_1 v}dvdG(u).$$

Thus,

$$(20)\qquad \pi_1(s,\ x) = \frac{\int_0^\infty e^{-st}\ \bar{G}(x+t)\ \int_0^t \lambda_1 e^{-\lambda_1 v}dvdH(t-v)}{1 - \frac{\lambda_2}{s+\lambda_2}\cdot g(s+\lambda_1) - \int_0^\infty e^{-st}\ \int_0^t \bar{H}(t-v)\lambda_1 e^{-\lambda_1 v}dvdG(t)}.$$

In a similar way, we can obtain the survival probability of the downtime, $\Pi_1(x)$, the mean downtime, $\int_0^\infty \Pi_1(x)dx$, and the first-passage time distribution to system down, $\pi_2(s,\ 0)$, although we omit here.

5. n-UNIT PARALLELED REDUNDANT SYSTEM

Consider an n-unit paralleled redundant system with a single repairman (see case A in Barlow and Proschan [2, p. 147]). All of units are good at time 0. The failure time of each unit is exponential with rate λ. If the repairman is busy, each failed unit waits for repair until the repairman is free. The repair discipline is 'first come first served.' The other assumptions are the same in the preceding sections. In such a model, we have

(21) $$\Pi_n(t, x) = \int_0^t \Pi_{n-1}(t-u, x)\, n\lambda e^{-n\lambda u} du,$$

(22) $$\Pi_k(t, x) = \int_0^t \bar{G}(x+u)\binom{k}{1}(1 - e^{-\lambda u})^{k-1}\, e^{-\lambda u} du$$
$$+ \sum_{j=1}^{k} \int_0^t \Pi_{j+1}(t-u, x)\binom{k}{j}(1 - e^{-\lambda u})^{k-j}\, e^{-j\lambda u}\, dG(u)$$
$$(k = 1, 2, \ldots, n-1).$$

It is evident that the above equations agree with (7) and (8), respectively, when $n = 2$, $\lambda_2 = 2\lambda$, and $\lambda_1 = \lambda$. The joint distribution $\Pi_k(t, x)$ can be calculated recursively from the above equations and is uniquely determined by them. For example, when $n = 3$, we have

(23) $$\pi_1(s, x) = \int_0^\infty \lambda e^{-(s+\lambda)t}\, \bar{G}(x+t)dt + \pi_2(s, x)g(s+\lambda),$$

(24) $$\pi_2(s, x) = \frac{\int_0^\infty 2\lambda e^{-(s+\lambda)t}\,(1 - e^{-\lambda t})\bar{G}(x+t)dt}{1 - \frac{3\lambda}{s+3\lambda}\cdot g(s+2\lambda) - 2[g(s+\lambda) - g(s+2\lambda)]},$$

(25) $$\pi_3(s, x) = \frac{3\lambda}{s+3\lambda}\cdot\pi_2(s, x).$$

In a similar fashion, we can obtain $\pi_k(s, x)$ for any k ($k = 1, 2, \ldots, n$). However, we omit here.

6. CONCLUSION

We have successfully derived the joint distrbutions of uptime and downtime for redundant repairable systems. The downtime distribution, which has easily derived as the marginal distribution of $\Pi_1(t, x)$, is of great importance because the system downtime is expensive in cost and/or dangerous. The results in this paper might be new so far as we know since the uptime and the downtime are dependent each other.

REFERENCES

[1] Barlow, R.E. and L.C. Hunter, "Reliability Anlysis of a One-Unit System," *Operations Research* 9 (1961), pp. 200-208.

[2] ——— and F. Proschan, *Mathematical Theory of Reliability*, Wiley, New York, 1965.

[3] Buzacott, J.A., "Reliability Analysis of a Nuclear Reactor Fuel Charging System," *IEEE Transactions on Reliability* R-22 (1973), pp. 88-91.

[4] Calabro, S.R., *Reliability Principles and Practices*, Wiley, New York, 1962.

[5] Downton, F., "The Reliability of Multiplex Systems with Repair," *Journal of the Royal Statistical Society, Ser.* B-28 (1966), pp. 459-476.

[6] Gaver, Jr., D.P., "Time to Failure and Availability of Paralleled Redundant Systems with Repair," *IEEE Transactions on Reliability* R-12 (1963), pp. 30-38.

[7] ———, "Failure Time for a Redundant Repairable System of Two Dissimilar Elements," *Ibid.* R-13 (1964), pp. 14-22.

[8] Gnedenko, B.V., Yu. K. Belyayev, and A.D. Solovyev, *Mathematical Methods of Reliability Theory*, Academic Press, New York 1969.

[9] Hosford, J.E., "Measures of Dependability," *Operations Research* 8 (1960), pp. 53-64.

[10] Kodama, M., J. Fukuta, and S. Takamatsu, "Mission Reliability for a 1-Unit System with Allowed Down-Time," *IEEE Transactions on Reliability* R-22 (1973), pp. 268-270.

[11] Nakagawa, T, and S. Osaki, "Stochastic Behaviour of a Two-Unit Standby Redundant System," *INFOR (Canadian Journal of Operational Research and Information Processing)* 12 (1974), pp. 66-70.

[12] Osaki, S., "System Reliability Analysis by Markov Renewal Processes," *Journal of the Operations Research Society of Japan* 12 (1970), pp. 127-188.

[13] Srinivasan, V.S., "The Effect of Standby Redundancy in System's Failure with Repair Maintenance," *Operations Research* 14 (1966), pp. 1024-1036.

[14] Takács, L., "On a Sojourn Time Problem in the Theory of Stochastic Processes," *Transactions of American Mathematical Society* 93 (1959), pp. 531-540.

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