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STOCHASTIC ORDER RELATIONS AMONG GI/G/1 QUEUES WITH A COMMON TRAFFIC INTENSITY

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Abstract. A class of queues in which the numbers of customers in systems are stochastically larger (smaller) than the one in *M/M/1* with a common traffic intensity is given. This fact ensures that *M/M/1* queues give safety bounds for a large class of queues. The case of $M/E_k/1$ ($k = 2,3,\cdots$) or *M/D/1* is treated in like manner. A new type of conservation law is derived to prove these results. Stochastic order relations among $M/E_k/1$ (or $E_k/M/1$) queues ($k = 1,2,\cdots$) are also obtained.

1. Introduction

The queueing process of a general *GI/G/1* queue cannot be solved analytically except for some cases, or even if it is solved, only a complicated solution is obtained, from which it is not easy to get numerical values of the characteristic quantities such as the steady state distributions of the waiting time and the queue length. From this and other reasons, it is useful to study closeness, continuity, order relations, and so on, among queueing models. In this paper, a stochastic order relation is considered, which is defined as follows (cf. [6]). For random variables X and Y, or their distributions F_X and F_Y, we denote $X \supset Y$ ($F_X \supset F_Y$) if $P\{ X > x \} \geqq P\{ Y > x \}$, or equivalently $F_X(x) \leqq F_Y(x)$ for any continuity point of $F(x)$. We say that X (F_X) is stochastically larger than Y (F_Y) if $X \supset Y$ ($F_X \supset F_Y$).

The purpose of this paper is to study stochastic order relations among *GI/G/1* queues, where a queue is called stochastically larger (smaller) if the steady state distribution of the number of customers in its system is stochastically larger (smaller). We only deal with the queues which have a common traffic intensity since such case is interested from a practical point of view but it has been scarecely studied.

Daley and Moran[4] showed that, if the service time and the inter-arrival time are stochastically smaller and larger, respectively, then the steady state distributions of the waiting time and the queue length in a $GI/G/1$ queue is stochastically smaller under some nonarithmetic condition. For a $GI/G/c$ queue, similar results were obtained under more restrictive conditions by Jacobs and Schach[6]. These results have large generality, but queues with a common traffic intensity cannot be compared with each other. Stoyan and Stoyan[14] also studied more general order relations and got similar results in $GI/G/1$ queues. By their method, the waiting time distributions of two queues with a common traffic intensity are compared with in some cases. But those order relations are always weaker than the stochastic one, and we cannot obtain stochastic order relations adapted to our purpose. These authors used monotone property of the waiting time process, and the difficulty lies in that we cannot use this monotonicity.

So we give attention to important but easily analysed queues such as $M/M/1$, $M/E_k/1$, $M/D/1$, $E_k/M/1$, and $D/M/1$, where M, E_k, and D denotes exponential, k-phase Erlang, and deterministic distributions, respectively. In this paper, these queues are called 'typical'. In Section 3, classes of queues which are stochastically larger (or smaller) than $M/M/1$, $M/E_k/1$ (k = 2,3, $\cdots$), and $M/D/1$ are given (Theorems 3.1 and 3.2). These imply, for example, that $M/M/1$ queues give safety bounds for a large class of queues at least concerning the queue length. The proofs are derived from a new type of conservation law. In Section 4, stochastic order relations among typical queues are obtained (Theorems 4.1 and 4.2). Our concern in this paper is only the number of customers in $GI/G/1$, but some remarks are given to the case of $GI/G/c$ and to the waiting time in $GI/G/1$.

2. Notation and preliminary results

We mainly consider a standard $GI/G/1$ queue, that is, a FCFS queue (first-come first-served) single-server queue with a recurrent input. Let T_n be the arrival time between the nth and $(n+1)$th customers and let S_n be the service time of the nth customer ($n = 1,2,\cdots$). Then, $\{T_n\}_{n=1}^{+\infty}$ and $\{S_n\}_{n=1}^{+\infty}$ are mutually independent, identically distributed sequences and independent of each other. Further, it is assumed for our $GI/G/1$ queue that

(i) The 1st customer arrives at time 0,

(ii) $ET < +\infty$ and $ES < +\infty$,

(iii) The traffic intensity $\rho = ES/ET < 1$, where the suffixes of T and S are omitted, this convention is always used if there is no confusion. The notation $F/G/1$ is used for a $GI/G/1$ queue if the distributions of T and S are F and G respectively. Sometimes a c-server FCFS queue with a stationary input defined by Loynes[8] is treated and it is denoted by $G/G/c$. Note that the notations $F/G/1$, $GI/G/1$, and $G/G/1$ must be distinguished. Throughout the paper, it is assumed that a $G/G/c$ queue is stable, that is, $ES/ET < c$.

Let $l(t)$, L_n, and L_n' be the numbers of customers in a system at time t, just before the nth arrival, and just after the nth departure of a customer, respectively. By letting the starting time of the operation of services to $-\infty$, we have stationary, limiting processes $\{l_*(t)\}$, $\{\tilde{L}_n\}$, and $\{\tilde{L}_n'\}$ corresponding to $\{l(t)\}$, $\{L_n\}$, and $\{L_n'\}$, respectively (cf. Miyazawa[11]). Note that l_*, $\tilde{L}$, and $\tilde{L}'$ are proper random variables since the queue is stable (cf. Theorem 4.1 of [11]). The next lemma is derived from Theorem 3.1 of Miyazawa[11].

Lemma 2.1 In a $G/G/c$ queue, we have

(2.1) $\lim_{t\to\infty} \frac{1}{t} \int_0^t f(l(s))ds = Ef(l_*(0))$ w.p.1,

(2.2) $\lim_{n\to\infty} \frac{1}{n} \Sigma_{i=1}^n f(L_i) = Ef(\tilde{L}_0)$ w.p.1,

(2.3) $\lim_{n\to\infty} \frac{1}{n} \Sigma_{i=1}^n f(L_i') = Ef(\tilde{L}_0')$ w.p.1,

for any nonnegative, nondecreasing (nonincerasing), lower (upper) semi-continuous(*) function f.

Remark Our main concern in this paper is a $GI/G/1$ queue, and stronger results are obtained for it. That is, there exists a proper random variable ν such that $L_n = \tilde{L}_n$ and $L_n' = \tilde{L}_n'$ for any $n \geqq \nu$ w.p.1 (cf. Loynes[8]). For a $GI/G/c$ queue, these facts are not true in general. But if $\{T - S > 0\} > 0$ then the system is empty infinitely often w.p.1 (cf. Whitt[15]), and it implies them.

By Lemma 2.1, the distributions of l_*, $\tilde{L}$, and $\tilde{L}'$ are considered to be the

(*) A real valued function f is called lower (upper) semi-continuous if $\underline{\lim}_{y\to x} f(y) \geqq (\leqq) f(x)$ for any x.

steady state ones of $l(t)$, L_n, and L'_n, respectively, although the limit distribution of $l(t)$ doesn't always exist. So those distributions are considered. Finch[5] showed for some restricted $GI/G/c$ queues that $\tilde{L} \overset{d}{\sim} \tilde{L}'$, where $\overset{d}{\sim}$ implies the equivalence of distributions. By Finch's argument and Lemma 2.1, this result is easily extended to:

Lemma 2.2 In a $G/G/c$ queue, $\tilde{L} \overset{d}{\sim} \tilde{L}'$.

Thus we study stochastic order relations of l_* and $\tilde{L}$. Now we denote an order relation of queues by $F_1/G_1/1 \supset F_2/G_2/1$ if $l_*^{(1)} \supset l_*^{(2)}$ and $\tilde{L}^{(1)} \supset \tilde{L}^{(2)}$, where $l_*^{(i)}$ and $\tilde{L}^{(i)}$ are l_* and $\tilde{L}$ of $F_i/G_i/1$ ($i = 1,2$), respectively. And $F_1/G_1/1$ is called stochastically larger than $F_2/G_2/1$. In particular, we denote $F_1/G_1/1 = F_2/G_2/1$ if $F_1/G_1/1 \supset F_2/G_2/1$ and $F_2/G_2/1 \supset F_1/G_1/1$.

We investigate stochastic order relations among queues with a common traffic intensity. In these queues, the arrival rate λ's($= (ET)^{-1}$) don't always agreed with one another. However, it is enough to consider queues with a common arrival rate and therefore with a common mean service time since we get

Lemma 2.3 For each fixed $a \geqq 0$, if $F_1(x) = F_2(ax)$ and $G_1(x) = G_2(ax)$ for any $x \geqq 0$, then $F_1/G_1/1 = F_2/G_2/1$.

This lemma is easily obtained by the change of time scale.

Finally, we introduce some classes of distributions for convenience. A distribution F is called γ-MRLA (γ-MRLB) if, for any $t \geqq 0$,

$$(2.4) \quad \int_t^{+\infty} (1-F(u))du/(1-F(t)) \leqq (\geqq) \ \gamma$$

(cf. Marshall[9]). In this paper, F is called A-type (B-type) if γ = the mean of F ($= \int u dF(u)$) and F is a γ-MRLA (γ-MRLB). F_A (F_B) in genaral denotes A-type (B-type) distribution. The important subclass of A-type (B-type) distributions is IFR (DFR). A distribution F is called IFR (DFR) if, for any fixed $t \geqq 0$, $F(t+x)-F(x)/(1-F(x))$ is not decreasing (increasing) function of x on the interval $\{ x \ ; \ F(x) < 1 \}$. The next distributions are all IFR or DFR and therefore A-type or B-type.

(i) A-type ··· M, E_k, D, uniform, Weibull, and truncated normal distributions.

(ii) B-type ··· M, H_k (kth order hyper-exponential) distributions.

In general, if X_1 and X_2 are independent of each other and have IFR distributions, then X_1+X_2 and $\min(X_1,X_2)$ also have IFR distributions. See Barlow and Proschan[1] for details of these facts and further results.

3. Stochastic order relations

Now we show main results of this paper. Firstly we prepare some lemmas, from which we obtain some interesting corollaries. Let t_n and t'_n be the nth arrival and departure times of customers respectively. And let $\tilde{t}'_n$ be the nth departure time of a customer after time 0 concerning the stationary, limiting process $\{l_*(t)\}_{t=-\infty}^{+\infty}$. We also introduce the following notations.

$$m_n \ (m'_n) = \sup \{ i \ ; \ t'_i < t_n \ (t_i < t'_n) \}$$

For example, m_n means the number of the customers who depart before the nth arrival. The next lemma can be obtained by Lemma 4.4 of Miyazawa[11].

Lemma 3.1 In a *G/G/c* queue, m_n/t_n, m_n/t'_n, n/t_n, and n/t'_n converges to λ ($= (ET)^{-1}$) w.p.1 as n tends to infinite.

We prove, so called, conservation laws concerning the queue length process (c.f. Brumelle[3]). At the first place, the process $\{l(t)\}_{t=0}^{+\infty}$ is compared with the process $\{L_n\}_{n=1}^{+\infty}$ in a *GI/G/c* queue. As illustrated in (i) of Figure 3.1, we have, for each sample path,

$$(3.1) \qquad \int_0^{t_n} \chi_{\{l(s)>j\}}ds = \Sigma_{i=1}^{n-1} T_i\cdot\chi_{\{L_i \geqq j\}} - \Sigma_{i=1}^{m_n} R_i^T\cdot\chi_{\{L'_i=j\}} \quad (j \geqq 0),$$

where χ is an indicator function, and $R_i^T = \inf_m \{ t_m - t'_i \ ; \ t'_i \leqq t_m \}$, that is, the time measured from the ith departure time to the arrival epoch of the next customer. Let $\tilde{R}_i^T = \inf \{ t_m - \tilde{t}'_i \ ; \ \tilde{t}'_i \leqq t_m \}$, then, under the assumption that $P\{ T-S>0 \} > 0$, we get, from (3.1), Lemma 2.1 and its Remark, and Lemma 3.1,

$$(3.2) \qquad P\{ l_* > j \} = \lambda E\{ T_1 \ ; \ \tilde{L}_1 \geqq j \} - \lambda E\{ \tilde{R}_1^T \ ; \ L'_1 = j \}$$

for any nonnegative integer j. By the assumptions of *GI/G/c*, T_1 and $\tilde{L}_1$ are independent of each other, so we obtain

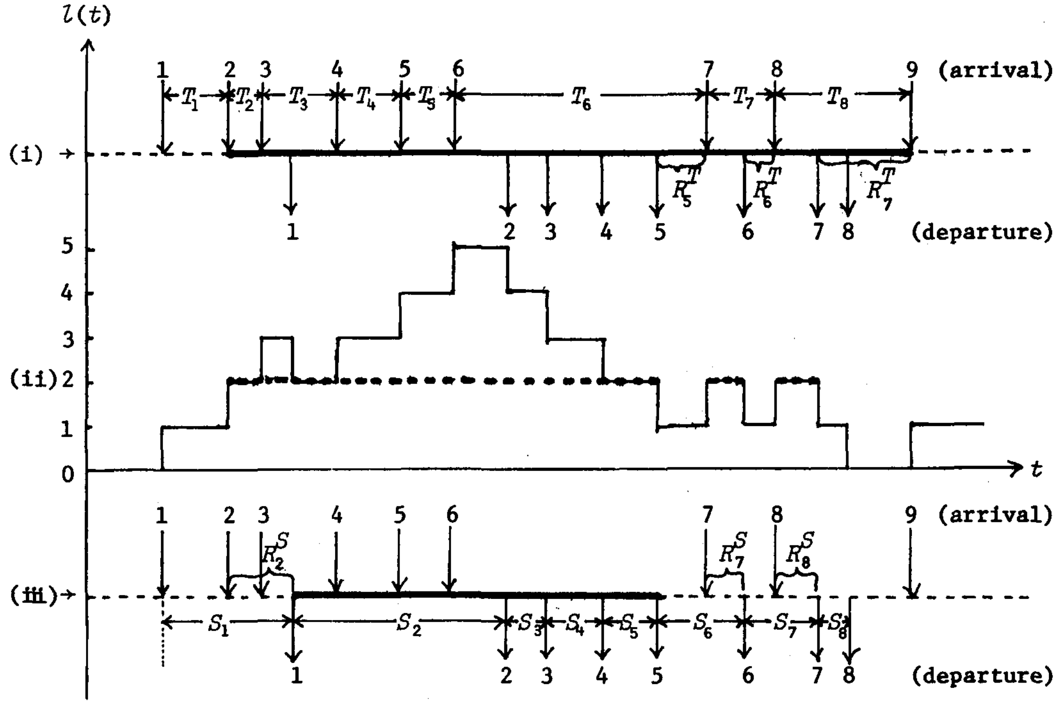

Fig. 3.1 The comparison of the time interval $\{ t ; l(t) > 1 \}$.

$$(3.3) \quad P\{ l_* > j \} = P\{ \tilde{L} \geqq j \} - \lambda E\{ \tilde{R}_1^T ; \tilde{L}_1' = j \} \qquad (j \geqq 0).$$

Now, (3.3), Lemma 2.2, and the definition of A-type (B-type) distribution yield

Lemma 3.2 In a $GI/G/c$ queue satisfying that $P\{ T-S > 0 \} > 0$, if the distribution of T is A-type (B-type), then

$$(3.4) \quad P\{ l_* > j \} \geqq (\leqq) P\{ \tilde{L} > j \} \qquad \text{for any } j \geqq 0,$$

where the equality of (3.4) holds if T has an exponential distribution.

Remark The equation (3.3) and therefore this lemma is true for a general $GI/G/c$ queue without any assumption. But it will be accompanied by lengthy proof (cf. Appendix of Miyazawa[12]). We don't deal with this general case since we are mainly concerned with a $GI/G/1$ queue, in which the assumption of this lemma is always satisfied.

In the second place, the process $\{l(t)\}$ is compared with $\{L'_n\}$ in a $GI/G/1$ queue. As shown in (iii) of Fig. 3.1, for each sample path,

$$(3.5) \quad \int_0^{t'_n} \chi_{\{l(s)>j\}} ds = \Sigma_{i=1}^{n-1} S_{i+1}\chi_{\{L'_i>j\}} + \Sigma_{i=1}^{m'_n} R_i^S \chi_{\{L_i=j\}} \quad (j \geqq 0),$$

where $R_i^S = \inf \{ t'_m - t_i \; ; \; t_i \leqq t'_m \}$. Note that (3.5) is not true for a $GI/G/c$ ($c \neq 1$), in the case of which a similar result is obtained, but it is more complicated (c.f. (5.1) of Sec. 5). Now, by the analogous argument in the proof of Lemma 3.2, we have, for any nonnegative integer j,

$$(3.6) \quad P\{ l_* > j \} = \rho P\{ \tilde{L} > j \} + \lambda E\{ \tilde{R}_1^S \; ; \; \tilde{L}_1 = j \},$$

where $\tilde{R}_i^S = \inf \{ \tilde{t}'_m - t_i \; ; \; t_i \leqq \tilde{t}'_m \}$. From (3.6), it follows that

Lemma 3.3 In a $GI/G/1$ queue, if the distribution of S is A-type (B-type), then

$$(3.7) \quad P\{ l_* > j \} \leqq (\geqq) \; \rho P\{ \tilde{L} \geqq j \} \qquad \text{for any } j \geqq 0,$$

where the equality of (3.7) holds if S has an exponential distribution.

Now we obtain two classes of queues which are stochastically larger and smaller than $M/M/1$ queues from Lemmas 3.2 and 3.3.

Theorem 3.1 $F_B/G_B/1 \supset M/M/1 \supset F_A/G_A/1$.

Proof. Assume that a queue is $F_A/G_A/1$. From (3.4) and (3.7), we have, for any nonnegative integer j,

$$(3.8) \quad P\{ \tilde{L} \geqq j+1 \} \leqq \rho P\{ \tilde{L} \geqq j \},$$

$$(3.9) \quad P\{ l_* \geqq j+1 \} \leqq \rho P\{ l_* \geqq j \},$$

where the equalities of (3.8) and (3.9) hold if $F_A = M$ and $G_A = M$. So, for an $M/M/1$ queue, the equalities of (3.8) and (3.9) hold, and the values of $P\{ \tilde{L} \geqq j \}$ and $P\{ l_* \geqq j \}$ ($j = 0,1,\cdots$) are determined recursively by them. It is clear that $P\{ \tilde{L} \geqq j \}$ and $P\{ l_* \geqq j \}$ satisfying (3.8) and (3.9) respectively are less than the ones in the case where the equalities hold. Thus we obtain a half of *Theorem*. For a $F_B/G_B/1$ queue, it is proved in like manner.

Corollary 3.1 In a $GI/G/1$ queue, we have

$$(3.10) \qquad \rho P\{ \tilde{L} > j \} \leqq P\{ l_* > j \} \leqq P\{ \tilde{L} \geqq j \} \qquad (j \geqq 0).$$

This corollary is easily obtained from (3.3) and (3.6). Notice that these results generalize some of results of Marshall and Wolff[10].

Next we proceed to the similar argument to an $M/E_k/1$ queue by using so called phase method. Suppose that each customers carries k phases of work each of which requires a service time subjected to a common distribution. Let $h(t)$, H_n, H'_n be phases in a system at time t, at the nth arrival time of a customer, and at the nth departure of a phase, respectively. For these processes, analogous results to Lemma 2.1 is also obtained. So, we let $h_*(t)$, $\tilde{H}_n$, $\tilde{H}'_n$ be their steady state versions. Lemma 2.2 is not true for $\tilde{H}_n$ and $\tilde{H}'_n$, but similar results can be obtained by the same idea of the proof as follows.

Lemma 3.4 In a $GI/G^{(k*)}/c$ queue, where $G^{(k*)}$ means that the distribution of S is the k-fold convolution of some distribution with itself,

$$(3.11) \qquad P\{ \tilde{H}' = j \} = \Sigma_{i=\max(0,j-k+1)}^{j} P\{ \tilde{H} = i \} \qquad (j \geqq 0).$$

In a similar way of getting (3.3) and (3.6), we obtain

Lemma 3.5 (i) In a $GI/G^{(k*)}/c$ queue satisfying that $P\{ T-S > 0 \} > 0$, we have, for any nonnegative integer j,

$$(3.12) \qquad P\{ h_* > j \} = P\{ \tilde{H} \geqq j-k+1 \} - \lambda E\{ \tilde{R}_1^{\hat{T}} ; \tilde{H}' = j \},$$

(ii) In a $GI/G^{(k*)}/1$ queue, we have, for any nonnegative integer j,

$$(3.13) \qquad P\{ h_* > j \} = \rho P\{ \tilde{H}' > j \}$$
$$+ k\lambda E\{ \tilde{R}_1^{\hat{S}} ; \tilde{H}_1 = \max(0,j-k+1),\cdots, j \},$$

where $\tilde{R}_i^{\hat{S}}$ ($\tilde{R}_i^{\hat{T}}$) is the time measured from the ith arrival epoch of a customer to the departure time of the next phase (from the ith departure time of a phase to the arrival epoch of the next customer).

Now we prove

Theorem 3.2 $F_B/G_B^{(k*)}/1 \supset M/E_k/1 \supset F_A/G_A^{(k*)}/1$, where $F^{(k*)}$ is the k-fold convolution of F with itself.

Proof. Assume that a queue is $F_A/G_A^{(k*)}/1$. Then each phase has the distribution G_A. So, combining (3.11), (3.12), and (3.13), we have

$$(3.14) \quad \frac{\rho}{k} \Sigma_{i=1}^{k} P\{ \tilde{H} \geqq \max(0,j-k+1) \} \geqq P\{ \tilde{H} \geqq j+1 \} \qquad (j \geqq 0),$$

$$(3.15) \quad \frac{\rho}{k} \Sigma_{i=1}^{k} P\{ h_* \geqq \max(0,j-k+1) \} \geqq P\{ h_* \geqq j+1 \} \qquad (j \geqq 0).$$

In particular, the equalities hold for an $M/E_k/1$ queue. By the similar reason in the proof of Theorem 3.1, the values $P\{ \tilde{H} \geqq j \}$ and $P\{ l_* \geqq j \}$ satisfying (3.14) and (3.15) are less than the ones in the case of an $M/E_k/1$ queue with a common traffic intensity. If we note that $P\{ \tilde{L} \geqq j \} = P\{ \tilde{H} \geqq k(j-1)+1 \}$ and $P\{ l_* \geqq j \} = P\{ h_* \geqq k(j-1)+1 \}$, we obtain a half of *Theorem*. Another half is proved in a similar fashion.

Corollary 3.2 $F_B/D/1 \supset M/D/1 \supset F_A/D/1$ for absolutely continuous functions F_A and F_B.

Corollary 3.3 $M/M/1 \supset M/E_k/1 \supset M/E_{2k}/1 \supset M/D/1$ ($k = 2,3,\cdots$).

Corollary 3.2 follows from Theorem 3.2 if we note that the distributions of $\tilde{L}$ and l_* of $F/E_k/1$ converge to the ones of $F/D/1$ as k tends to infinite. This fact is easily implied by a continuity theorem of the steady state distribution of the waiting time (Theorem 4 of Borovkov[2]) and the relations between the queue length and the waiting time (cf. [4], [10], and [11]). Corollary 3.3 is a simple consequence of Theorem 3.2 and Corollary 3.2. Next we try to study special cases.

4. Special cases

In this section, we restrict the problem to typical queues such as *M/M/1*, $M/E_k/1$, *M/D/1*, $E_k/M/1$, and *D/M/1*. These queues are classified into *M/G/1* and *GI/M/1* queues. Note that stochastic order relations of $\tilde{L}$ imply those of l_* in these queues by Lemma 3.2 and 3.3. So we are concerned only with $\tilde{L}$. Now we study the two groups of queues separately.

(1) GI/M/1 queues

In this case, it is well known that the distribution of $\tilde{L}$ is given as follows (cf. p 126 of Prabhu[12]).

$$(4.1) \qquad P\{ \tilde{L} > j \} = \zeta^j \qquad (j \geqq 0),$$

where ζ is the smallest positive root of the equation:

$$(4.2) \qquad \zeta = E\{e^{-(1-\zeta)T/ES}\}.$$

For an $E_k/M/1$ queue, (4.2) is equal to

$$(4.3) \qquad \zeta^{-1} = \{ 1 + \frac{1-\zeta}{\rho k} \}^k.$$

It is easily examined that (4.3) has only one root in the interval (0,1) and the right hand side of (4.3) is increasing in k. Thus the root ζ is decreasing in k. So we get that $E_k/M/1 \underset{\sim}{\supset} E_{k+1}/M/1$ ($k = 1,2,\cdots$). And, by the continuity of the distribution of $\tilde{L}$ in *GI/M/1* queues, we obtain that $E_k/M/1 \supset D/M/1$ for any $k \geqq 1$. Further, for a *D/M/1* queue, a stronger result can be obtained. From (4.2) and Jensen's inequality,

$$(4.4) \qquad \zeta = E\{e^{-(1-\zeta)T/ES}\} \geqq e^{-(1-\zeta)/\rho} \qquad (0 < \zeta < 1).$$

The last two terms of (4.4) are convex functions of ζ ($0 < \zeta < 1$), and so it is obtained that $GI/M/1 \supset D/M/1$. Now the results are summed up.

Theorem 4.1 $E_k/M/1 \supset E_{k+1}/M/1 \supset D/M/1$ ($k = 1,2,\cdots$). Moreover, a *D/M/1* queue is the smallest in stochastic order relations among *GI/M/1* queues.

(2) M/G/1 queues

We study stochastic order relations among $M/E_k/1$ queues ($k = 1,2,\cdots$) and an $M/D/1$ queue. For these queues, we have already obtained partial answer in the previous section (Corollary 3.3). Now we strengthen this result to the relation that $M/E_k/1 \supset M/E_{k+1}/1$ ($k = 1,2,\cdots$). For a general $M/G/1$ queue, the distribution of $\tilde{L}$ is also known (cf. Prabhu[13]). But it is difficult to study stochastic ordering by using those results since they are not so simple as in a $GI/M/1$ queue. Here we devise another method. We use the relation (3.14) satisfied by the number of phases $\tilde{H}$, where the equality of (3.14) holds for an $M/E_k/1$ queue. Let a positive integer k be fixed. And let

$$a_i = 1 \qquad (i = 1,2,\cdots,k),$$
$$a_i = P\{ \tilde{H} \geqq i-k \} \qquad (i = k+1,k+2,\cdots),$$

where $\tilde{H}$ is the number of phases in an $M/E_k/1$ queue. Similarly, we define the sequence $\{b_i\}_{i=1}^{+\infty}$ for an $M/E_{k+1}/1$ queue. Note that these two sequences are not increasing in i. The relation (3.14) gives the next equations.

$$(4.5) \qquad a_{k+j} = \frac{\rho}{k} \{ a_j + \cdots + a_{j+k-1} \},$$

$$(4.6) \qquad b_{(k+1)+j} = \frac{\rho}{k+1} \{ b_j + \cdots + b_{k+j} \},$$

for any integer $j \geqq 1$. It is clear that, if $a_{nk+1} \geqq b_{n(k+1)+1}$ for any integer $n \geqq 0$, then $M/E_k/1 \supset M/E_{k+1}/1$. Now we prove inductively the next inequalities.

$$(4.7) \qquad a_{nk+j} \geqq \frac{k-j+1}{k} b_{n(k+1)+j} + \frac{j-1}{k} b_{n(k+1)+j} \qquad (j = 1,2,\cdots,k),$$

for $n = 0,1,2,\cdots$. By the very definitions of the two sequences, these inequalities are satisfied for $n = 0$. We show that these inequalities for $n = 1$ are derived only from (4.5), (4.6), and (4.7) for $n = 0$. If these are shown, then we can get (4.7) for any positive integer n inductively in like manner. From (4.5), (4.6), and (4.7) for $n = 0$,

$$(4.8) \qquad a_{k+1} = \frac{\rho}{k} \{ a_1 + \cdots + a_k \} \qquad \text{(by (4.5))}$$

$$\geqq \frac{\rho}{k} \{ \frac{k-1}{k} (b_1 + \cdots + b_{k+1}) + \frac{1}{k} b_1 \} \qquad \text{(by (4.7))}$$

$$= b_{(k+1)+1} + \frac{1}{k^2} \{ \rho b_1 - b_{(k+1)+1} \} \qquad \text{(by (4.6))}$$

$$\geq b_{(k+1)+1}$$

where the last inequality is obtained since

$$(4.9) \qquad b_{(k+1)+1} = \frac{\rho}{k+1}\{ b_1 + \cdots + b_{k+1}\} \leq \rho b_1,$$

by the monotonicity of $\{b_i\}_{i=1}^{+\infty}$. Let $\{ 1,2,\cdots,i \}$ denote values of j for which (4.7) is true for $n = 1$. Then, from these assumptions,

$$\begin{aligned}(4.10) \qquad a_{k+i+1} &= \frac{\rho}{k}\{ a_{i+1} + \cdots + a_{k+i} \} \\ &\geq \frac{\rho}{k}\{ \frac{k-i}{k}(b_{i+1} + \cdots + b_{k+i+1}) \\ &\qquad + \frac{i-1}{k}(b_{i+2} + \cdots + b_{k+i+2}) + \frac{1}{k} b_{(k+1)+1} \} \\ &= \frac{k-i}{k} b_{(k+1)+i+1} + \frac{i}{k} b_{(k+1)+i+2} \\ &\quad + \frac{1}{k^2}\{(k-i)b_{(k+1)+i+1} - (k-i+1)b_{(k+1)+i+2} + \rho b_{(k+1)+1}\} \\ &\geq \frac{k-i}{k} b_{(k+1)+i+1} + \frac{i}{k} b_{(k+1)+i+2},\end{aligned}$$

where the last inequality is obtained since

$$\begin{aligned}(4.11) \qquad &\frac{k+1}{\rho}\{(k-i)b_{(k+1)+i+1} - (k-i+1)b_{(k+1)+i+2} + \rho b_{(i+1)+1}\} \\ &= (k-i)\{ b_{i+1} + \cdots + b_{(k+1)+i} \} + (k+1)b_{(i+1)+1} \\ &\quad - (k-i+1)\{ b_{i+2} + \cdots + b_{(k+1)+i+1}\} \geq 0,\end{aligned}$$

by the monotonicity of $\{b_i\}_{i=1}^{+\infty}$. Thus we get (4.7) for $n = 1$ by induction on i in (4.10). Therefore we obtain

Theorem 4.2 $M/E_k/1 \supset M/E_{k+1}/1 \supset M/D/1$ ($k = 1,2,\cdots$).

Remark Contrasted with the case of *GI/M/1* queues, it is questioned whether an *M/D/1* queue is the smallest in stochastic ordering among *M/G/1* queues or not ? It seems to be affirmative from the numerical computations of several models (see Table 4.1), but the author cannot yet prove it.

j =	1	2	3	4	5	6	7
M/D/1	0.6	0.27115	0.10926	0.04260	0.01652	0.00640	0.00248
M/G'/1	0.6	0.30783	0.14172	0.06304	0.02786	0.01231	0.00544

Table 4.1 The values of $P\{\tilde{L} \geqq j\}$ ($\rho = 0.6$, $\lambda = 1$, and G' has point mass 0.2 at 0.0 and 0.8 at 0.75.)

Using a well known formula of the Laplace transform of the probability distribution of $\tilde{L}$ of an *M/G/1* queue (cf. [13]), it is easily obtained that, in *M/G/1* queues, $P\{\tilde{L} \geqq 1\} = \rho$ and $P\{\tilde{L} \geqq 2\}$ attains minimum for *M/D/1*. But, it is not easy to get further results from the formula.

5. Summary of results and further problems

Now we aggregate the results in the preceding sections in the following diagram of stochastic order relations. In the diagram, it is assumed that queues have a common traffic intensity less than 1.

(1) $M/M/1 \supset \cdots \supset E_k/M/1 \supset \cdots \supset D/M/1$

(2)
$$
\begin{array}{ccccc}
F_B/G_B/1 & \supset & M/M/1 & \supset & F_A/G_A/1 \\
 & & \cup & & \\
 & & \vdots & & \\
 & & \cup & & \\
F_B/G_B^{(k*)}/1 & \supset & M/E_k/1 & \supset & F_A/G_A^{(k*)}/1 \\
 & & \cup & & \\
F_B/G_B^{(k+1*)}/1 & \supset & M/E_{k+1}/1 & \supset & F_A/G_A^{(k+1*)}/1 \\
 & & \cup & & \\
 & & \vdots & & \\
 & & \cup & & \\
F_B/D/1 & \supset & M/D/1 & \supset & F_A/D/1
\end{array}
$$

For example, we see that an *M/M/1* queue is stochastically larger than a large class of queues from a practical point of view since A-type distributions are often encountered in practice (cf. examples of Sec. 2). That is, an *M/M/1* queue gives a safety bound for these queues at least concerning the queue length in the steady state. Since $M/M/1$, $M/E_k/1$, and $M/D/1$ queues are only easily and explicitly solved ones among $GI/G/1$, these diagram will also have practical usefulness to estimate tail probabilities.

It seems to be rather surprised that a strong relation such as stochastic ordering holds for many queues with a common traffic intensity. But, the obtained results and numerical calculations may insist that stochastic order relations are effective in larger extent. These problems remain to be solved. Here, we give some remarks to the queue length of a many-server queue and to the waiting time.

The extensions of our method to many-server queues are difficult except for special cases since the relation (3.6) must be replaced by the complicated one for $GI/G/c$ such as:

$$(5.1)\quad cP\{ l_* > j \} = \rho P\{ \tilde{L} > j \} + \lambda[E\{ \Sigma_{i=1}^{c} \tilde{R}_{i,1}^{S} ; \tilde{L} = j \} - E\{ \Sigma_{i=1}^{c} \tilde{R}_{i,2}^{S} ; \tilde{L}' = j \} \quad \text{for any } j \geqq c,$$

where $\tilde{R}_{i,1}^{S}$ ($\tilde{R}_{i,2}^{S}$) is the time measured from an arrival (departure) epoch of a customer to the departure time of the next customer being served by the ith server ($i = 1,2,\cdots,c$). Special cases such that the distribution of the service time is M or E_k are dealt with in the same way as single server queues. For example, we can easily get, concerning the phase number in a system,

$$(5.2)\quad F_B/E_k/c \supset M/E_k/c \supset F_A/E_k/c \qquad (k = 1,2,\cdots).$$

Now we question if Theorem 4.1 is also true for a many-server queue, that is,

$$(5.3)\quad M/E_k/c \supset M/E_{k+1}/c \qquad (k = 1,2,\cdots).$$

Our numerical calculations show that (5.3) is not true for the number of customers in the system, but it is conjenctures that (5.3) holds for the queue length (see Table 5.1).

Next we note on the waiting time. The waiting time process wouldn't be dealt with by the method in Sec. 3. But, for $GI/M/1$ queues, similar results as in Sec. 5 is easily shown. For $M/G/1$ queues, if $M/G_1/1 \supset M/G_2/1$ in our

$k \backslash j$	1	2	3	4	5	6	7	8
1	0.59654	0.23343	0.07003	0.02101	0.00630	0.00189	0.00056	0.00016
2	0.59710	0.23377	0.06913	0.01815	0.00447	0.00106	0.00024	0.00005
3	0.59746	0.23393	0.06861	0.01712	0.00387	0.00082	0.00017	0.00003
4	0.59771	0.23401	0.06828	0.01658	0.00358	0.00071	0.00013	0.00002
5	0.59790	0.23407	0.06803	0.01624	0.00341	0.00065	0.00012	0.00002

Table 5.1 The values of $P\{ l_* \geqq j \}$ in $M/E_k/c$ ($c = 3$, $\rho = 0.6$, and l_* is the number of customers in the system).

original sence, then, by moment relations got by Marshall and Wolff[9], we have

(5.4) $Ew_1^n \geqq Ew_2^n$ for any positive integer n,

where w_1 and w_2 are the waiting times in the steady state of $M/G_1/1$ and $M/G_2/1$ respectively. From these facts, it is inferred that the stochastic or-relations obtained in Sections 3 and 4 are also true for the waiting time. These problems also wait for further studies.

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