Journal of the Operations Reseach
Society of Japan
Vol. 19, No. 2, June, 1976

MARGINAL CHECKING OF A MARKOVIAN DEGRADATION UNIT WHEN CHECKING INTERVAL IS PROBABILISTIC

HISASHI MINE and HAJIME KAWAI, *Kyoto University*

(Received May 29, 1975; Revised January 12, 1976)

Abstract

Marginal checking of a Markovian degradation unit is treated when time interval to the next checking is not fixed but obeys a certain general distribution. The problem of determing the optimal set of the states at which the unit is replaced with a new one (marginal set) is discussed. It is solved by using Markov-renewal programming with modified policy iteration cycle. It is showed that control limit rule holds for the optimal policy. The expected cost associated with preventive maintenance and corrective maintenance when the unit is operated in an infinite time span (cost rate) is derived. The unimodality of the cost rate with respect to the control limit is discussed, and a necessary and sufficient condition for preventive replacement to be effective is given.

1. Introduction

We consider a unit which is capable of assuming many states. Flehinger [3] considered a model where it is assumed that a unit may be in any one of states $0, 1, \ldots, n, n+1$ (0 : good state, $1, \ldots, n$: degraded state, $n+1$: failed state) and during a normal operation these states constitute a continuous parameter Markov process in which $n+1$ is the absorbing state. (Such unit is called a Markovian degradation unit.) She determined the operating characteristics of policy which is called control limit rule when the checking interval is specified and it is constant. A control limit rule is of the simple form: Replace the unit if and only if the observed state is one of the states $m, m+1, \ldots, n, n+1$ for some m. The set of states $m, \ldots, n+1$ is called marginal states and state m is called control limit. For the case that the checking interval, the time for preventive maintenance and corrective maintenance are constant and have the same length, the optimality of control limit rule can be verified [2], by using a Markovian sequential decision process. In such case the policy, which minimizes the expected cost per one transition of the process constituted of the states of the unit at checking time and at the time instant of the completion of maintenance, is the same as the policy which minimizes the expected cost per unit time. However, in more general case where the checking time interval is probabilistic, they are not necessarily identical. In this paper, we consider almost the same unit as that of [3] but the checking interval is not constant. That is, we treat the case where the checking interval is distributed for some reason even if it was predtermined. For example, if we have two machines and only one repair man. When one machine is under repair, the other machine can not be checked at the predtermined checking time and must wait until repair completes. Therefore, checking time of operating unit is disturbed. For a Markovian degradation unit in such case, we consider the problem of determing the optimal set of states at which preventive maintenance (PM) is performed which minimizes the expected cost per unit time in an infinite time span (cost rate).

Note that maintenance in any degraded state is called PM and in the failed state it is called corrective maintenance (CM).

In the following, we explain the Markovian degradation in more detail. The unit is subject to random failure from any state; the failed unit is maintained. As a unit degrades more, its failure rate increases. The unit has the following properties in the absence of maintenance.

(i) The transition rates from one state to another are independent of time, i.e., are constant.

(ii) The transition rates from state i to $n+1$ are lower as i is lower. In the absence of maintenance, the state number can not decrease.

(iii) From state i, only a transition to $i+1$ or $n+1$ is possible.

For such unit, at each checking time we have the following three actions.

(PM) We perform PM.

(W) We don't perform PM and wait the next checking time.

(CM) We perform CM.

In our model, the problem is to determine the set of states for which the action *(PM)* is to be selected.

In this optimization problem the following assumptions are made.

(i) The states of the unit $(0, 1, \ldots, n, n+1)$ can not be determined without checking.

(ii) Each checking needs negligibly small time and associates with negligibly small cost. (Even if it is relatively large, it gives no affection to our optimization problem.)

(iii) Checking is perfect. That is, each checking accumulates no damage to our unit.

(iv) The time to the next checking is destributed according to the distribution function $H(t)$ with expected value H. This time is measured from the time instant at which checking, PM or CM is completed.

(v) The cost incurred by the failure of the unit is c_d per unit time.

(vi) PM time and CM time are distributed according to $M(t)$ with mean value M and $R(t)$ with mean value R, respectively.

(vii) PM and CM are associated with cost c_p and c_r per unit time, respectively.

(viii) Considering that the unit is in failed state under CM,

(1.1) $$c_d < c_r \ .$$

(ix) For PM to have meaning,

(1.2) $$c_p \le c_r \ ,$$

(1.3) $$M \le R \ ,$$

(x) If the unit is in $n+1$ at checking time, it is immediately maintained.

(xi) Immediately after PM or CM, the unit is in state 0.

Under the above assumptions, we shall derive the optimal policy which is given by some combination of actions *(PM)*, *(W)* and *(CM)*, using the theory of Markov-renewal programming (section 3). And we shall show that in the optimal policy, control limit rule holds (section 4). Moreover, we shall determine the operating characteristic (cost rate) of the policy (control limit rule) and give the necessary and sufficient condition that PM is useful (section 5). In section 6, some numerical examples will be given.

2. Transition Probability

α_i transition rate from i to $n+1$ (failure rate), $\alpha_i < \alpha_{i+1}$.

β_i transition rate from i to $i+1$ (degradation rate), $\beta_n = 0$.

$\lambda_i \equiv \alpha_i + \beta_i$.

$P_{ij}(t) \equiv \text{Prob}\{\text{unit is in state } j \text{ at time t} \mid \text{it was in state } i \text{ at time } 0\}$.

$P_{ij}(t)$, are easily derived; see [5]

$$P_{ij}(t) = \begin{cases} 0 & \text{for } i > j , \\ 1 & \text{for } i = j = n+1 . \end{cases} \tag{2.1}$$

$$P_{ii}(t) = e^{-\lambda_i t} . \tag{2.2}$$

$$P_{ij}(t) = \beta_i \beta_{i+1} \cdots \beta_{j-1} \sum_{k=i}^{j} \{e^{-\lambda_k t} / \prod_{\ell=i,\neq k}^{j} (\lambda_\ell - \lambda_k)\} , \tag{2.3}$$

$$\text{for} \quad 0 \le i < j \le n .$$

$$P_{i,n+1}(t) = \sum_{j=i}^{n} \alpha_j \int_0^t P_{ij}(x)dx , \quad i=0,1,\ldots,n . \tag{2.4}$$

These are the transition probabilities of the unit with no maintenance.

3. Markov-renewal Programming Formulation

Noting that if the unit is in $n+1$ at checking time, it is always maintained, we consider the following states:

E_0 the time instant at which PM or CM has been completed, or at which checking has been completed and the unit is in state 0,

E_i the time instant at which checking has been completed and the unit is in state i, $i=1,\ldots,n+1$.

States E_0, $E_1,\ldots$, E_{n+1} constitute a semi-Markov process [7] since each checking time instant is a regeneration point (regeneration point is the time instant at which the process is considered to start, e.g., in Markov process, every time instant is a regeneration point). And the time instants at which we can make some decisions are only E_0, $E_1,\ldots$, E_n, E_{n+1}. Moreover, it is easily seen that for every policy, the imbedded Markov chain of $E_0,\ldots,E_n,E_{n+1}$ is ergodic. Therefore, we can formulate our optimization problem by Markov-renewal programming [5]. However, we use a Policy Improvement Routine (PIR) which is different from the ordinary PIR in the Policy Iteration Cycle (PIC). We start our PIC by first guessing the initial policy and second going to Value Determination Routine VDR.

Notation

P_{ij} probability that when the unit was in state i at a checking time, it is in state j at the next checking time, $i,j=0,1,\ldots,n+1$.

These are given by

$$(3.1) \qquad P_{ij} = \int_0^\infty P_{ij}(t)dH(t) .$$

$P_i \equiv P_{ii}$.

$f_i \equiv P_{i,n+1}$.

μ_i the expected time for the unit to be in state $n+1$ to the next checking time, when it was in state i at a checking time and action (W) was selected.

These are given by

$$(3.2) \qquad \mu_i = \int_0^\infty P_{i,n+1}(t)[1 - H(t)]dt .$$

g cost rate for a given policy

v_i the so-called relative values [5] for a given policy, $i=0,1,\ldots,n+1$.

It is usually practical to set one of v_i equal to zero. We put $v_0=0$.

D_i the action to be selected for E_i.

Note that we always select the action (CM) for E_{n+1}.

Value Determination Routine (VDR)

The following three quantities play important roles in Markov-renewal programming:

the expected time to the next transition

H if $D_i = (W)$, M if $D_i = (PM)$, R if $D_{n+1} = (CM)$,

the expected cost to the next transition

$c_d\mu_i$ if $D_i = (W)$, c_pM if $D_i = (PM)$, c_rR if $D_{n+1} = (CM)$,

the transition probability in imbedded Markov chain

P_{ij}	from E_i to E_j	if $D_i = (W)$,
1	from E_i to E_0	if $D_i = (PM)$,
1	from E_{n+1} to E_0	if $D_{n+1} = (CM)$.

Using the above quantities, we have the following VDR.

VDR

For the current policy, solve the following set of equations with respect to g, where $v_0 = 0$.

For E_i, $i=0,1,\ldots,n$,

$$(3.3) \qquad Hg + v_i = c_d\mu_i + \sum_{j=i}^{n+1} P_{ij}v_j \,, \qquad \text{if} \quad D_i = (W)$$

$$(3.4) \qquad Mg + v_i = c_pM + v_0 \,, \qquad \text{if} \quad D_i = (PM).$$

For E_{n+1}

$$(3.5) \qquad Rg + v_{n+1} = c_rR + v_0 \,.$$

If the cost rate g obtained in this VDR is equal to the one obtained in the previous VDR, we have the optimal policy (current policy). Otherwise, go to PIR.

Policy Improvement Routine (PIR)

We define the following quantities, which play important roles and which have no special physical meaning.

$$(3.6) \qquad v_{n+1}(g) = c_rR - gR \,,$$

$$(3.7) \qquad v_i(g) = \begin{cases} [c_d\mu_i + \sum_{j=i+1}^{n+1} P_{ij}v_j(g) - Hg] \,/\, [1 - P_i] \,, & \\ \qquad\qquad \text{if} \quad D_i = (W), & \\ c_pM - gM \qquad \text{if} \quad D_i = (PM), \; i=0,1,\ldots,n. & \end{cases}$$

Moreover, we define

$$(3.8) \qquad v^*_{n+1}(g) = v_{n+1}(g) \,,$$

$$(3.9) \qquad v^*_i(g) = \min \begin{cases} [c_d\mu_i + \sum_{j=i+1}^{n+1} P_{ij}v_j{}^*(g) - Hg] \,/\, [1 - P_i] \,, \\ c_pM - gM \,. \end{cases}$$

Note that in VDR, we get g and v_i by giving a certain policy and putting $v_0=0$. On the other hand, $v_i(g)$ is determined by giving certain policy and g. Therefore, $v_0(g)$ is not necessarily zero. (however, if g is the cost rate for a given policy, $v_0(g) = 0$.)

PIR

For using cost rate g which has been obtained in the VDR, we find the policy that minimizes $v_0(g)$. Then this policy becomes the new policy. This policy is gotten by finding the action for each E_i which gives $v_i{}^*(g)$. That is, if $c_p M - gM < [c_d \mu_i + \sum_{j=i+1}^{n+1} P_{ij} v_j{}^*(g) - Hg] / [1 - P_i]$, then we select (PM) and otherwise (W) is selected. Then go to VDR.

It should be noted that, in ordinary Markov Renewal Programming [5], the action to be selected is determined by comparing $[c_d \mu_i + \sum_{j=i+1}^{n+1} P_{ij} v_j - v_i] / H$ and $[c_p M + v_0 - v_i] / M$ at each step of PIR. That is, the action which minimizes the quantity just like cost rate is selected in PIR. On the other hand, in our PIR the action which minimizes the quantily just like relative value is selected.

In the following, we shall show that our PIC leads us to the optimal policy.

Proof of PIC

We shall show that our PIC gives us the optimal policy. Suppose that we have evaluated a policy A at some PIC and the PIR has produced a policy B that is different from A. Use superscript A and B to indicate the quantities relevant to policy A and B. We seek to prove $g^B \le g^A$.

Case (i) $\quad D_0{}^B = (W)$.

For policy B, from (3.3), we have,

$$Hg^B = c_d \mu_0 + \sum_{j=1}^{n+1} P_{0j} v_j{}^B(g^B) \; . \tag{3.8}$$

It follows from the definition of PIR, since B was chosen over A,

$$[c_d \mu_0 + \sum_{j=1}^{n+1} P_{0j} v_j{}^B(g^A) - Hg^A] / [1 - P_0] \tag{3.9}$$

$$\leq \begin{cases} [c_d\mu_0 + \sum_{j=1}^{n+1} P_{0j}v_j^A(g^A) - Hg^A] / [1 - P_0] = 0 \ , & \text{if } D_0^A = (W), \\ c_pM - g^AM = 0 \ , & \text{if } D_0^A = (PM). \end{cases}$$

From (3.8) and (3.9), we have

$$H(g^B - g^A) \leq \sum_{j=1}^{n+1} P_{0j}[v_j^B(g^B) - v_j^B(g^A)] \ . \tag{3.10}$$

In the following, by induction, we shall show that

(3.11) $v_i^B(g^B) - v_i^B(g^A) = (g^A - g^B)M_i^B$, where M_i^B is a positive

number independent of g^A, g^B.

From (3.6), we have

$$v_{n+1}^B(g^B) - v_{n+1}^B(g^A) = (g^A - g^B)R \ . \tag{3.12}$$

Since R is positive, (3.11) holds for $i=n+1$. We assume that (3.11) holds for $i=j+1,\ldots,n+1$. Then from (3.7), we have

$$v_j^B(g^B) - v_j^B(g^A) = \begin{cases} (g^A - g^B)M_j^B \ , \\ \quad \text{where} \quad M_j^B = \sum_{k=j+1}^{n+1} P_{jk}M_k^B/[1 - P_j] \ , \\ \quad \text{if} \quad D_j^B = (W) \ , \\ (g^A - g^B)M \ , \quad \text{if} \quad D_j^B = (PM) \ , \end{cases} \tag{3.13}$$

using the assumption of induction. Therefore (3.11) holds for $i=j$, which implies that (3.11) holds for $i=0,1,\ldots,n+1$. Applying this to (3.10), we have

$$(g^A - g^B)[H + \sum_{j=1}^{n+1} P_{0j}M_j^B] \geq 0 \ , \tag{3.14}$$

which implies $g^B \leq g^A$.

Case (ii), $D_0^B = (PM)$.

For policy B, we have

$$g^B M = c_p M . \tag{3.15}$$

It follows from the definition of PIR, since B was chosen over A,

$$c_p M - g^A M \le \begin{cases} [c_d\mu_0 + \sum_{j=1}^{n+1} P_{0j}v_j^A(g^A) - Hg^A]/[1 - P_0] = 0 , & \text{if } D_0^A = (W), \\ c_p M - g^A M = 0 , & \text{if } D_0^A = (PM), \end{cases} \tag{3.16}$$

From (3.15) and (3.16), we have

$$g^B M = c_p M \le g^A M , \tag{3.17}$$

which implies $g^B \le g^A$.

For both cases (i) and (ii), $g^B \le g^A$ holds. Since g is positive for all policicies and since our PIC reduces the cost rate g, it converges to a certain limiting value. Moreover, in a similar discussion to the above one, it is easily shown that it is impossible for a better policy to exist and not to be found at any time by PIR. Q.E.D.

4. Proof of Control Limit Rule

Since the optimal policy exists in the policies constructed from the actions which give $v_i^*(g)$ (for brevity, we let $D_i^*(g)$ to be such action), it is sufficient to discuss the property of such policies for investigating the property of the optimal policy.

We have the following lemmas. The proof of lemma 1 is easily done, and is omitted.

Lemma 1.

$$P_{i,n+1}(t) < P_{j,n+1}(t) \quad \text{for} \quad i < j . \tag{4.1}$$

Lemma 2.

$$(4.2) \qquad \mu_i < \mu_j \quad \text{and} \quad f_i < f_j \quad \text{for} \quad i < j\ .$$

Proof. From the definition of μ_i, f_i and lemma 1, (4.2) is easily shown.

Theorem 1. For the optimal policy, control limit rule holds.

Proof. We define,

$$(4.3) \qquad A_i(g) \equiv [c_d\mu_i + \sum_{j=i+1}^{n+1} P_{ij}v_j{}^*(g) - Hg] \,/\, [1 - P_i]\ ,$$

$$(4.4) \qquad X(g) = (c_r - g)R\ , \qquad Y(g) = (c_p - g)M\ .$$

Note that if $A_i(g) > Y(g)$, then $D_i{}^*(g) = (PM)$. If we start our PIC with the policy that for all $i=0,1,\ldots,n$, we perform PM, then the cost rate g is not larger than c_p at any VDR. Therefore, we have

$$(4.5) \qquad X(g) \geq 0\ , \qquad Y(g) \geq 0\ .$$

Noting that

$$(4.6) \qquad v_i{}^*(g) \leq Y(g) \leq X(g)\ ,$$

and from lemma 2, we have

$$(4.7) \qquad (1 - P_i)[Y(g) - A_i(g)] - (1 - P_n)[Y(g) - A_n(g)]$$

$$\geq c_d(\mu_n - \mu_i) > 0\ .$$

The above relation implies that if $D_n{}^*(g) = (W)$, then $D_i{}^*(g) = (W)$ for $i=0,1,\ldots,n-1$. In this case control limit rule holds in the special form, that is, control limit is state $n+1$. In the following, we consider the case where $D_n{}^*(g) = (PM)$. We assume that $D_i{}^*(g) = (PM)$. Then we seek to show $D_{i+1}^*(g) = \cdots = D_n{}^*(g) = (PM)$, which will be proved by induction. We assume that $D_{j+1}^*(g) = \cdots = D_n{}^*(g) = (PM)$, where $j > i$ (note that $D_n{}^*(g) = (PM)$ holds), then we have

$$(4.8)\quad (1-P_j)[A_j(g)-Y(g)]-(1-P_i)[A_i(g)-Y(g)]$$

$$= c_d(\mu_j-\mu_i)+(f_j-f_i)X(g)+\sum_{k=j+1}^{n} P_{j,k}v_k^*(g)-\sum_{k=i+1}^{n} P_{i,k}v_k^*(g)+(P_j-P_i)Y(g)$$

$$\geq c_d(\mu_j-\mu_i)+Y(g)[f_j-f_i+P_j-P_i+\sum_{k=i+1}^{n} P_{j,k}-\sum_{k=i+1}^{n} P_{i,k}]$$

$$= c_d(\mu_j-\mu_i) > 0 ,$$

which implies that if $D_i^*(g) = (PM)$, then $D_j^*(g) = (PM)$. Therefore, if $D_i^*(g) = (PM)$, then for all $j=i+1,\dots,n$, $D_j^*(g) = (PM)$, which implies theorem 1 holds.

5. Operating Characteristic

In this section, we shall give the cost rate when we operate the unit under a policy of control limit rule. When control limit is $m+1$ $(m=0,1,\dots,n)$, we have the following set of equations (see (3.3) - (3.5)), where cost rate is g^m and $v_{n+1} = 0$.

$$(5.1)\quad Hg^m + v_i = c_d\mu_i + \sum_{j=i}^{m} P_{ij}v_j + \sum_{k=m+1}^{n} P_{ik}v_k , \quad i=0,1,\dots,m ,$$

$$(5.2)\quad Mg^m + v_k = c_pM + v_0 , \quad k=m+1,\dots,n ,$$

$$(5.3)\quad Rg^m = c_rR + v_0 .$$

We define the following quantities to unclutter the equations.

$$(5.4)\quad a \equiv c_rR - c_pM \geq 0 , \qquad b \equiv R - M \geq 0 ,$$

$$(5.5)\quad P_i^m \equiv \sum_{k=m+1}^{n} P_{ik} ,$$

$$(5.6)\quad A_i^m \equiv [c_d\mu_i - aP_i^m]/[1-P_i] , \quad B_i^m \equiv [H - bP_i^m]/[1-P_i] ,$$

$$(5.7)\quad q_{ij} \equiv P_{ij} / [1-P_i] .$$

Then, by solving (5.1) - (5.3), with respect to g^m, we have

$$(5.8) \qquad g^m = \frac{c_r R + \sum_{j=0}^{m} Q_{0j} A_j^m}{R + \sum_{j=0}^{m} Q_{0j} B_j^m} ,$$

where Q_{ij} is given by

$$(5.9) \qquad Q_{ii} = 1 , \qquad Q_{ij} = \sum_{k=i+1}^{j} q_{i,k} Q_{k,j} .$$

Note that

$$(5.10) \qquad Q_{ij} = \sum_{k=i}^{j-1} Q_{i,k} q_{k,j} .$$

For example,

$$(5.11) \qquad Q_{03} = q_{03} + q_{01}q_{13} + q_{02}q_{23} + q_{01}q_{12}q_{23}$$

$$= q_{03}+q_{01}(q_{13}+q_{12}q_{23})+q_{02}q_{23} = q_{03}Q_{33}+q_{01}Q_{13}+q_{02}Q_{23}$$

$$= q_{03}+q_{01}q_{13}+(_{02}+q_{01}q_{12})q_{23} = Q_{00}q_{03}+Q_{01}q_{13}+Q_{02}q_{23} .$$

Here, we have had the cost rate g.

We can also obtain the optimal policy by using (5.8) instead of using PIC. It is done by comparing $(g^0, g^1, \ldots, g^n)$. In this procedure, the theorem 2 is very useful.

The following lemma is evident, but since it plays an important role in the proof of theorem 2, we state it here.

Lemma 3. Let A_0, A_1, A_2, B_0, B_1 and B_2 be positive numbers.

$$(5.12) \qquad \frac{A_0 + A_1}{B_0 + B_1} > \frac{A_0}{B_0} \quad \text{if and only if} \quad \frac{A_1}{B_1} > \frac{A_0}{B_0} .$$

$$(5.13) \qquad \frac{A_0 + A_1 + A_2}{B_0 + B_1 + B_2} > \frac{A_0 + A_1}{B_0 + B_1} \quad \text{if} \quad \frac{A_2}{B_2} > \frac{A_1}{B_1} > \frac{A_0}{B_0} .$$

Theorem 2. If $g^{m+1} \geq g^m$, then $g^{m+2} > g^{m+1}$, $m=0,1,\ldots,n-2$.

Proof. Using the relations,

$$(5.14)\qquad \sum_{j=0}^{m+1} Q_{0,j}A_j^{m+1} = \sum_{j=0}^{m} Q_{0,j}A_j^{m} + (a + A_{m+1}^{m+1})Q_{0,m+1},$$

$$(5.15)\qquad \sum_{j=0}^{m+1} Q_{0,j}B_j^{m+1} = \sum_{j=0}^{m} Q_{0,j}B_j^{m+1} + (b + B_{m+1}^{m+1})Q_{0,m+1},$$

we have,

$$(5.16)\qquad g^{m+1} = \frac{c_rR + \sum_{j=0}^{m} Q_{0,j}A_j^{m} + (a + A_{m+1}^{m+1})Q_{0,m+1}}{R + \sum_{j=0}^{m} Q_{0,j}B_j^{m} + (b + B_{m+1}^{m+1})Q_{0,m+1}},$$

$$(5.17)\qquad g^{m+2} = \frac{c_rR + \sum_{j=0}^{m} Q_{0,j}A_j^{m} + (a + A_{m+1}^{m+1})Q_{0,m+1} + (a + A_{m+2}^{m+2})Q_{0,m+2}}{R + \sum_{j=0}^{m} Q_{0,j}B_j^{m} + (b + B_{m+1}^{m+1})Q_{0,m+1} + (b + B_{m+2}^{m+2})Q_{0,m+2}},$$

In order to verify theorem 2, we have only to show that

$$(5.18)\qquad \frac{a + A_{m+2}^{m+2}}{b + B_{m+2}^{m+2}} > \frac{a + A_{m+1}^{m+1}}{b + B_{m+1}^{m+1}},$$

considering lemma 3 and equations (5.8), (5.16) and (5.17).

$$(5.19)\qquad (1 - P_{m+1})(1 - P_{m+2})[(a + A_{m+2}^{m+2})(b + B_{m+1}^{m+1}) - (a + A_{m+1}^{m+1})(b + B_{m+2}^{m+2})]$$

$$= c_dH(\mu_{m+2} - \mu_{m+1}) + aH(f_{m+2} - f_{m+1}) + c_db(f_{m+1}\mu_{m+2} - f_{m+2}\mu_{m+1})$$

By the assumption (viii) in section 1 and lemma 2, we can easily show that the right-hand side of equation (5.19) is positive. Here, theorem 2 has been proved.

Note that if control limit is state 0, then the cost rate is c_p. It is possible for such case to be optimal when c_p is very small. In this case our unit is of no use economically. Further note should be done on that there exists the case where the optimal control limit is state $n+1$. In such case, PM has no meaning. Such policy is to be optimal when c_d and/or c_r is relatively small. The following corollary is derived directly from theorem 2.

Corollary PM is effective if and only if

$$\text{(5.20)} \qquad \frac{c_d\mu_n + af_n}{H + bf_n} > \frac{c_r R + \sum_{j=0}^{n-1} Q_{0j} A_j^{n-1}}{R + \sum_{j=0}^{n-1} Q_{0j} B_j^{n-1}} .$$

Theorem 2 and corollary will be very useful to search the optimal control limit and for sensitivity analysis with respect to c_d, c_p, c_r .

6. Example

As a simple example, we consider the unit which has three degraded states, i.e., $n=3$ and the time interval to the next checking obeys a negative exponential distribution.

Let, $\lambda_0=1.00$, $\alpha_0=0.20$, $\lambda_1=1.50$, $\alpha_1=0.50$, $\lambda_2=2.00$, $\alpha_2=1.00$, $\lambda_3=\alpha_3=2.50$ and $H=1.50$, $R=M=0.50$, $c_r=10.00$, $c_d=3.00$. For c_p, we consider the five cases, that is, $c_p=1.00$, 5.00, 7.00, 9.00 and $10.00(=c_r)$. By using the operating characteristic, we have the following table.

c_p	1.00	5.00	7.00	9.00	10.00
g^0	1.85	2.15	2.25	2.37	2.45
g^1	2.06	2.17	2.23	2.28	2.31
g^2	2.08	2.24	2.26	2.27	2.29
g^3	2.29	2.29	2.29	2.29	2.29
optimal control limit	*0*	*1*	*2*	*3*	*3* or *4*

From the above examples we have the following properties, though they are thought to be natural. (i) The larger c_p becomes, the greater cost rate we have and the higher the number of control limit state becomes. That is, when c_p is large, we should not perform PM while the degradation does not progress so much. (ii) When c_p is very small we should perform PM for even a new unit. That is, in such case, our unit has no meaning in operating it. (iii) When c_p is very large, we should not necessarily perform PM. That is, it is to be optimal to perform only CM. (iv) Theorem 2 holds.

7. Conclusion

We have considered a marginal checking model where a Markovian degradation unit is treated and checking interval is not regular but probabilistic. The problem of an optimal preventive maintenance has been discussed and we have showed that it can be solved by using Markov-renewal programming with a little different policy improvement routine from the ordinary one. This PIR is applied to the process where all states except only one state (in our model, it is E_0) are irreversible, that is, only the transition from state i to state j ($>i$) possible.

By using this PIC, we have showed that in the optimal policy, control limit rule holds, though the ordinary PIC may do. We should note that there exists the case where the policy that control limit is state 0 or state $n+1$ is optimal. We have showed the condition that the policy that control limit is state $n+1$ is optimal, but we have not so much refered to the case where the optimal control limit is state 0.

If we put $H(t)=0$ for $t < T$, 1 for $t \geq T$, then our model becomes the one where checking interval is constant. And if we let $T \to 0$, then it becomes the one where the unit is observed continuously in time.

References

[1] Barlow, R. and Proschan, F.,"Mathematical theory of reliability," J. Wiley, 1965.

[2] Derman, C., "Finite State Markovian Decision Process," Academic Press, 1970.

[3] Flehinger, B.J., "A Markovian model for the analysis of the effects of marginal testing on system reliability," Ann. Math. Statist., Vol.33 (1962) 754-766.

[4] Howard, R.A., "Dynamic programming and Markov process," M. I. T. Press., 1960.

[5] Jewell, W.S., "Markov-Renewal programming, I and II," Operations Res., Vol.11 (1963) 938-971.

[6] Mine, H. and Kawai, H. "An optimal maintenance policy for a 2-unit parallel system with degraded states," IEEE Trans. Rel., Vol.23 (1974) 81-86.

[7] Pyke, R., "Markov renewal processes; Definitions and preliminary Proper-

ties," Ann. Math. Statist., Vol.32 (1961) 1231-1242.

[8] _____, "Markov renewal processes with finitely many states," Ann. Math. Statist., Vol.32 (1961) 1243-1259.

(Authors' address: Department of Applied Mathematics and Physics, Faculty of Engineering, Kyoto University, Kyoto Japan.)