EFFECTS OF MULTIPLICITY IN ARRIVAL PROCESSES

ON QUEUE LENGTHS AND WAITING TIMES

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Abstract

This paper treats of some queueing systems with multiple inputs and expo-
nential service times with a common rate μ. The queue discipline for all
systems in this paper is first-come, first-served. The systems of equations
for steady-state probabilities of queueing systems are of the similar forms
and hence the methods for solving these systems are also similar to each
other. Under the steady-state condition the relations between the state prob-
ability distribution at arrival epoch of a type-1 customer and that of a
type-2 customer are investigated.

1. Introduction

The queueing systems with multiple inputs have potential applicability in
modeling the real systems in which there are significant differences in
interarrival times. Such systems are encountered in the theories of telephone
traffic and queueing networks. The queueing process with such differences
may be expressed in terms of a semi-Markov process with a number of states.
Let there be k customer types and let J_n denote the type of the n-th arrival.
In addition, let τ_n be the n-th arrival epoch. Then the sequence $\{t_n , J_n\}$,
where $t_n = \tau_n - \tau_{n-1}$ with $\tau_0 = 0$, is called the arrival process. The arrival pro-
cess is often assumed to form a semi-Markov process over a finite state space
$\{1, 2, \ldots, k\}$.
Then we usually assume that

$$P\left\{ t_n \leq x , J_n = j \mid (t_m , J_m ; m \leq n-2) ; t_{n-1} , J_{n-1} = i\right\}$$
$$= P\left\{ t_n \leq x , J_n = j \mid J_{n-1} = i \right\} = A_{ij}(x)$$

for n=1,2,...; i, j=1,2,...,k. In particular, when the arrival process is composed of k independent Poissonian streams with arrival rates λ_i, i=1,2,... ,k, we know that

$$A_{ij}(x)=\lambda_j(1-e^{-\lambda x}) \qquad (i,j=1,2,\ldots,k)$$

where $\lambda=\lambda_1+\lambda_2+\ldots+\lambda_k$.

Some investigators have studied queueing systems with multiple inputs, but from viewpoints different from a point of view of the author [see, for example, Sahin(1971) and Neuts et al.(1972)]. They have studied each such a system with thier respective approaches.

We attempt to study, in a unified way, the relationships between two state probabilities at arrival epochs of two types of customers for a family of queueing systems with multiple inputs. Now we assume that the arrival process is composed of two independent streams, one of which (type-1 arrival) is general independent and the other (type-2 arrival) is Poissonian with arrival rate λ_2.

In this case, we will consider the sequence of arrival epochs $\{\tau_n, n \geq 0\}$ as that of the type-1 customers. Then we rewrite $A_{11}(x)$ as $A(x)$, which denotes the interarrival time distribution of type-1 customers. Let $1/\lambda_1$ be the mean interarrival time of type-1 customers. In addition, let the S_n denote the service time of the n-th customer. We assume that the S_n are independent, identically distributed positive random variables with a common distribution defined as

$$B(x)= \begin{cases} 1-e^{-\mu x} & \text{if} \quad x \geq 0 , \\ \\ 0 & \text{if} \quad x<0 , \end{cases}$$

and that the sequence $\{S_n\}$ is independent of the arrival process.
That is, throughout this paper the distribution of service times is assumed to be independent of the type of customer to be served.

Moreover we assume that the queue discipline is first-come, first-served.

2. Notations and Definitions

In addition to notations introduced in the preceding section, we make a list of notations which should be employed in the following sections.

$\xi(t)$; the queue length (total number of type-1 and type-2 customers in the system) at time t.

$$P_{i+1,j}(t)=P\left(\xi(\tau_{n-1}+t)=j \mid \xi(\tau_{n-1}+0)=i+1\right) \quad (\tau_{n-1}\leqq \tau_{n-1}+t < \tau_n) .$$

$\xi_n=\xi(\tau_n-0)$: the queue length just before the arrival of the n-th type-1 customer.

$$q_{ij}=P(\xi_n=j \mid \xi_{n-1}=i)= \int_0^\infty P_{i+1,j}(t)dA(t) .$$

$\pi_j=\lim_{n\to\infty} P(\xi_n=j)$: the steady-state probability of there being j customers in the system at arrival epoch of a type-1 customer ; $\pi_j=\sum_{i=0}^\infty \pi_i q_{ij}$ (j=0, 1,2,...) .

$P_j=\lim_{n\to\infty} P(\xi(t)=j)$ ($\tau_{n-1}\leqq t<\tau_n$) ; the steady-state probability of there being j customers in the system at arrival epoch of a type-2 customer.

$\Pi(z)$: the generating function of π_j ; $\Pi(z)= \sum_{j=0}^\infty \pi_j z^j$.

$P_i(z,t)$: the generating function of $P_{ij}(t)$, (i=1,2,...) ;

$$P_i(z,t)= \sum_{j=0}^\infty P_{ij}(t)z^j .$$

$P(z)$: the generating function of P_j ; $P(z)= \sum_{j=0}^\infty p_j z^j$.

L_k : the mean number of customers present in the system at arrival epoch of a type-k customer (k=1,2) ;

$$L_1= \sum_{j=0}^\infty j\pi_j \text{ and } L_2= \sum_{j=0}^\infty j p_j .$$

$A*(s)$: the Laplace-Stieltjes transform of A(t) ;
$$A*(s)= \int_0^\infty e^{-st}dA(t) \text{ for } Re(s)>0.$$

$F_k(t)$: the waiting-time distribution function of a type-k customer (k=1, 2).

$F_k^*(s)$: the Laplace-Stieltjes transform of $F_k(t)$ (k=1,2).

W_k : the mean waiting time of a type-k customer (k=1,2).

$c_0=1$ and $c_j=\prod_{m=1}^{j}A^*(m\mu)\big/[1-A^*(m\mu)]$ (j$\geqq$1).

$a_m=\sum_{j=m}^{\infty}\binom{j}{m}(-1)^{j-m}c_j$ (m$\geqq$0).

$\rho_k=\lambda_k/\mu$ (k=1,2).

3. Results

In view of the assumptions made in section 1, following results can be obtained for the systems : multiple inputs/M/c, multiple inputs/M/∞ and multiple inputs / M / c (c). The sketch of proof of these results is given in the next section.

(i) Queue Lengths

$$(3.1)\quad P_j=\begin{cases}\dfrac{\rho_2^{j}}{j!}p_0+\rho_1\dfrac{\rho_2^{j}}{j!}\sum_{i=0}^{j-1}i!\,\pi_1\big/\rho_2^{i+1} & (1\leqq j\leqq c),\\[2ex]\left(\dfrac{\rho_2}{c}\right)^{j-c}p_c+\dfrac{\rho_1}{c}\left(\dfrac{\rho_2}{c}\right)^{j}\sum_{i=0}^{j-c-1}\pi_{c+1}\left(\dfrac{c}{\rho_2}\right)^{c+i+1} & (j>c),\end{cases}\quad (\text{---}/M/c)*$$

where p_0 can be found by the boundary condition $\sum_{j=0}^{\infty}p_j=1$.
In particular, for c=1 we have

$$(3.2)\quad p_j=\begin{cases}1-\rho_1-\rho_2 & (j=0),\\[2ex]\rho_2^{j}(1-\rho_1-\rho_2)+\rho_1\,\rho_2^{j}\sum_{i=0}^{j-1}\pi_1\big/\rho_2^{i+1} & (j\geqq1).\end{cases}\quad (\text{---}/M/1)$$

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* ---/M/c denotes c-server queueing system with multiple inputs and expo-
 nentially distributed service times.

$$(3.3) \qquad P_j = \begin{cases} e^{-\rho_2} - \rho_1 e^{-\rho_2} \sum\limits_{K=1}^{\infty} \dfrac{\rho_2^K}{K!} \sum\limits_{i=0}^{K-1} i! \, \pi_i / \rho_2^{i+1} & (j=0), \\[12pt] \dfrac{\rho_2^j}{j!} P_0 + \rho_1 \dfrac{\rho_2^j}{j!} \sum\limits_{i=0}^{j-1} i! \, \pi_i / \rho_2^{i+1} & (j \geqq 1). \end{cases} \qquad (-\!/M/\infty)$$

The above P_0 can also be written as

$$(3.4) \qquad P_0 = e^{-\rho_2} - \rho_1 \sum_{i=0}^{\infty} \pi_i \, \Gamma_{\rho_2}(i+1) / \rho_2^{i+1}$$

where $\Gamma_{\rho_2}(i+1)$ is the Incomplete Γ-function, namely,

$$(3.5) \qquad \Gamma_t(i+1) = \int_0^t x^i e^{-x} dx.$$

$$(3.6) \qquad P_j = \begin{cases} [1 - \rho_1 \sum\limits_{m=1}^{c} \dfrac{\rho_2^m}{m!} \sum\limits_{i=0}^{m-1} i! \, \pi_i / \rho_2^{i+1}] / [\sum\limits_{K=0}^{c} \dfrac{\rho_2^K}{K!}] & (j=0), \\[12pt] \dfrac{\rho_2^j}{j!} P_0 + \rho_1 \dfrac{\rho_2^j}{j!} \sum\limits_{i=0}^{j-1} i! \, \pi_i & (j \geqq 1). \end{cases} \qquad (-\!/M/c(c))$$

If we let $\rho_1=0$ in the above expressions, it is found that these results are identical to those of queueing systems with a single Poisson input source. Thus, the second term on the right-hand side of each of equations (3.1),(3.2) (3.3),(3.4) and (3.6) for $j \geqq 1$ is considered to be increment due to the presence of the type-1 customers. Note that P_0 is smaller than that of the system with a single input source in each case. Before proceeding with the determination of $\{\pi_j\}$, we next give the relationship between $P(z)$ and $\Pi(z)$ and from which we derive the relationship between L_1 and L_2 for each queueing system :

$$(3.7) \qquad P(z) = [\rho_1 z \Pi(z) + \sum_{j=0}^{c-1} (c-j) p_j z^j] / (c - \rho_2 z) \qquad (-\!/M/c),$$

$$(3.8) \qquad P(z) = [\rho_1 z \Pi(z) + p_0] / (1 - \rho_2 z) \qquad (-\!/M/1),$$

$$(3.9) \qquad dP(z)/dz - \rho_2 P(z) = \rho_1 \Pi(z) \qquad (-\!/M/\infty),$$

$$(3.10) \qquad dP(z)/dz - \rho_2 P(z) = \rho_1 \Pi(z) - \rho_1 \pi_c z^c - \rho_2 \, p_c z^c \quad [-\!/M/c(c)] ;$$

$$(3.11) \qquad L_2 = [\rho_1(1+L_1) + \rho_2 + \sum_{j=0}^{c-1} j(c-j) P_j] / (c - \rho_2) \qquad (\text{from } (3.7)),$$

(3.12) $\qquad L_2 = \rho_1(1+L_1) \Big/ (1-\rho_2) + \rho_2 \Big/ (1-\rho_2)$ $\qquad$ (from (3.8)),

(3.13) $\qquad L_1 = A^*(\mu) \Big/ [1-A^*(\mu)] + \rho_2$ and $L_2 = \rho_1 + \rho_2$ $\qquad$ (from (3.9)).

Thus, an interesting observation can be made about (3.12) and (3.13). That is, the first term on the right-hand side of each expression illustrates effect of the presence of the type-1 customers. We now note that an equality $L_1 = L_2$ holds if and only if $A'(t) = \lambda_1 e^{-\lambda_1 t}$ ($t>0$). Furthermore, we note that when $c \geqq 2$, $\sum_{j=0}^{c-1}(c-j)p_j z^j$ on the right-hand side of (3.7) may be treated in a manner similar to that by Saaty [5], which is accompanied by some algebraic complexity. By employing the assumption that $A'(t) = (k\lambda_1)^k t^{k-1} e^{-k\lambda_1 t} / (k-1)!$, it is found, from the first equation of (3.13), that

(3.14) $\qquad L_1 = \left\{ [1+1 \big/ (k\rho_1)]^k - 1 \right\}^{-1} + \rho_2 < \rho_1 + \rho_2 = L_2$ (k>1)

for the queueing system with infinitely many servers. In particular, if k approaches infinity in (3.14), then we have

(3.15) $\qquad L_1 = \left\{ (e^{1/\rho_1} - 1) \right\}^{-1} + \rho_2 < \rho_1 + \rho_2 = L_2$

which corresponds to the case where $A(t)=1$ ($t \geqq 1/\lambda_1$) ; $=0$ ($t < 1/\lambda_1$).

(ii) Waiting Times

We now give the stationary waiting-time distribution functions for multiple inputs/M/1 queue. Our emphasis is on observing the relationship between the waiting times of type-1 and type-2 customers.

(3.16) $\quad F_1(0) = \pi_0$,

(3.17) $\quad dF_1(t) = \sum_{j=1}^{\infty} \mu^j t^j e^{-\mu t} \pi_j dt / (j-1)!$ $\qquad$ ($t>0$),

(3.18) $\quad F^*_1(s) = \Pi[\mu/(s+\mu)]$,

(3.19) $\quad F^*_2(s) = P[\mu/(s+\mu)]$,

(3.20) $F*_2(s)=[(s+\mu)(1-\rho_1-\rho_2)+\lambda_1 F*_1(s)]/(s+\mu-\lambda_2)$ (from(3.8)),

(3.21) $W_1=L_1/\mu$, $W_2=L_2/\mu$,

(3.22) $F_2(t)=1-\rho_2 e^{-\mu(1-\rho_2)t}-\lambda_1\left\{1/\mu+\int_0^t[\rho_2-F_1(t-x)]e^{-\mu(1-\rho_2)x}\right\}dx,$

(3.23) $W_2=\rho_1(1+\mu W_1)/[\mu(1-\rho_2)]+\rho_2/[\mu(1-\rho_2)].$

We note that (3.21),(3.22) and (3.23) can easily be derived from (3.18), (3.19) and (3.20). Furthermore the following is noteworthy.

The relation (3.20) can be extended to the case where the interarrival-time distribution function for type-1 customers is $A(x)=1-\exp[-\int_0^x\lambda(t)dt]$ for $x>0$ and type-K customers have a general service-time distribution function, $B_k(x)$, with mean $1/\mu_k$. Then, using the notation of a virtual waiting-time process we get

(3.24) $F*_2(s)=\left\{s(1-\rho_1-\rho_2)+[1-B*_1(s)]\lambda_1 F*_1(s)\right\}/\bar{s}$

where $\bar{s}=s-\lambda_2+\lambda_2 B_2^*(s)$ (see[8]).

In addition, we note that in the queueing system with infinitely many servers each customer starts being served as soon as he arrives, that is, there is no waiting time because there are a sufficient number of servers. Also there is no waiting time in the loss system.

4. Sketch of the Proof

For the queueing systems with a single Poisson input of intensity λ_2 and exponential service-time distribution with mean $1/\mu$, the balance equations for three systems in statistical equilibrium are of the form

$$z_j=0 \text{ for } j\geq1,$$

where $z_j=\lambda_2 p_{j-1}-\mu_j p_j$ and

$$\mu_j=\begin{cases} j\mu & (1\leq j\leq c)\\ \\ c\mu & (j\geq c+1) \end{cases} \quad (M/M/c),$$

$$\mu_j=j\mu \qquad (j\geq1) \qquad (M/M/\infty)$$

and

$$\mu_j=j\mu \qquad (1\leq j\leq c) \qquad (M/M/c(c)).$$

In any queueing system for which the realizations of the state process are step functions with only unit jumps (positive and negative) the equilibrium state distribution just before arrival epochs is the same as that just following departure epochs. The essential characteristic is simply that the number of customers present can increase and decrease by at most one customer at a time. Using the above facts, we can write the "rate in equals rate out" state equations for our queueing systems with multiple inputs :

(4.1) $\lambda_1 \pi_{j-1} + \lambda_2 p_{j-1} = \mu_j p_j$ ($j \geq 1$),

where μ_j is defined above. This yields equations (3.7), (3.8), (3.9) and (3.10).

We next derive the expressions for $\pi(z)$ in the above-mentioned queueing systems. For multiple inputs/M/1 queue we have a functional equation

$$(4.2) \quad \pi(z) = \int_0^\infty \sum_\ell \pi_\ell P_{\ell+1}(z,t) dA(t)$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{z^2 \pi(z) - (1-z)f(s)}{z[s-g(z)]} A^*(-s) ds \qquad (i=\sqrt{-1})$$

where $f(s) = z_i^2 \pi(z_i)/(1-z_i)$, $g(z) = (1-z)(\mu - \lambda_2 z)/z$ and
$z_1 \equiv z_1(s) = [\lambda_2 + \mu + s - \sqrt{(\lambda_2 + \mu + s)^2 - 4\lambda_2 \mu}]/(2\lambda_2)$ ($|z_i| < 1$)

(see[5,pp.8-9]). Here we used a relation

$$(4.3) \quad \int_0^\infty e^{-\alpha t} P_K(z,t) dA(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} P^*_K(z,s) A^*(\alpha-s) ds \qquad (\alpha > 0).$$

To determine fully $\pi(z)$ for some special cases of $A(t)$, we can make use of the Cauchy integral formula.

For multiple inputs/M/∞, we again have a functional equation

$$(4.4) \quad \pi(z) = e^{-\rho_2(1-z)} \int_0^\infty (q+pz) \pi(q+pz) e^{-p\rho_2(z-1)} dA(t),$$

where $p = e^{-\mu t}$, $q = 1-p$ and $\rho_2 = \lambda_2/\mu$ (see[5,pp.22-23]). Noting that the right-hand side of (4.4) is equal to the product of the generating function of the stationary system-size distribution for type-2 customers, namely, $e^{-\rho_2(1-z)}$ and

$$(4.5) \quad \pi_1(z) \equiv \int_0^\infty (q+pz) \pi(q+pz) e^{-p\rho_2(z-1)} dA(t),$$

we can find that

$$(4.6) \quad \Pi(z) = e^{-\ell_2(1-z)} \sum_{j=0}^{\infty} c_j (z-1)^j = e^{-\ell_2(1-z)} \sum_{j=0}^{\infty} a_j z^j$$

where

$$(4.7) \quad c_0 = \Pi(0) = 1, \quad c_j = \prod_{m=1}^{j} \frac{A^*(m\mu)}{1 - A^*(m\mu)} \quad (j \geq 1).$$

Here the coefficients $\{a_j\}$ and $\{c_j\}$ in the series expansions of $\Pi(z)$ satisfy the following relation

$$(4.8) \quad a_m = \sum_{j=m}^{\infty} \binom{j}{m} (-1)^{j-m} c_j .$$

Thus (4.6) yields

$$(4.9) \quad \pi_j = \sum_{m=0}^{j} a_m \frac{e^{-\ell_2} \ell_2^{j-m}}{(j-m)!} \quad ,$$

which is the convolution of two probability distributions. In this case, we have

$$(4.10) \quad P(z) = e^{-\ell_2(1-z)} \left\{ 1 - \ell_1 \sum_{j=0}^{\infty} (1-z^{j+1}) a_j / (j+1) \right\} .$$

If the variances of ξ_n and $\xi(t)$ are denoted by σ_1^2 and σ_2^2 respectively, we can see that

$$(4.11) \quad \sigma_1^2 = c_1 + \ell_2 - c_1^2 + 2c_2 , \quad \sigma_2^2 = \ell_1 + \ell_2 + \ell_1(c_1 - \ell_1) .$$

For $A'(t) = \lambda_1 e^{-\lambda_1 t}$ these reduce to

$$\sigma_1^2 = \sigma_2^2 = \ell_1 + \ell_2 ,$$

as we should have expected.

We note that if type-1 customers arrive as a renewal process in groups of random size B, where B has the probability distribution $P(B=m) = b_m$ $(m \geq 1)$ and its generating function $\beta(z) = E(z^B)$ $(|z| \leq 1)$, then functional equations (4.2) and (4.4) may be written as

(4.12) $\Pi(z) = \dfrac{1}{2\pi i} \displaystyle\int_{c-i\infty}^{c+i\infty} \dfrac{z\beta(z)\;\Pi(z)-(1-z)\widetilde{f}(s)}{z[s-g(z)]}\; A*(-s)\,ds$

and

(4.13) $\Pi(z) = e^{-\rho_2(1-z)} \displaystyle\int_{t-i\infty}^{c+i\infty} \beta(q+pz)\;\Pi(q+pz)\,e^{-p\rho_2(z-1)}\,dA(t)$

respectively. If $b_m = \delta_{1m}$ (Kronecker delta) then these completely agree with the results obtained earlier. Here we set $p=e^{-\mu t}$, $q=1-p$ and $\widetilde{f}(s)=z_1\beta(z_1)/(1-z_1)$.

For exapmle, if $A'(t)=\lambda_1 e^{-\lambda_1 t}$ $(t>0)$ then by (4.12) and Cauchy integral formula we have

(4.14) $\Pi(z) = \dfrac{1-\rho_1^{*}-\rho_2}{1-z\left\{\rho_2+\rho_1[1-\beta(z)]/(1-z)\right\}}$

where $\rho_1^{*}=\lambda_1 E(B)/\mu$, and $E(B)$ denotes the mean batch size. Similarly (4.14) would be solve without much diffeculty in the same manner as before. We now turn our attention to the simplest loss system with c servers and a single Poisson input stream. The transition probabilities for this system have complex forms such that

(4.15) $P_{ij}(t)=p_j^{*}+\displaystyle\sum_{k=1}^{c}\gamma_k(i,j)e^{z_k\mu t}$

where

$P_j^{*}=\rho_2^{j}/j! \;\Big/\; [\displaystyle\sum_{k=0}^{c}\rho_2^{k}/k!]$ (stationary state probabilities),

$\gamma_k(i,j)=\dfrac{c!}{j!}\rho_2^{c-i}\dfrac{D_i(z_k)D_j(z_k)}{z_k D_c(z_k)D_c'(z_k+1)}$,

$D_0(z)=1,\quad D_j(z)=\displaystyle\sum_{m=0}^{j}\binom{j}{m}\rho_2^{j-m}z(z+1)\ldots\ldots(z+m-1)\quad (j\geq 1),$

$z_k D_c'(z_k+1)=\displaystyle\sum_{m=0}^{c}\rho_2^{m}c(c-1)\ldots\ldots(c-m+1)D_{c-m}^{2}(z_k)\Big/D_c(z_k)$

and z_1, z_2, $\ldots\ldots$, z_c are the c roots of $D_c(z+1)=0$.

For example, if c=1 then (4.15) reduces to

$$(4.16) \qquad P_{ij}(t) = \frac{\rho_2^j}{j!\,(1+\rho_2)} + \frac{(-1)^{i+j}\,\rho_2^{1-i}}{j!\,(1+\rho_2)}\, e^{-(1+\rho_2)\mu t}$$

and for c=2 we have

$$(4.17) \qquad P_{ij}(t) = \frac{\rho_2^j/j!}{1+\rho_2+\rho_2^2/2} + \frac{\rho_2^{2-i}}{j!} \sum_{k=1}^{2} \frac{D_i(z_k)D_j(z_k)}{z_k(z_k+1-\rho_2)}\, e^{z_k\mu t}$$

where $z_1 = [2\,\rho_2 + 3 - \sqrt{1+4\rho_2}\,]/(-2)$, $z_2 = [2\rho_2 + 3 + \sqrt{1+4\rho_2}\,]/(-2)$ and $D_2(z_k) = -2(\rho_2+1+z_k)$ for $k=1,2$ (see[6,pp.81-86]).

Especially, for multiple inputs/M/1(1) we have the following:

$$q_{01} = \frac{\rho_2}{1+\rho_2} + \frac{1}{1+\rho_2}\, A^*[(1+\rho_2)\mu] \qquad (q_{11}=q_{01})$$

$$\pi_1 = \frac{\rho_2}{1+\rho_2} + \frac{1}{1+\rho_2}\, A^*[(1+\rho_2)\mu] \quad \text{and} \quad \pi_0 = 1 - \pi_1 .$$

In this case, we have

$$p_1 = \frac{\rho_2}{1+\rho_2} + \frac{\rho_1}{(1+\rho_2)^2}\left\{1 - A^*[(1+\rho_2)\mu]\right\} \quad \text{and} \quad p_0 = 1 - p_1 .$$

In general, we find from the definition of q_{ij} that

$$(4.18) \qquad q_{ij} = p_j^* + \sum_{k=1}^{c} \gamma_k(i+1,j)A^*(-\mu z_k) \qquad (0 \leq i,\ j \leq c)$$

where $\gamma_k(c+1,j) = \gamma_k(c,j)$. Equation (4.18), together with the balance equations will yield the required results.

5. Concluding Remarks

In this paper we have attempted to derive, in a unified way, the relationships between two probabilities for a family of queueing systems with multiple inputs, that is, the stationary probability π_j of there being j customers in the system at arrival epoch of a type-1 customer (a regenerative point) on one hand, and the stationary probability p_j of there being j customers in the system at arrival epoch of a type-2 customer on the other hand, assuming the

common distribution of service times for both types of customers to be exponential of rate μ. However, if there are two distributions of service times, one for each type of customers, then the assumption of a common service rate will be undesirable. The analysis of such systems, though needed in practice, will be complicated. Under such conditions we were concerned here only with a relationship between distributions of waiting times for two types of customers in a single-server queueing system. More generally, it will be necessary to study a system having two mutually dependent input processes in the case of the analysis of queueing networks.

Acknowledgement

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