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MINIMAX POLICIES FOR THE PRODUCTION OF SEASONAL STYLE-GOODS

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ABSTRACT

The problem considered in this note deals with the optimal minimax policies for the sequential production problem in which seasonal atyle-goods are produced in anticipation of the actual demand in future, when the production cost and the anticipated demand in each period follow arithmetic random walks. Also after the actual demand is known, terminal costs representing overage and underage costs are incurred at the terminal period.

1.INTRODUCTION

Hausman [1] has considered the related problem for style-goods which has a forecasting mechanism. In general, as mentioned in [1], if a minimization of expected costs is used as an optimality criterion, the problem under consideration here becomes more intractable. Then the form of the optimal policy may be obtained, but the entire optimal policy can be obtained only in tabular form. Now, this note considers the model in which an explicit expression for the optimal policy is obtained, using the minimax criterion and arguments given by Pye [2], [3].

2. MINIMAX POLICY

To derive a minimax policy for producing seasonal style-goods we will use the following notations and assumptions. Assume that there are n+1 periods which production decisions can be made at the beginning of. The first such period is numbered n and the terminal period zero. Let p_i, $n \geqq i \geqq 0$, denote the unit production cost in period i. Let x_i, $n \geqq i \geqq 0$, represent the anticipated demand for goods, with the anticipation made at the beginning of period i, and x_0 be specially the actual demand for goods. In this note we assume that $\{x_i\}$, $\{p_i\}$ follow arithmetic random walks, i.e.,

(1) $$x_i = x_{i+1} + \Delta x_{i+1}, \quad p_i = p_{i+1} + \Delta p_{i+1} \quad (i = n-1,\cdots,1,0),$$

where $\{\Delta x_i\}$, $\{\Delta p_i\}$ are sequences of random variables, and whose changes satisfy $-\underline{\Delta x} \leqq \Delta x_i \leqq \overline{\Delta x}$, $-\underline{\Delta p} \leqq \Delta p_i \leqq \overline{\Delta p}$. The parameters $\underline{\Delta x}$, $\underline{\Delta p}$ and $\overline{\Delta x}$, $\overline{\Delta p}$ are taken to correspond to the maximum possible decreases and increases respectively, and we assume that $p_n - n\underline{\Delta p} \geqq 0$. Let z_i be the amount of inventory before production is made at period i ($z_n \leqq z_{n-1} \leqq \cdots \leqq z_0$). Also to define terminal costs let:

$c_o(z)$ = overage costs (i.e., the cost of the excess amount z of inventory over demand after the actual demand is known and the production is made at the terminal period).

$c_u(z)$ = underage costs (i.e., the cost of the excess amount z of demand over inventory after the actual demand is known and the production is made).

And we assume that $c_o(\cdot)$ and $c_u(\cdot)$ are convex, increasing, continuous and vanishing at the orizin.

The problem is to characterize the production decision procedure which minimizes the maximum possible value of total costs. Now define the usual recursive function in dynamic programming as follows:

$f_i(z,x,p)$ = minimax costs incurred from the beginning of period i through period 0, given that the current state $(z_i,x_i,p_i) = (z,x,p)$ and that optimal minimax policies are used henceforth.

Then following relation holds at the terminal period when ginen state is (z,x,p):

$$f_0(z,x,p) = \min_{w \geq z} [\ p(w-z) + c_u((x-w)^+) + c_o((w-x)^+)\ ]$$

where $x^+ = \max(0,x)$. This equation indicates that an optimal policy in period 0 is simply obtained by producing the smallest amount of goods so as to minimize the terms in brackets since the actual demand has been written as a perfect anticipation x. And that it is to produce $(x+\tilde{a}(p)-z)^+$ where $-\tilde{a}(p)$ is the largest value with that satisfies $c_u'(-\tilde{a}(p)) = p$. For an inductional convenience, we express $f_0(z,x,p)$ as follows:

(2) $$f_0(z,x,p) = \int_0^{(x-z)^+} \min[p, c_u'(t)]dt + \int_0^{(z-x)^+} c_o'(t)dt\ .$$

By the principle of optimality, the general recurrence relation for period i when given state is (z,x,p) would be written as :

(3) $$f_i(z,x,p) = \min_{w \geq z} g_i(w|z,x,p)$$

$$g_i(w|z,x,p) = \max_{\substack{-\underline{\Delta x} \leq \Delta x \leq \overline{\Delta x} \\ -\underline{\Delta p} \leq \Delta p \leq \overline{\Delta p}}} [\ f_{i-1}(w,\ x+\Delta x,\ p+\Delta p) + p(w-z)\ ]$$

$$(i = 1,2,\cdots,n)\ .$$

Our objective of this note is to show that the optimal policy in each period is characterized by a single critical level (see Theorem 2) and to develop quantitative results describing the variation of the critical level as a function of the anticipated demand and the unit production cost in its period. Then we give two theorems without proof, which are familiar to us in inventory problem.

<u>Theorem 1</u>. (a) $f_i(z,x,p)$ is convex in z and x given p for all i.
(b) $f_i(z,x,p)$ is nondecreasing function of p given z and x for all i.

Proof. Induction. ((a): Similar proof is given in [2]).

<u>Theorem 2</u>. At any period i optimal minimax policies are of the form:

Produce $(z_i^*(x,p)-z)^+$, where $z_i^*(x,p)$ is the smallest value w that minimizes $g_i(w|z,x,p)$.

Proof. A simple consequence of the theorem 1 (a).

In section 3,4 we will determine the critical level $z_i^*(x,p)$ explicitly, which characterizes the optimal minimax policy given by the theorem 2, and examine its dependence on x,p.

And notice that if we use as an optimality criterion a minimization of expected cost instead of a minimax criterion then we would define $f_i(z,x,p)$ which satiafies the theorem 1 (a), i.e., the theorem 2. So that the form of an optimal policy would be same as given by the theorem 2.

3. ONE-PERIOD PROBLEM

By (2), (3),

$$(4) \qquad g_1(w|z,x,p) = \max_{\substack{\Delta x \\ \Delta p}} \Big[\int_0^{(x+\Delta x-w)^+} \min(p+\Delta p, c_u'(t))dt + \int_0^{(w-x-\Delta x)^+} c_o'(t)dt + p(w-z) \Big]$$

$$= \max \Big[\int_0^{x+\overline{\Delta x}-w} \min(p+\overline{\Delta p}, c_u'(t))dt \ , \int_0^{w-x+\underline{\Delta x}} c_o'(t)dt \Big] + p(w-z) \qquad \text{(by the theorem 1).}$$

Define three numbers $a_1(p)$, $c_1(p)$ and $\tilde{a}_1(p)$ by:

$$\int_0^{\overline{\Delta x}-a_1(p)} \min(p+\overline{\Delta p}, c_u'(t))dt = \int_0^{\underline{\Delta x}+a_1(p)} c_o'(t)dt$$

$$c_1(p) = c_o(\underline{\Delta x}+a_1(p)) \ , \qquad c_u'(\overline{\Delta x}-\tilde{a}_1(p)) = p \ ,$$

and if $c_u'(0) > p$ then $\tilde{a}_1(p) = \overline{\Delta x}$, if $c_u'(\infty) < p$, $\tilde{a}_1(p) = -\infty$. Where $-\tilde{a}_1$ is the largest value of above equation. And notice that $a_1(p)$, $c_1(p)$ are nondecreasing in p and $\tilde{a}_1(p)$ is nonincreasing in p. We illustrate those numbers by Fig.1.

Case 1, $a_1 \leqq a_1$. By the definition of numbers, $g_1(w|z,x,p)$ reaches its minimum at $w = x+a_1$, then $z_1^*(x,p) = x+a_1$.
For $z < z_1^*$; by (4),

$$f_1(z,x,p) = c_1 + p(a_1+x-z),$$

for $z \geqq z_1^*$;

$$f_1(z,x,p) = \int_0^{z-x+\underline{\Delta x}} c_o'(t)dt = c_1 + \int_o^{z-x-a_1} c_o'(t+\underline{\Delta x}+a_1)dt \ .$$

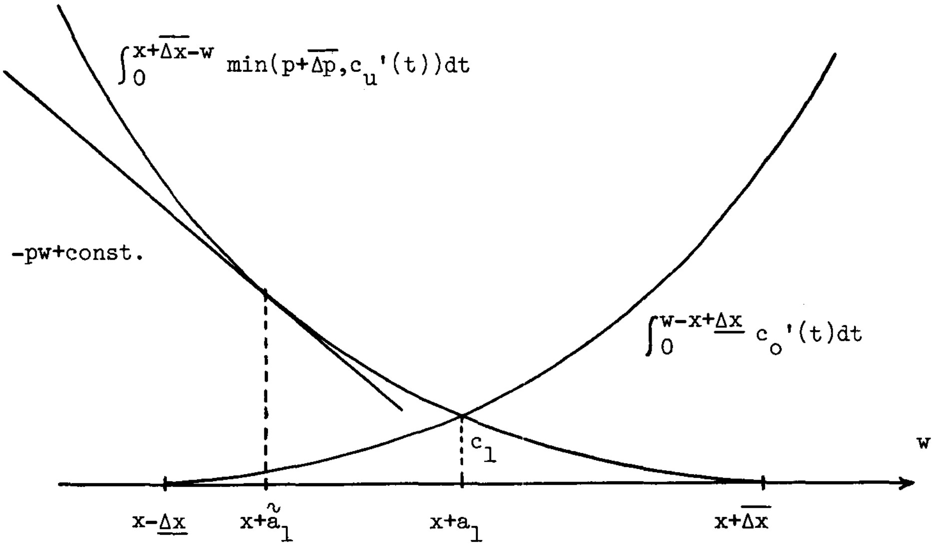

Fig.1 Definition of numbers $a_1, \tilde{a}_1$ and c_1: Case 2, $a_1 > \tilde{a}_1$.

Case 2, $a_1 > \tilde{a}_1$. By the definition of numbers, $g_1(w|z,x,p)$ reaches its minimum at $w = x+\tilde{a}_1$, then $z_1^*(x,p) = x+\tilde{a}_1$.

For $z < z_1^*$; by (4),

$$f_1(z,x,p) = \int_0^{\overline{\Delta x}-\tilde{a}_1} \min(p+\overline{\Delta p}, c_u'(t))dt + p(\tilde{a}_1+x-z)$$

$$= c_1 + \int_{\overline{\Delta x}-a_1}^{\overline{\Delta x}-\tilde{a}_1} c_u'(t)dt + p(\tilde{a}_1+x-z) \quad (\text{since } c_u'(t)<p \text{ for } t<\overline{\Delta x}-\tilde{a}_1)$$

$$= c_1 + \int_0^{x+a_1-z} \min(p, c_u'(t+\overline{\Delta x}-a_1))dt \quad (\text{since } c_u'(t)>p \text{ for } t>\overline{\Delta x}-\tilde{a}_1),$$

for $z_1^* \leqq z < x+a_1$;

$$f_1(z,x,p) = \int_0^{x+\overline{\Delta x}-z} \min(p+\overline{\Delta p}, c_u'(t))dt$$

$$= c_1 + \int_{\overline{\Delta x}-a_1}^{x+\overline{\Delta x}-z} c_u'(t)dt \quad (\text{since } c_u'(t)<p \text{ for } t<\overline{\Delta x}-\tilde{a}_1)$$

$$= c_1 + \int_0^{x+a_1-z} \min(p, c_u'(t+\overline{\Delta x}-a_1))dt,$$

for $a_1+x \leqq z$;

$$f_1(z,x,p) = \int_0^{z-x+\underline{\Delta x}} c_o'(t)dt = c_1 + \int_0^{z-x-a_1} c_o'(t+\underline{\Delta x}+a_1)dt .$$

Therefore, summarizing Case 1 and 2, we have;

$$z_1^*(x,p) = x + \min(a_1(p), \tilde{a}_1(p))$$

$$f_1(z,x,p) = \int_0^{(x+a_1(p)-z)^+} \min[p, c_u'(t+\overline{\Delta x}-a_1(p))]dt + \int_0^{(z-x-a_1(p))^+} c_o'(t+\underline{\Delta x}+a_1(p))dt + c_o(\underline{\Delta x}+a_1(p))$$

Notice that f_1 is same type of function as f_0.

4. THE n-PERIOD PROBLEM

In this section no proofs are presented since they are similar to proofs stated before. In order to describe the critical level $z_n^*(x,p)$ in an n-period problem, define the real numbers $a_n(p)$, $\tilde{a}_n(p)$ by:

$$\int_0^{\overline{\Delta x}-a_n(p)} \min[p+\overline{\Delta p}, c_u'(t+(n-1)\overline{\Delta x}-A_{n-1}(p+\overline{\Delta p}))]dt$$
$$= \int_0^{\underline{\Delta x}+a_n(p)} c_o'(t+(n-1)\underline{\Delta x}+A_{n-1}(p+\overline{\Delta p}))dt ,$$

$$c_u'(n\overline{\Delta x}-A_{n-1}(p+\overline{\Delta p})-\tilde{a}_n(p)) = p$$

and if $c_u'(0) > p$ then $\tilde{a}_n(p) = n\overline{\Delta x}-A_{n-1}(p+\overline{\Delta p})$, if $c_u'(\infty) < p$, $\tilde{a}_n(p) = -\infty$, and where $-\tilde{a}_n$ is the largest value of equation, $A_n(p) = \sum_{i=1}^n a_i(p+(n-i)\overline{\Delta p})$.

Now we can express the critical lenel $z_n^*(x,p)$ and the minimax cost $f_n(z,x,p)$ in the n-period problem as follows;

$$z_n^*(x,p) = x + A_{n-1}(p+\overline{\Delta p}) + \min(a_n(p), \tilde{a}_n(p)) , \tag{5}$$

$$f_n(z,x,p) = \int_0^{(x+A_n(p)-z)^+} \min(p, c_u'(t+n\overline{\Delta x}-A_n(p))dt + \int_0^{(z-x-A_n(p))^+} c_o'(t+n\underline{\Delta x}+A_n(p))dt + c_o(n\underline{\Delta x}+A_n(p)) . \tag{6}$$

We must not conjecture that the critical levels are nondecreasing functions of p as well as x simply. On the basis of examples it appears that they are much sensitive to the form of the terminal cost function.

Then three straightforward but interesting examples of above results are given by the following:

Example 1. (Positive influence) If $c_u'(\cdot) = c_u$, $c_o'(\cdot) = c_o$, put

$$a(p) = \frac{\min(p+\overline{\Delta p}, c_u)\overline{\Delta x} - c_o\underline{\Delta x}}{\text{mim}(p+\overline{\Delta p}, c_u) + c_o}, \qquad c(p) = \frac{\min(p+\overline{\Delta p}, c_u)c_o(\overline{\Delta x}+\underline{\Delta x})}{\min(p+\overline{\Delta p}, c_u) + c_o},$$

$$A_n(p) = \textstyle\sum_{i=1}^{n} a(p + (n-i)\overline{\Delta p}), \qquad C_n(p) = \textstyle\sum_{i=1}^{n} c(p + (n-i)\overline{\Delta p}).$$

We have

$$(5)' \qquad z_n^*(x,p) = x + A_n(p) \qquad \text{if } p < c_u,$$
$$= -\infty \qquad \text{if } p \geqq c_u.$$

$$(6)' \qquad f_n(z,x,p) = \min(p,c_u)(x+A_n(p)-z)^+ + c_o(z-x-A_n(p))^+ + C_n(p).$$

Note that the critical levels are nondecreasing in p on $(0,c_u)$, and at any period n an optimal minimax policy is to produce $(x+A_n(p)-z)^+$ if $p < c_u$ or not to produce if $p \geqq c_u$.

Example 2. (Indifferent influence) If $c_u'(\cdot) = c_u$, $c_o(\cdot) = 0$.

We have

$$(5)'' \qquad z_n^*(x,p) = x + n\overline{\Delta x} \qquad \text{if } p < c_u,$$
$$= -\infty \qquad \text{if } p \geqq c_u.$$

$$(6)'' \qquad f_n(z,x,p) = \min(p,c_u)(x+n\overline{\Delta x}-z)^+.$$

Note that the critical levels are constant in p on $(0,c_u)$, and at any period n an optimal minimax policy is to produce $(x+n\overline{\Delta x}-z)^+$ if $p<c_u$ or not to produce if $p \geqq c_u$.

Example 3. (Negative influence) If $c_o(\cdot) = 0$, and as defined in section 2, $-\tilde{a}(p)$ is the largest value with that satisfies $c_u'(-\tilde{a}(p))=p$.

We have

$$(5)''' \qquad z_n^*(x,p) = x + n\overline{\Delta x} + \tilde{a}(p),$$

$$(6)''' \qquad f_n(z,x,p) = \int_0^{(x+n\overline{\Delta x}-z)^+} \min(p,c_u'(t))dt.$$

Note that the critical levels are nonincreasing in p, and at any period n an optimal minimax policy is to produce $(x+n\overline{\Delta x}+\tilde{a}(p)-z)^+$.

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