AN OPTIMIZING METHOD

IN SYSTEM RELIABILITY WITH FAILURE-MODES

BY IMPLICIT ENUMERATION ALGORITHM

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ABSTRACT

In this paper we propose an effective method for solving the optimization problem of the redundant allocation and unit selection in system reliability with several failure-modes by using the implicit enumeration algorithm.   The quantitative evaluation for the proposed method is indicated clearly.   This shows that the number of constraints and variables in the proposed one are few than those of the integer programming method, respectively. Recently, McLeavey points out an example in which an algorithm reported by Ghare and Taylor for determining optimum redundancy in a series system dose not produce an optimal solution.   We also report a new optimal solution of the numerical example in which the objective function to be minimized is smaller than and the system reliability is higher than that of the integer programming method, respectively.   Consequently, the computer CPU time is shorter than that of the integer programming method with the same computer and implicit enumeration as a solution algorithm.

1. INTRODUCTION AND SUMMARY

Abe [1] and Makabe [17] reported that the methodologies developed in the field of operations research are being applied to some branches in the reliability theory. Especially, Barlow, Hunter, and Proschan [2] determind the optimum number of redundant units for a system where the units are subject to two types of failures. Tillman and Liittschwager [23] solved the problem of optimizing system reliability subject to nonlinear constraints by the integer programming method. Mizukami [19] solved this problem subject to linear constraints by using the convex and integer programming method with the different approach of [23]. In [19] and [23] only one failure-mode was considered and the cutting plane method was used as a solution algorithm. Ghare and Taylor [12,22] solved the same problem with Mizukami's by using the branch and bound method. Kondo [16] reported an application of the mathematical programming to system design for solving the optimum allocation of redundancy. Recently, Misra and Sharma [18] solved the linear formulation problem of Ghare and Taylor by using Christofides's zero-one programming [4]. In addition, Gen and Okuno [7,10] proposed an effective algorithm for solving the optimiztion problem of the redundant allocation and selection in system reliability by zero-one programming with the different approach of [14,23] and [18].

Recently, Henin [13] solved the problem of optimizing system reliability with two failure-modes subject to linear constraints by using the branch and bound method. Kolesar [15] formerly solved the same problem by using the linear programming. Fan, Hwang, and Tillman [5,14,24] solved the optimal redundant allocation problem by using the integer programming when the system is subject to nonlinear constraints, and when the subsystem and units within the sub-

system are subject to more than two failure-modes. A system reliability approach to the linear programming is developed by Sengupta [21] when the restrictions are chance-constrained. Gen and Okuno [8,9] solved the same problem with Fan, et al. by using zero-one programming.

In this paper we shall propose an effective method for solving the optimization problem of the redundant allocation and unit selection in system reliability with several failure-modes by using the implicit enumeration algorithm as an extension of [8,9]. The implicit enumeration algorithm used in the paper is a class of the branch and bound method. Narihisa [20] reported a survey of the integer programming including the implicit enumeration algorithm. In section 2, the mathematical models of system reliability are indicated with the general form as a nonlinear integer programming (NIP) problem. In section 3, the NIP problems are linearly formulated into zero-one linear programming (ZOLP) problems by introducing O-1 variables. We also remark that there are one-to-one correspondences between the NIP problems and the ZOLP problems. In section 4, we demonstrate that the numerical example treated by Fan, et al. [5,14,24] is linearly formulated into the ZOLP problem and the optimal solution is obtained by using the implicit enumeration algorithm on the computer NEAC 2200/500. In section 5, we report a new optimal solution of the numerical example treated by Fan, et al. in which the objective function to be minimized is smaller than and the system reliability is higher than that of the integer programming method. In addition, we indicate quantitatively that the coefficients of objective function and those of constraints in the proposed method are direct and simple, and the number of constraints and variables in the proposed one are fewer than those of the integer programming method, respectively.

Consequently, the computer CPU time is shorter than that of the integer programming method with the same computer and implicit enumeration as a solution algorithm.

2. MODELS

2.1   REDUNDANT ALLOCATION MODEL WITH FAILURE-MODES

The structure of any digital system may be represented in the form of a set of subsystems such as the logical combinatorial system and the computer system.   The failures of the such subsystems are staistically independent and the failure of a subsystem causes that of the entire system.   Then the complex hierachy of an irredundant structure is represented by a series or multistage system in a reliability sense.

As an example we consider a switching circuit subsystem with m_i redundant units in series.   In this situation all the redundant units must remain open for the subsystem to operate and can fail by only one of s_i failure-modes at any given time.   Their failure-modes are exclusively divided into either "A" ($e=1,2,\ldots,h_i$) or "O" ($e=h_i+1,h_i+2,\ldots,s_i$) class.

The "A" failures are those where all switches close when they shoud not, causing the subsystem to fail.   For example, these can occur from the following:

1) vibration of the subsystem,

2) the subsystem being subject to a surge in voltage and current.

Let q_{ier} be a probability of the e-th failure-mode of the r-th redundant unit (r=1; basic unit, r=2,3,$\ldots$,m_i+1; redundant units) at the i-th subsystem.   The failure probability of the i-th subsystem subject to the "A" failure-modes is given by

(1) $$Q^A_{1\sim h_i}(m_i) = \sum_{e=1}^{h_i} \prod_{r=1}^{m_i+1} q_{ier}.$$

Now let for all i and e (e=1,2,...,h_i) be $q_{ie} = q_{ier}$ for r=1,2,...,m_i+1.
Then, the equation (1) is as follows:

(2) $\quad Q_{1\sim h_i}^A (m_i) = \sum\limits_{e=1}^{h_i} (q_{ie})^{m_i+1}$.

The "O" failures are those where one switch fails by not closing
when it shoud, causing the subsystem to fail.   For example, these
can occur from the following:

  1) a bad connection due to oxidation in a moist environment,

  2) the subsystem's receiving too weak a signal.

The failure probability of the i-th subsystem subject to the "O"
failure-modes is given by

(3) $\quad Q_{h_i+1\sim s_i}^O (m_i) = s_i - h_i - \sum\limits_{e=h_i+1}^{s_i} \prod\limits_{r=1}^{m_i+1} (1 - q_{ier})$.

Now let for all i and e (e=h_i+1,h_i+2,...,s_i) be $q_{ie} = q_{ier}$ for r=1,2,
...,m_i+1.   Then, the equation (3) is as follows:

(4) $\quad Q_{h_i+1\sim s_i}^O (m_i) = s_i - h_i - \sum\limits_{e=h_i+1}^{s_i} (1 - q_{ie})^{m_i+1}$

The unreliability of the i-th subsystem is given by adding the
failure probability subject to the "A" failure-modes to that subject
to the "O" failure-modes, that is,

(5) $\quad Q_i (m_i) = s_i - h_i + \sum\limits_{e=1}^{h_i} (q_{ie})^{m_i+1} - \sum\limits_{e=h_i+1}^{s_i} (1 - q_{ie})^{m_i+1}$

The system reliability function with the "A" and "O" failure-modes
is given as

(6) $\quad R_S (m) = \prod\limits_{i=1}^{n} (1 - Q_i (m_i))$

$\quad\quad = \prod\limits_{i=1}^{n} (1 - s_i + h_i - \sum\limits_{e=1}^{h_i} (q_{ie})^{m_i+1} + \sum\limits_{e=h_i+1}^{s_i} (1 - q_{ie})^{m_i+1})$.

This function is a nonlinear one with respect to the unknown positive
integer m_i for each subsystem, where m=(m_1,m_2,...,m_n) is a vector.

Associated with each unit at the i-th subsystem, there is resource requirement $g_{ti}(m_i)$ for each system resource t (t=1,2,...,T), where is nonlinear and separable constraints,

(7) $G_t(m) = \sum_{i=1}^{n} g_{ti}(m_i) \leqq b_t$,   (t=1,2,...,T),

where b_t is the amount of the t-th system resource available.   The optimization problem is to determine the redundant allocation $m=(m_1, m_2,...,m_n)$ which maximizes the nonlinear system reliability (6) subject to the T nonlinear constraints (7).   This is a nonlinear integer programming problem and we call it the NIP-1 problem.

2.2   UNIT SELECTION AND REDUNDANT ALLOCATION MODEL
WITH FAILURE-MODES

Let a_i ($a_i=1,2,...,\alpha_i$) represent the design alternatives available for the i-th subsystem with a specified inherent unit reliability. As a special case if $\alpha_i=1$ for all i, it is reduced to the NIP-1 problem.   Let q_{iea_i} be a probability of the e-th failure-mode of the a_i-th unit at the i-th subsystem.   Then, the system reliability function with the design alternatives and several failure-modes is given by

(8) $R_S(a,m) = \prod_{i=1}^{n} (1-s_i+h_i- \sum_{e=1}^{h_i} (q_{iea_i})^{m_i+1} + \sum_{e=h_i+1}^{s_i} (1-q_{iea_i})^{m_i+1})$.

Associated with each unit at the i-th subsystem, there are resource requirements $g_{tia_i}(m_i)$ for each system resource t and alternative unit a_i which are separable and nonlinear function with respect to the redundant unit m_i.   The T nonlinear constraints are given as

(9) $G_t(a,m) = \sum_{i=1}^{n} g_{tia_i}(m_i) \leqq b_t$,   (t=1,2,...,T).

The problem is to determine the optimal simultaneous unit selection $a=(a_1,a_2,...,a_n)$ and redundant allocation $m=(m_1,m_2,...,m_n)$ which maximizes the nonlinear system reliability (8) subject to the T

nonlinear constraints (9).   This is a nonlinear integer programming problem and we call it the NIP-2 problem.

3. LINEAR FORMULATION

When we introduce a 0-1 variable to the nonlinear system reliability of the NIP-1 problem, we can get the following linearized objective function:

$$(10) \qquad f_1(X) = \sum_{i=1}^{n} \sum_{k=r_i}^{u_i} c_{ik} \, x_{ik} \; ,$$

where $X=(x_{ik})$, x_{ik} is 1 if the k redundant units are allocated at the i-th subsystem and 0 if otherwise, r_i and u_i are lower and upper bounds of the redundant unit m_i, respectively, and the coefficient c_{ik} is

$$(11) \qquad c_{ik} = Ln(1-s_i+h_i- \sum_{e=1}^{h_i} (q_{ie})^{k+1} + \sum_{e=h_i+1}^{s_i} (1-q_{ie})^{k+1}).$$

When we also introduce the 0-1 variable into the T nonlinear constraints of the NIP-1 problem, we can get the following T linearized constraints:

$$(12) \qquad g_t(X) = \sum_{i=1}^{n} \sum_{k=r_i}^{u_i} a_{tik} \, x_{ik} \leqq b_t \; , \quad (t=1,2,\ldots,T),$$

where the coefficient a_{tik} is

$$(13) \qquad a_{tik} = g_{ti}(k).$$

From the nature of the 0-1 valiable we must add the following n linear constraints to the equation (12):

$$(14) \qquad g_{T+i}(X) \equiv 1- \sum_{k=r_i}^{u_i} x_{ik} = 0 \; , \quad (i=1,2,\ldots,n).$$

By introducing the 0-1 variable, we have therefore formulated the NIP-1 problem into a zero-one linear programming problem which maximizes the linear objective function (10) subject to the T+n linear constraints (12) and (14), and we call it the ZOLP-1 problem.

REMARK 1: If the equations (11) and (13) are satisfied, there is

a one-to-one correspondence between the feasible solutions to the

NIP-1problem and the feasible solutions to the ZOLP-1 problem.

As a conclusion, $R_s(m)$ of the NIP-1 problem is maximum when $f_1(X)$

of the ZOLP-1 problem is maximum, or the optimal solution to the

ZOLP-1 problem corresponds to the optimal solution to the NIP-1

problem.

Now we present an algorithm for computing the upper bound u_i in

the ZOLP-1 problem (see [26] for the treatment of the algorithm).

ALGORITHM :

$$\text{Input } \left\{ g_{ti}(m_i) \right\}_{T \times n} \ , \ (b_t)_T$$

For $i_0 = 1, 2, \ldots, n$:

L_1:   For $t = 1, 2, \ldots, T$:

     Set $m_{ti_0} = 1$

$$B_t(i_0) = b_t - \sum_{\substack{i=0 \\ i \neq i_0}}^{n} g_{ti_0}(m_{ti_0})$$

L_2:   If $g_{ti_0}(m_{ti_0}) \leqq B_t(i_0) \leqq g_{ti_0}(m_{ti_0})$, Go to L_1

     .....else $m_{ti_0} = m_{ti_0} + 1$, Go to L_2

$$\ldots u_{i_0} = \underset{t}{\text{MIN}} \left\{ m_{ti_0} \right\}$$

$$\text{Output } \left\{ u_{i_0} \right\}_n$$

The flowchart of the algorithm is shown in Fig. 1.

When we introduce a 0-1 variable to the nonlinear system reli-

ability of the NIP-2 problem, we can get the following linearized

objective function:

$$(15) \qquad f_2(X) = \sum_{i=1}^{n} \sum_{j=1}^{a_i} \sum_{k=r_i}^{u_i} c_{ijk} x_{ijk} \ ,$$

where $X = (x_{ijk})$, x_{ijk} is 1 if the j-th unit is selected and the k

redundant units are allocated at the i-th subsystem and 0 if otherwise,
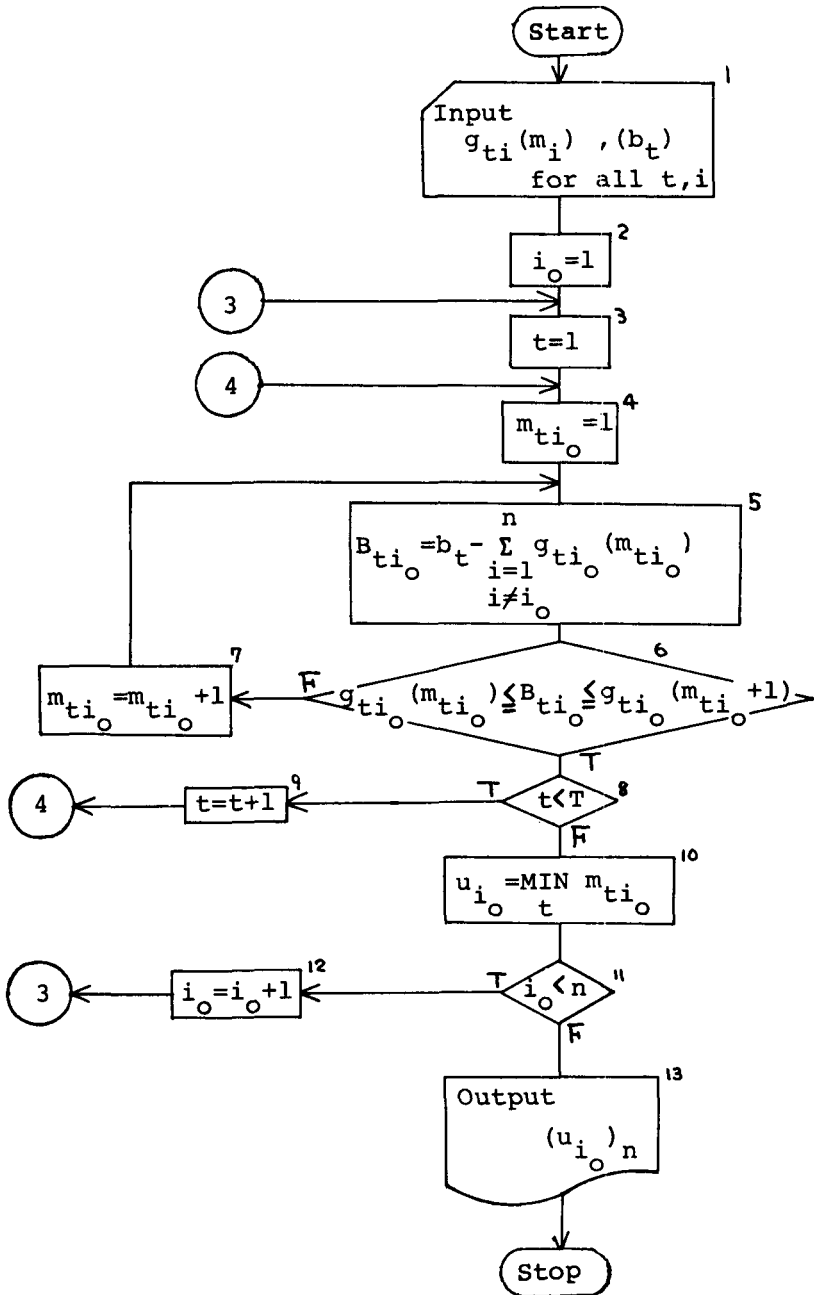
and the coefficient c_{ijk} is

Fig. 1. Flowchart for computing the upper bound u_i in each subsystem

(16) $c_{ijk} = Ln(1-s_i+h_i- \sum\limits_{e=1}^{h_i} (q_{iej})^{k+1} + \sum\limits_{e=h_i+1}^{s_i} (1-q_{iej})^{k+1})$.

When we also introduce the 0-1 variable to the T nonlinear constraints of the NIP-2 problem, we can get the following T linearized constraints:

(17) $g_t(X) = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{\alpha_i} \sum\limits_{k=r_i}^{u_i} a_{tijk} x_{ijk} \leqq b_t$, $(t=1,2,\ldots,T)$,

where the coefficient a_{tijk} is

(18) $a_{tijk} = g_{tij}(k)$.

From the nature of the 0-1 variable we must add the following n linear constraints to the equation (17):

(19) $g_{T+i}(X) \equiv 1- \sum\limits_{j=1}^{\alpha_i} \sum\limits_{k=r_i}^{u_i} x_{ijk} = 0$, $(i=1,2,\ldots,n)$.

By introducing the 0-1 variable, we have therefore formulated the NIP-2 problem into the zero-one linear programming problem which maximizes the linear objective function (15) subject to the T+n linear constraints (17) and (19), and we call it the ZOLP-2 problem.

REMARK 2: If the equations (16) and (18) are satisfied, there is a one-to-one correspondence between the feasible solutions to the NIP-2 problem and the feasible solutions to the ZOLP-2 problem.

As a conclusion, $R_S(a,m)$ of the NIP-2 problem is maximum when $f_2(X)$ of the ZOLP-2 problem is maximum, or the optimal solution to the ZOLP-2 problem corresponds to the optimal solution to the NIP-2 problem.

4.   EXAMPLE

In this section, we demonstrate that the numerical example treated by Fan, et al. as a special case of the NIP-2 problem is linearly formulated into the proposed ZOLP problem and obtained a new optimal solution by using the implicit enumeratiom algorithm on the computer NEAC 2200/500.   The numerical example which minimizes

the nonlinear cost function subject to the 4 nonlinear constraints is following:

min. $C_S(m) = 5 \, Ln(m_1+1) + (m_2)^2 + m_3 \, exp(m_3)$,

subj. to $G_1(m) = (m_1+3)^2 + (m_2)^2 + (m_3+2)^2 \leqq 65$,

$$G_2(m) = 20(m_1+e^{-m_1})+20(m_2+e^{-m_2})+20(m_3+e^{-m_3}) \geqq 120,$$

$$G_3(m) = 20(m_1 e^{-m_1/4})+20(m_2 e^{-m_2/4})+20(m_3 e^{-m_3/4}) \geqq 50,$$

$$R_S(m) = \prod_{i=1}^{3} (1-s_i+h_i - \sum_{e=1}^{h_i} (q_{ie})^{m_i+1} + \sum_{e=h_i+1}^{s_i} (1-q_{ie})^{m_i+1}) \geqq 0.74,$$

where $m=(m_1,m_2,m_3)$, m_i; positive integer for i=1,2,3.

The subsystems are subject to four failure-modes $s_i=4$, with one "A" failure-mode $h_i=1$ and three "O" failure-modes for i=1,2,3. For each subsystem the failure probability is shown in Table 1.

Table 1. Class of the failure-modes and its failure
probability

subsystem i	class of the failure-modes e	failure probability q_{ie}
1	A	0.01
	O	0.04
	O	0.05
	O	0.02
2	A	0.02
	O	0.08
	O	0.02
	O	0.05
3	A	0.08
	O	0.01
	O	0.05
	O	0.01

The example is illustrated in Table 2 in the required integer programming formulation by Fan, et al. The optimal solution is $m_1^*=4$, $m_2^*=1$, $m_3^*=1$ [24, pp.53]. Then, the minimum cost $C_S(m^*)$ is

Table 2 Integer programming formulation

OBJECTIVE FUNCTION

		X 1 / X11	X 2 / X12	X 3 / X13	X 4 / X14	X 5 / X15	X 6	X 7	X 8	X 9	X10
		.0 / .0	3.5 / 2.7	2.0 / 12.2	1.4 / 45.5	1.1 / 158.1	.0	1.0	3.0	5.0	7.0

CONSTRAINTS

	CONSTANT	X 1 / X11	X 2 / X12	X 3 / X13	X 4 / X14	X 5 / X15	X 6	X 7	X 8	X 9	X10
G 1	=120.0	20.0 / 20.0	7.4 / 7.4	15.3 / 15.3	18.3 / 18.3	19.4 / 19.4	20.0	7.4	15.3	18.3	19.4
G 2	-50.0	.0 / .0	15.6 / 15.6	8.7 / 8.7	4.1 / 4.1	1.1 / 1.1	.0	18.6	8.7	4.1	1.1
G 3	.0	1.0 / .0	-1.0 / .0	.0 / .0	.0 / .0	.C / .C	.0	.0	.0	.0	.0
G 4	.0	.0 / .0	1.0 / .0	-1.0 / .0	.0 / .0	.C / .C	.0	.0	.0	.0	.0
G 5	.0	.0 / .0	.0 / .0	1.0 / .0	=1.0 / .0	.C / .C	.0	.0	.0	.0	.0
G 6	.0	.0 / .0	.0 / .0	.0 / .0	1.0 / .0	-1.0 / .0	.0	.0	.0	.0	.0
G 7	.0	.0 / .0	.0 / .0	.0 / .0	.0 / .0	.C / .C	1.0	-1.0	.0	.0	.0
G 8	.0	.0 / .0	.0 / .0	.0 / .0	.0 / .0	.C / .0	.0	1.0	-1.0	.0	.0
G 9	.0	.0 / .0	.0 / .0	.0 / .0	.0 / .0	.C / .0	.0	.0	1.0	-1.0	.0
G10	.0	.0 / .0	.0 / .0	.0 / .0	.0 / .0	.0 / .C	.0	.0	.0	1.0	-1.0
G11	.0	.0 / 1.0	.0 / -1.0	.0 / .0	.0 / .0	.C / .C	.0	.0	.0	.0	.0
G12	.0	.0 / .0	.0 / 1.0	.0 / -1.0	.0 / .0	.C / .C	.0	.0	.0	.0	.0
G13	.0	.0 / .0	.0 / .0	.0 / 1.0	.0 / -1.0	.C / .C	.0	.0	.0	.0	.0
G14	.0	.0 / .0	.0 / .0	.0 / .0	.0 / 1.0	.C / =1.0	.0	.0	.0	.0	.0
G15	65.0	-9.0 / -4.0	-7.0 / -5.0	-9.0 / -7.0	-11.0 / -9.0	-13.C / -11.0	.0	-1.0	-7.0	-19.0	-37.0
G16	30.1	18.6 / 16.2	-13.6 / .7	1.1 / 8.0	1.9 / 8.3	2.C / 8.3	12.7	-10.3	.5	.9	1.0

11.77 and the system reliability $R_S(m^*)$ is 0.744.

Now, the proposed ZOLP example is to minimize the linear objective function (10) in the case of n=3 subject to the linear constraints (12) and (14) in the case of n=3 and T=4, where the coefficient c_{ik} of (10) and a_{tik} of (12) for all k are as follows:

$$c_{1k} = 5 \ln(k+1), \quad c_{2k} = k^2, \quad c_{3k} = k \exp(k),$$

$$a_{11k} = -(3+k)^2, \quad a_{12k} = -k^2, \quad a_{13k} = -(2+k)^2,$$

$$a_{2ik} = 20(k+\exp(-k)), \quad a_{3ik} = 20 k \exp(-k/4),$$

$$a_{4ik} = \ln(1-s_i+h_i - \sum_{e=1}^{h_i} (q_{ie})^{k+1} + \sum_{e=h_i+1}^{s_i} (1-q_{ie})^{k+1}), \quad i=1,2,3,$$

$$b_1 = 65, \quad b_2 = -120, \quad b_3 = -50, \quad b_4 = 301.1.$$

The example problem is illustrated in Table 3 in the required ZOLP formulation which is $1000 \times a_{4ik}$ for all i and k. The 0-1 variables between the ZOLP formulation and the computer used are as follows:

$$x_{1k} = x_k, \quad x_{2k} = x_{4+k} ; \ k=1,2,3,4,$$

$$x_{3k} = x_{9+k} ; \ k=0,1,2,3.$$

The feasible and optimal solutions are shown in Table 4 by using the implicit enumeration algorithm. From Table 4 we can get a new optimal solution $x_3=1$, $x_6=1$, $x_9=1$, that is,

$$m_1^*=3, \quad m_2^*=2, \quad m_3^*=0.^\#$$

Table 4   Feasible and optimal solutions by the proposed method

PEASIBLE SOLUTION

	X 1	X 2	X 3	X 4	X 5	X 6	X 7	X 8	X 9	X10	X11	X12
STEP 7	0	0	1	0	0	0	1	0	1	0	0	0
STEP 13	0	0	1	0	0	1	0	0	0	1	0	0
STEP 15	0	0	1	0	0	1	0	0	1	0	0	0

OPTIMAL SOLUTION

	X 1	X 2	X 3	X 4	X 5	X 6	X 7	X 8	X 9	X10	X11	X12
	0	1	0	0	1	0	0	1	0	0	0	

[#] Recently, Gen and Okuno [11] obtained that this result is agreed with one by using Lawler-Bell algorithm.

Table 3 ZOLP formulation

OBJECTIVE FUNCTION

X 1 / X11	X 2 / X12	X 3	X 4	X 5	X 6	X 7	X 8	X 9	X10
3.5	5.5	6.9	8.0	1.0	4.0	9.0	16.0	.0	2.7
14.8	60.3								

CONSTRAINTS

CONSTANT

		X 1 / X11	X 2 / X12	X 3	X 4	X 5	X 6	X 7	X 8	X 9	X10
1	65.0	-16.0 / -16.0	-25.0 / -25.0	-36.0	-49.0	-1.0	-4.0	-9.0	-16.0	-4.0	-9.0
2	-120.0	27.4 / 42.7	42.7 / 70.0	70.0	80.4	27.4	42.7	70.0	80.4	20.0	27.4
3	-50.0	15.6 / 24.3	24.3 / 28.3	28.3	29.4	15.6	24.3	28.3	29.4	.0	15.6
4	301.1	-24.3 / -249.7	-30.5 / -334.1	-40.8	-50.2	-50.2	-60.8	-81.2	-100.9	-162.0	-169.6
G 5	-1.0	1.0 / .0	1.0 / .0	1.0	1.0	.0	.0	.0	.0	.0	.0
6	-1.0	.0 / .0	.0 / .0	.0	.0	1.0	1.0	1.0	1.0	.0	.0
7	-1.0	.0 / 1.0	.0 / 1.0	.0	.0	.0	.0	.0	.0	1.0	1.0
8	3.0	-1.0 / -1.0	-1.0 / -1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0

Then, the minimum cost $C_S(m^*)$ is 9.045 and the system reliability $R_S(m^*)$ is 0.769.　Recentry, Mcleavey [25] pointed out an example in which Ghare and Taylor algorithm [12,22] did not produce an optimal solution.

5. QUANTITATIVE EVALUATION

In [7] we reported the detailed quantitative evaluations between the integer programming method and the proposed method.　Now we summarize in Table 5 the quantitative evaluation and show in Table 6 a sufficient condition as a new optimal solution by the proposed method.

Table 5. Comparision with integer programming method
and the proposed method

compared factors	integer programming method	the proposed method
no. of constraints	$M_T = T + u \cdot n$, if $u = u_i$ for all i $M_T = 16$ in Table 2	$M_H = T + n + 1$ $M_H = 8$ in Table 3
no. of variables	$N_T = u \cdot n + n$, if $u = u_i$ for all i $N_T = 15$ in Table 2	$N_H = u \cdot n + n - n \cdot r$, if $u = u_i$, $r = r_i$ for all i $N_H = 12$ in Table 3
no. of coefficients	2	1
used computer	unpublished* NEAC 2200/500**	NEAC 2200/500
solution algorithm	cutting plane algorithm* implicit enumeration algorithm**	implicit enumeration algorithm
CPU time	unpublished* 14.892 sec.**	1.858 sec.

*: by [5] and [24],　**: by the authors.

Table 6　Sufficient condition for a new optimal solution

methods	Integer programming	The proposed method
system reliability $R_S(m^*)$	0.744	0.769
minimum cost $C_S(m^*)$	11.77	10.93
optimal solution m^*	$m_1^*=4, m_2^*=1, m_3^*=1$	$m_1^*=3, m_2^*=2, m_3^*=0$

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