

# OPTIMAL BATCH (s, S) POLICIES FOR THE MULTIPLE SET-UP ORDERING COSTS IN INVENTORY PROBLEM

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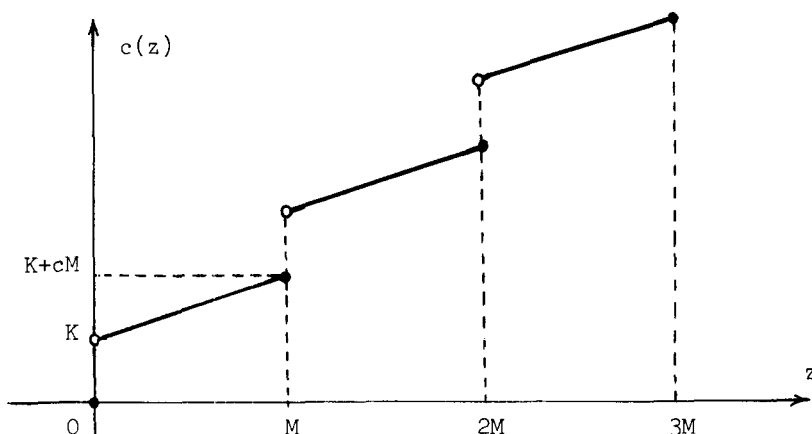
## ABSTRACT

Consider the dynamic inventory problem when the ordering cost function is linear with multiple set-up costs. In general, an optimal inventory policy is sensitive to the form of the ordering cost, so that the purpose of this note is to define a new policy, ie, a batch (s,S) policy and to show the sufficient conditions under which this policy is optimal.

## 1. INTRODUCTION

Consider the single item, periodic review, stochastic and dynamic inventory model when the ordering cost function is linear with multiple set-up costs rather than one with a single set-up cost. This type of cost is neither convex nor concave, but has a practical meaning when the ordered quantity in each period is delivered by transportation's vehicle which has certain limited capacity.

In general, an optimal inventory policy is sensitive to the form of the ordering cost, so that until now some types of inventory policies have examined and studied for several authors. Scarf [4] proved that an (s,S) policy is optimal for a linear cost with a single set-up, and this case was investigated in detail by Iglehart [1], Veinott [5] et al. Recently Porteus [3] proved that a generalized (s,S) policy is optimal for a concavely increasing cost. The purpose of this note is to discuss the Lippman [2]'s model in which the ordering cost has multiple set-ups, and show the optimality of "batch (s,S) policies".

Fig.1 Graph of  $c(z)$ 

## 2. FORMULATION OF THE MODEL

In this section we discuss the model which explicitly allows for the uncertainty in demands and make assumptions to keep the notation simple. Less restrictive assumptions under which the results of this paper still valid are given in Section 4.

Let  $c(z)$  denote the ordering cost function with multiple set-up as follows:

$$(1) \quad c(z) = K\left\{\frac{z}{M}\right\} + cz \quad \text{for } z \geq 0,$$

where  $c \geq 0$ ,  $K, M > 0$  and  $\{z\}$  is the minimum integer not smaller than  $z$ .

We illustrate this by Fig. 1. When we interpret  $M$  as the capacity of a transportation vehicle,  $K$  as the cost of its use and  $c$  as the unit cost of the treated item, then  $c(z)$  is more reasonable for if vehicles of the transportation are trucks the ordering cost is a function only of the number of trucks required to satisfy the order and not of the fraction of truck space used. (if excess space cannot be used.) It is specifically assumed that orders are delivered immediately, shortages are backlogged and the objective is to minimize the total expected cost attributed to ordering, holding and penalty for shortages over  $n$  periods. The quantities demanded in each period are independent, identically distributed, nonnegative random variables with common p.d.f.  $\phi(\cdot)$ . Costs to be incurred  $n$  periods in the future are discounted by the factor  $\alpha^n$ , where  $0 \leq \alpha \leq 1$ . Let holding and shortages costs charged on ending inventory in each period be denoted by  $l(\cdot)$ , then the one-period expected holding and shortage cost for the level  $y$  of inventory after

ordering is denoted by

$$(2) \quad g(y) = \int_0^\infty l(y-\xi)\phi(\xi)d\xi.$$

We assume  $g(y)$  exists for each  $y$ . As usual, let  $f_n(x)$  be the minimum expected cost over  $n$  periods as a function of the level  $x$  of inventory before ordering. We have,

$$(3) \quad f_n(x) = \inf_{y \geq x} [ c(y-x) + g(y) + \alpha \int_0^\infty f_{n-1}(y-\xi)\phi(\xi)d\xi ]$$

(  $n = 1, 2, \dots; f_0(x) \equiv 0$  ).

Let

$$(4) \quad G_n(x) = g(x) + cx + \alpha \int_0^\infty f_{n-1}(x-\xi)\phi(\xi)d\xi .$$

Then

$$(5) \quad f_n(x) = \inf_{y \geq x} [ G_n(y) + K\{\frac{y-x}{M}\} ] - cx$$

(  $n = 1, 2, \dots; f_0(x) \equiv 0$  ).

Let  $Y_n(x)$  denote the optimal inventory policy in the period  $n$ , ie, the optimal level of inventory after ordering in the first of  $n$  periods when the level of inventory before ordering is  $x$ . Then we have

$$f_n(x) = G_n(Y_n(x)) + K\{\frac{Y_n(x)-x}{M}\} - cx \quad \text{for every } x.$$

### 3. OPTIMALITY OF BATCH (s,S) POLICIES

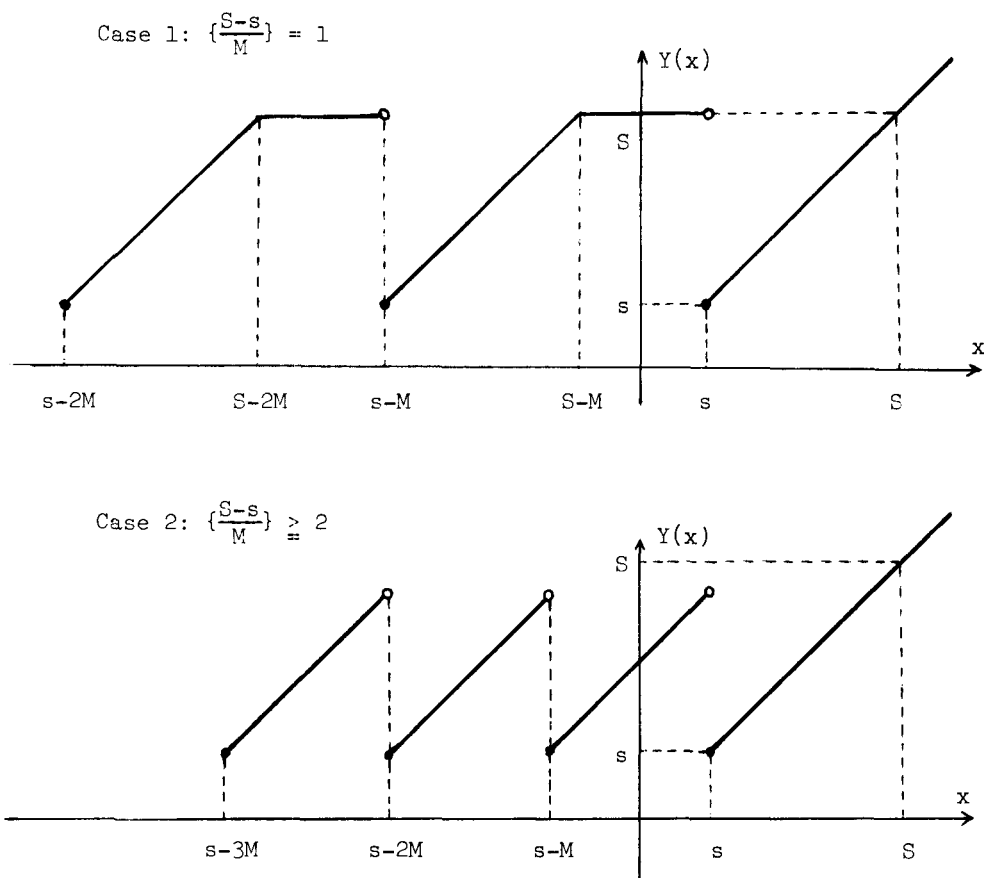
In this section we shall give a definition of batch (s,S) policy and some sufficient conditions under which this policy is optimal in the finite horizon problem.

Definition 1. A batch (s,S) policy is an inventory policy defined by parameters  $s, S$  with  $s < S$  and  $M (>0)$ , such that

$$Y(x) = x \quad \text{for } x \geq s$$

$$Y(x) = \min ( S, x + M\{\frac{S-x}{M}\} ) \quad \text{for } x < s.$$

We illustrate this by Fig. 2. A batch (s,S) policy has a following economic interpretation. In case 2, where  $\{\frac{S-s}{M}\} \geq 2$ ,  $M$  is smaller than  $S-s$ , ie, the manager has small-sized trucks for transportation use as compared with a satisficing level region (s,S]. then he orders a minimum amount of the item with full-loaded trucks so as to raise the inventory level upto the

Fig. 2 Batch  $(s,S)$  policy

region  $(s,S]$  if the initial level is less than  $s$ ; batch policy. In case 1, where  $\{\frac{S-s}{M}\} = 1$ , the manager has large-sized trucks, then he cannot order the item with full-loaded trucks so as to raise the inventory level into the region  $(s,S]$ . So that he raise the inventory level not to exceed  $S$  with trucks which are not always full-loaded if the initial level is less than  $s$ ; batch policy +  $(s,S)$  policy. If  $M$  is sufficiently large this policy is identical to the well-known  $(s,S)$  policy.

**Theorem 1.** If  $G_n(x)$  is convex and bounded below, then a batch  $(s,S)$  policy is optimal in period  $n$ .

**Proof.** From notational convenience, we abbreviate the subscript  $n$ . By our assumption on  $G(x)$ , there exist the smallest real numbers  $s$  and  $S$  with  $s < S$  (which may be  $+\infty$ ), such that

(6)  $G(S) \leq G(x)$  for all  $x$ ,

(7)  $\Delta_M G(x) \equiv \max_{0 \leq m \leq M} [G(x) - G(x+m)] \leq K$  for all  $x \geq s$ .

Case 1,  $\{\frac{S-s}{M}\} = 1$  :

(i) We have, for any  $x$  and  $y$  with  $s \leq x < y$

$$G(y) + K\{\frac{y-x}{M}\} \geq G(y) + K \geq G(x) \quad \text{[by (7)]}.$$

Thus it follows that

(8)  $Y(x) = x$  on  $[s, \infty)$ .

(ii) We have, for any  $x$  and  $y$  with  $S-M \leq x < s$ ,  $x < y$ ,

$$G(y) + K\{\frac{y-x}{M}\} \geq G(y) + K \geq G(S) + K \quad \text{[by (6)]}.$$

Hence we get

(9)  $Y(x) = S = \min [ S, x+M\{\frac{S-x}{M}\} ]$  on  $[S-M, s)$ .

For any  $x$  and  $y$  with  $S-2M \leq x < S-M$ ,  $x < y \leq x+M$ , we have

$$G(y) + K \geq G(x+M) + K \quad (\text{equality holds iff } y = x+M).$$

Thus it is easily shown by induction that for any  $x$  with  $S-(d+1)M \leq x < S-dM$ ,  $d = 1, 2, \dots$ ,

(10)  $\min_{x+dM \leq y \leq x} [ G(y) + K\{\frac{y-x}{M}\} ] \geq G(x+dM) + dK$ .

(iii) Therefore we have for any  $x$  with  $S-(d+1)M \leq x < S-dM$ ,  $d=1,2,\dots$ ,

$$\begin{aligned} \min_{y \geq x} [ G(y) + K\{\frac{y-x}{M}\} ] &= \min_{y \geq x+dM} [ G(y) + K\{\frac{y-x-dM}{M}\} + dK ] \\ &= G(S) + (d+1)K \quad \text{if } S-M \leq x+dM < s \quad \text{[by (9)]} \\ &= G(x+dM) + dK \quad \text{if } s \leq x+dM < S \quad \text{[by (8)]}. \end{aligned}$$

Hence, if  $S-M \leq x+dM < s$  then  $\{\frac{S-x}{M}\} = d+1$ , so that  $Y(x) = S \leq x + M\{\frac{S-x}{M}\}$ , and if  $s \leq x+dM < S$  then  $\{\frac{S-x}{M}\} = d$ , so that  $Y(x) = x + M\{\frac{S-x}{M}\} < S$ . Then,

$$Y(x) = \min [ S, x + M\{\frac{S-x}{M}\} ] \quad \text{on } (-\infty, s).$$

Case 2,  $\{\frac{S-s}{M}\} \geq 2$  :

Similarly to Case 1, we can prove that  $Y(x) = x$  on  $[s, \infty)$ ,  $Y(x) = x + M\{\frac{s-x}{M}\} < S$  on  $(-\infty, s)$ , and  $Y(x) = \min [ S, x + M\{\frac{s-x}{M}\} ]$  on  $(-\infty, s)$ , and hence a batch  $(s, S)$  policy is optimal in period  $n$ , which completes the proof of the theorem.

It is clear that an inventory policy which is optimal to use in the first of  $n$  periods is also optimal to use when there are  $n$  periods left in an  $m$  periods problem. We will therefore only be interested in deriving conditions which insure that  $G_n(\cdot)$  is convex for every  $n$  and hence that a batch  $(s, S)$  policy is optimal for the first of  $n$  periods.

Definition 2. A density  $\phi(\cdot)$  is called  $M$ -indifferent, if it satisfies

$$\sum_{i=0}^{\infty} \phi(\xi+iM) = \text{constant} \left( = \frac{1}{M} \right) \quad \text{for } 0 \leq \xi < M.$$

If we divide the demanded quantities by  $M$ , the  $M$ -indifferent densities give no information about which quantities left are likely to occur, that is, such densities are indifferent (ignorant) of the remaining quantities.

For example, let, for  $\eta \geq \frac{M}{2}$ ,  $v \geq 0$ ,

$$\begin{aligned} \psi(\eta, v, \xi) &= v & \bullet & \quad \text{if } \eta - \frac{M}{2} \leq \xi < \eta + \frac{M}{2} \\ &= 0 & & \quad \text{otherwise,} \end{aligned}$$

then an  $M$ -indifferent density  $\phi(\xi)$  is given by,

$$\phi(\xi) = \int \psi(\eta, v(\eta), \xi) v(\eta) d\mu(\eta),$$

where  $v(\cdot)$  is a generalized probability density defined on  $[\frac{M}{2}, \infty)$  and  $\int dv(\eta) = \frac{1}{M}$ .

Theorem 2. If  $g(\cdot)$  is convex and bounded below and  $\phi(\cdot)$  is  $M$ -indifferent, then a batch  $(s, S)$  policy is optimal for any finite horizon problem.

Proof. Here we will show by induction that  $G_n(\cdot)$  is convex for all  $n$ . For  $n = 1$ ,  $G_1(x) = g(x) + cx$  is convex. Assume that  $G_k(\cdot)$  is convex. Then by Theorem 1 there exist two levels  $s_k, S_k$  with  $s_k < S_k$  such that

$$\begin{aligned} (11) \quad f_k(x) &= G_k(\min(S_k, x+dM)) + dK - cx \\ &\quad \text{on } [s_k - dM, s_k - (d-1)M], \quad d = 1, 2, \dots, \\ &= G_k(x) - cx \quad \text{on } [s_k, \infty). \end{aligned}$$

Hence we have

$$(12) \quad f_k(x) - f_k(x-dM) = -dK \quad \text{if } x \leq \min(S_k, s_k+M) \equiv s^*,$$

and  $f_k(x) - f_k(x-dM)$  is nondecreasing in  $x$ .

Now we examine  $G_{k+1}(\cdot)$  defined by (4).

$$G_{k+1}(x) = g(x) + cx + \alpha \int_0^\infty f_k(x-\xi) \phi(\xi) d\xi.$$

We have from continuity and piecewise convexity of  $f_k(\cdot)$  given by (11)

$$(13) \quad \frac{d}{dx} \int_0^\infty f_k(x-\xi) \phi(\xi) d\xi = \int_0^\infty f'_k(x-\xi) \phi(\xi) d\xi.$$

For  $x \leq s^*$  we have

$$\begin{aligned} (13) &= \sum_{i=0}^\infty \int_0^{M^-} f'_k(x-\xi-iM) \phi(\xi+iM) d\xi \\ &= \sum_{i=0}^\infty \int_0^{M^-} f'_k(x-\xi) \phi(\xi+iM) d\xi && \text{[by (12)]} \\ &= \frac{1}{M} \int_0^{M^-} f'_k(x-\xi) d\xi && \text{[by the } \phi(\cdot) \text{'s } M\text{-indifference]} \\ &= \frac{1}{M} \int_{s_k}^{s^*} G'(\xi) d\xi - c = \text{constant } (= -C) && \text{[by (11)].} \end{aligned}$$

And for  $x > s^*$  let  $d^* = \{\frac{x-s^*}{M}\}$ , we have

$$\begin{aligned} (13) &= \int_0^{x-s^*} f'_k(x-\xi)\phi(\xi)d\xi + \int_{x-s^*}^\infty f'_k(x-\xi)\phi(\xi)d\xi \\ &- \int_0^{x-s^*} f'_k(x-\xi-d^*M)\phi(\xi)d\xi - \int_0^{x-s^*} f'_k(x-\xi-d^*M)\phi(\xi)d\xi \\ &= \int_0^{x-s^*} [f'_k(x-\xi) - f'_k(x-\xi-d^*M)]\phi(\xi)d\xi \\ &+ \int_0^\infty f'_k(x-\xi-d^*M)\phi(\xi)d\xi && \text{[by (12)].} \end{aligned}$$

Then the second term is  $-C$ , it is therefore sufficient to show that the integrand of the first term is non-negative and non-decreasing in  $x$ . For

$x_1 < x_2$ , let  $d_1^* = \{\frac{x_1-s^*}{M}\}$ ,  $d_2^* = \{\frac{x_2-s^*}{M}\}$  respectively, then  $d_1^* \leq d_2^*$ .

$$\begin{aligned} 0 &\leq f'_k(x_1-\xi) - f'_k(x_1-\xi-d_1^*M) && \text{[by (12)]} \\ &= f'_k(x_1-\xi) - f'_k(x_1-\xi-d_2^*M) && \text{[} x_1-\xi-d_1^*M \leq s^* \text{]} \\ &\leq f'_k(x_2-\xi) - f'_k(x_2-\xi-d_2^*M) && \text{[by (12)].} \end{aligned}$$

Thus the proof of Theorem 2 is complete.

## 4. CONCLUDING REMARK

In this note we have shown that a batch  $(s,S)$  policy is optimal in a standard inventory model when the ordering cost is linear with multiple set-ups. The most crucial assumption is that the demand pdf is to be M-indifferent. Our results will also valid for the nonstationary case;  $c_n(z) = c_n z_n + K_n \{\frac{z}{M}\}$  where  $c_n \geq 0$ ,  $K_n, M > 0$ . If  $G_n(\cdot)$  is convex and bounded below and  $\phi_n(\cdot)$  is M-indifferent, then we have

$$f_n(x) = \inf_{y \geq x} [ c_n(y-x) + g_n(y) + \alpha_n \int_0^{\infty} f_{n-1}(y-\xi) \phi_n(\xi) d\xi ]$$

where  $f_0(x) \equiv$  lower-bounded convex function, for  $n = 1, 2, \dots$

Unfortunately convexity of  $f_n(x)$ 's will not be obtained for general demand pdf and hence any batch  $(s,S)$  policy may not be optimal. However, in many cases the demand pdf is not determined precisely, and an M-indifferent pdf gives a good approximation to the demand density by exploiting a least-square method. For instance, when the actual demand pdf is  $\tilde{\phi}(\cdot)$ , the M-indifferent pdf  $\phi(\cdot)$  is given by

$$\phi(\xi) = \int \psi(\eta, v^*(\eta), \xi) v^*(\eta) d\mu(\eta)$$

in which  $v^*(\eta)$  is the minimizing  $v(\eta)$  of the integral  $\int (\phi(\xi) - \tilde{\phi}(\xi))^2 d\xi$ . Although it is a rough approximation it is useful to put  $v^*(\cdot) = \tilde{\phi}(\cdot)/M$ .

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