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TRAFFIC QUEUE CAUSED BY A SINGLE INTERRUPTION

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Abstract

Traffic queue formed when a stochastic vehicle stream is interrupted by a single duration of blockage, or of red signal, is analysed. It is assumed that the delayed vehicles depart with constant intervals when the interruption is terminated. The probability distribution of the number of delayed vehicles, the moments of that distribution and the expected total delay are obtained using Wald's identity in sequential analysis. The expressions for those quantities are rigorous with a Poisson input stream and are approximated with the general input stream. In addition, the influence of initial condition of interruption is discussed with relation to the bounds on mean number of delayed vehicles.

1. Introduction

Delay of vehicles at a fixed-cycle traffic signal has been investigated based on various models. Since Clayton [3] has presented an expression for mean delay with a non-stochastic vehicle stream, many attempts were made to obtain the expression with a stochastic stream. But no exact representation has been given explicitly even with a Poisson stream. As for the approximated mean delay with constant departure headways, the expressions were obtained by Webster [9] with a Poisson stream, by Miller [5] and Newell [7] respectively with the general input stream. Recently, Allsop [1] reviewed nearly forty works on this problem.

The case of non-cyclic signal, that is the case where a stochastic vehicle stream is interrupted by a single duration of red signal, was analysed by Buckley and Wheeler [2]. They found the probability distribution of the number of delayed vehicles for a Poisson input and gave the formal representation of expected delay. Though this case is rather simple, it is worthy to be investigated. Because the expressions for quantities in this case are, if acquired, the limiting solutions for cyclic signal at low traffic density. Furthermore, at any density the queue at a cyclic red period consists of two components; the overflow which is the residue of queue in the preceding red period and the queue newly formed in the period. And the latter is regarded as the queue caused by a single interruption.

In this paper, adopting the model used by Buckley and Wheeler but by a different approach, the explicit representations of characteristic quantities are given. The results are exact with a Poisson input and are approximated with the general inputs.

2. Model

We consider the situation where a stochastic vehicle flow on a single lane road is interrupted by a single duration r of blockage. This blockage causes N vehicles to be stopped a total delay W. When the interruption terminates, delayed vehicles start one after another with constant velocity which is equal to that on the approach. And they pass through a point, where the interruption arose, at constant time separation δ. After the all delayed vehicles have departed and the queue has been exhausted for the first time, no queueing occurs.

The interarrival time between the $(n-1)$th and the nth vehicles on the approach is represented by mutually independent random variable X_n, with mean μ and variance σ^2. The variable X_n is assumed to be identically distributed for all n. As for X_1, the time interval from the commencement of interruption till the first arrival of vehicle, this assumption is inevitable only for a Poisson input. The influence of this initial condition of blockage will be clarified in section 3.4, where a reasonable distribution for X_1 is assumed and the bounds on expected queue length are given.

The model stated above is the same one as Buckley and Wheeler's. In Fig. 1 the queueing process is represented in a flow diagram.

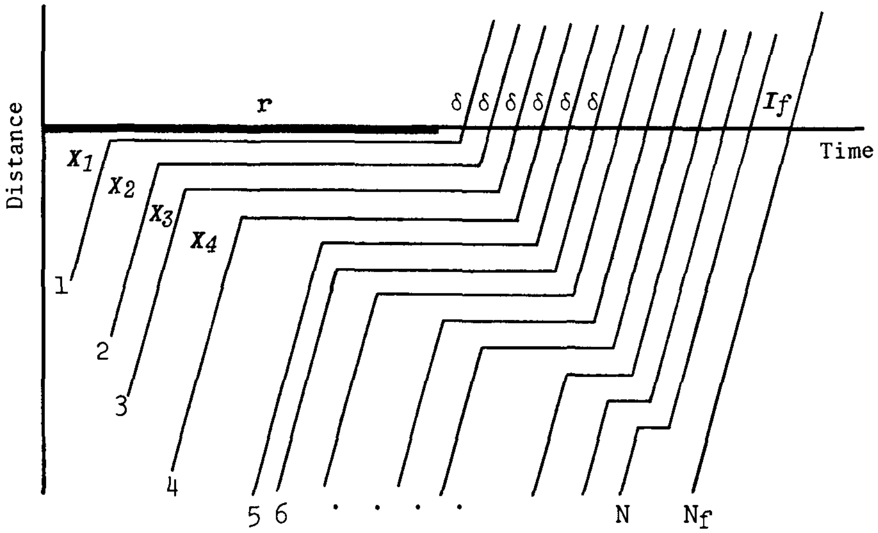

Fig.1. Flow diagram of vehicle stream interrupted by a single blockage.

3. Number of Delayed Vehicles

3.1 Basic Equation

The basic equation is formulated using Wald's identity in sequential analysis. The statistical quantities for number of delayed vehicles will be derived from the equation.

The number N of delayed vehicles and the ordinal number N_f of the first undelayed vehicle are defined as random variables by

$$(3.1) \qquad N_f = N + 1 = \min \{ n \mid T_n \geqq T'_{n-1} \},$$

where the arrival time T_n of the nth vehicle and the departure time T'_{n-1} of the $(n-1)$th delayed vehicle are given by

$$(3.2) \qquad T_n = \sum_{i=1}^{n} X_i , \qquad T'_{n-1} = r + (n-1)\delta .$$

Introducing the new random variable Y_n and it's partial sum S_n, we rewrite equation *(3.1)* as

$$N_f = N + 1 = \min \{ n \mid S_n \geqq r-\delta \} \tag{3.3}$$

where

$$Y_n = X_n - \delta , \tag{3.4}$$

$$S_n = \sum_{i=1}^{n} Y_i . \tag{3.5}$$

It should be noticed that, if the interarrival time is restricted to be larger than δ, N becomes the number of renewal during the time $r-\delta$ in usual renewal process $\{Y_i\}$ where $Y_i \geqq 0$. Although the restriction mentioned may exist in real situation, we treat the problem without such a limitation.

Equation *(3.3)* shows that N_f is the first step at which the partial sum of mutually independent random variables exceeds the barrier at $r-\delta$. This type of problem was fully analysed in sequential analysis or in random walks. And it is known, [6], [8], that Wald's identity exists with relation to N_f, S_{N_f} and the moment generating function $\phi(\theta)$ for Y_i as follows,

$$\mathrm{E}[\ e^{\theta S_{N_f}} (\ \phi(\theta)\ )^{-N_f}\ ] = 1 \qquad \theta_t < \theta < \theta_a , \tag{3.6}$$

provided that $r-\delta>0$ and $\mathrm{E}[Y_i] > 0$. The parameters θ_a, θ_t and $\phi(\theta)$ are given by

$$\phi(\theta) = \mathrm{E}[\ e^{\theta Y_i}\ ] = \mathrm{E}[\ e^{\theta(X_i-\delta)}\ ], \tag{3.7}$$

θ_a = upper bound of region where $\phi(\theta)$ exists,

θ_t = unique real root of $\phi'(\theta) = 0$ which is negative when $\mathrm{E}[Y_i] > 0$.

Though Wald's identity holds in broader sense with a complex variable θ, as discussed in detail by Miller [6], the one stated above is enough to our purpose. Equation *(3.6)* may be differentiated any times with θ under the expectation sign. In the following $\phi(\theta)$ is assumed to be differentiable in the region surrounding $\theta=0$. Hence the interarrival time has any moment required.

Now introducing the first idle time I_f and substituting $N+1$ for N_f in *(3.6)*, we have

$$\mathrm{E}[\ e^{\theta(r-\delta+I_f)} (\ \phi(\theta)\ )^{-N-1}\ ] = 1, \tag{3.8}$$

where

(3.9) $$I_f = T_{Nf} - T'_N = S_{Nf} - (r-\delta).$$

The basic equation (3.8) exists in region of θ including $\theta=0$ provided that $r>\delta$ and $\mathbf{E}[Y_i]>0$, i.e. $\mu>\delta$. If we differentiate (3.8) with θ one or two times and put $\theta=0$, we obtain for $\mu>\delta$

(3.10) $$\mathbf{E}[N] = \frac{1}{\mu-\delta}(r - \mu + \mathbf{E}[I_f]),$$

$$\mathbf{E}[N^2] = \frac{1}{(\mu-\delta)^2}\mathbf{E}[N(\sigma^2+2(\mu-\delta)(r-\mu+I_f))-(r-\mu+I_f)^2+\sigma^2].$$

3.2 Moments of N

With a Poisson input stream the distribution of X_i is negative exponential. In this case, the first idle time I_f is independent on N and has the same exponential distribution as is shown easily. Then the left hand side of equation (3.8) simplifies to

$$\mathbf{E}[e^{\theta(I_f-\delta)}]\,\mathbf{E}[e^{\theta r}(\phi(\theta))^{-N-1}] = \mathbf{E}[e^{\theta r}(\phi(\theta))^{-N}].$$

And we have

(3.11) $$\mathbf{E}[e^{\theta r}(\phi(\theta))^{-N}] = 1.$$

From this equation the exact moment of N is derived for a Poisson input.

If the distribution of X_i is general, the first idle time I_f depends on N. And the distribution of I_f will be difficult to be represented explicitly. However, if the duration r of interruption is large compared with μ and σ, the contribution of I_f in equation (3.8) will be small. Then we may neglect the influence of I_f and may approximate I_f by X_i; equation (3.11) may be considered to exist approximately. Using that equation the approximated moment is derived for general inputs.

Differentiating (3.11) one or two times and putting $\theta=0$, we obtain for $\mu>\delta$

(3.12) $$\mathbf{E}[N] = \frac{r}{\mu-\delta},$$

(3.13) $$\mathbf{E}[N^2] = \frac{\sigma^2 r}{(\mu-\delta)^3} + \frac{r^2}{(\mu-\delta)^2} \quad \text{or} \quad \mathrm{Var}[N] = \frac{\sigma^2 r}{(\mu-\delta)^3},$$

rigorously with a Poisson input and approximately with general inputs. Moments of higher order would be given if necessary.

Above results are similar to what given in renewal theory for the number of renewal during long time r. The meaning and validity of apppoximation for general inputs will be realized in later section with relation to the bounds on $\mathbf{E}[N]$.

If transformed to customary representation in references, the results are

$$\mathrm{E}[N] = \frac{q\,r}{1 - q/s} \qquad q < s,$$

$$\mathbf{Var}[N] = \frac{Iqr}{(1 - q/s)^3} \qquad q < s,$$

where

$q = 1/\mu$, i.e., the average arrival rate of traffic,

$s = 1/\delta$, i.e., the saturation flow of traffic,

$$I = \frac{\text{variance of the number of vehicles arriving in } r}{\text{mean number of vehicles arriving in } r}.$$

In the transformation, the asymptotic relation $I=\sigma^2/\mu^2$ for large r was used. For a Poisson input this relation holds exactly, then the rigorousness was conserved.

3.3 Probability Distribution of N

The probability $p_n = Pr\{N=n\}$ may be derived from *(3.11)* if $\phi(\theta)$ is known. For a Poisson arrival stream with parameter λ, $\phi(\theta)$ is given from *(3.7)* by

$$\phi(\theta) = e^{-\theta\delta}\frac{\lambda}{\lambda-\theta} \qquad \theta < \lambda.$$

Using this $\phi(\theta)$ the left hand side of equation *(3.11)* is developed in power series of $\lambda-\theta$ as follows,

$$\begin{aligned} \mathrm{E}[e^{\theta r}(\phi(\theta))^{-N}] &= \sum_{n=0}^{\infty} e^{\theta(r+n\delta)}\lambda^{-n}(\lambda-\theta)^n p_n \\ &= \sum_{n=0}^{\infty} \lambda^{-n} e^{\lambda(r+n\delta)} p_n \sum_{k=0}^{\infty} \frac{(-(r+n\delta))^k}{k!}(\lambda-\theta)^{n+k} \\ &= \sum_{m=0}^{\infty}\{\sum_{n=0}^{m}\lambda^{-n}e^{\lambda(r+n\delta)}p_n\frac{(-(r+n\delta))^{m-n}}{(m-n)!}\}(\lambda-\theta)^m. \end{aligned}$$

This must be equal to unity, when $s>\lambda-\theta>0$. Then we have for coefficients of $(\lambda-\theta)^m$ following equations.

$$(3.14)\quad \begin{cases} \sum_{n=0}^{m} \lambda^{-n} e^{\lambda(r+n\delta)} p_n \dfrac{(-(r+n\delta))^{m-n}}{(m-n)!} = 0 & m \geqq 1, \\ e^{\lambda r} p_0 = 1. \end{cases}$$

The system of equations *(3.14)* gives the p_m in terms of $p_{m-1}, p_{m-2}, \cdots, p_0$. And it can be solved by a mathematical induction, making use of the relation

$$\sum_{n=0}^{m} \frac{(x+ny)^h}{n!(m-n)!} (-1)^{m-n} = 0 \qquad 0 \leqq h \leqq m-1, \quad h\text{:integer},$$

which follows from

$$\left[\frac{d^h}{dz^h}(1-e^z)^m\right]_{z\to 0} = 0 \qquad 0 \leqq h \leqq m-1, \quad h\text{:integer}.$$

After all, we know that

$$p_n = \lambda^n e^{-\lambda(r+n\delta)} \frac{r(r+n\delta)^{n-1}}{n!} \qquad n \geqq 0.$$

This distribution of N with a Poisson input stream is equivalent to the result given in Buckley and Wheeler [2].

For the general input flow, the approximated p_n may be induced with the same procedure. But in general, it is not easy to obtain it explicitly.

3.4 Bounds on E[N]

The upper and lower bounds on E[N] will be discussed in consideration of initial condition of interruption. In this section only, the distribution of X_1 is noted by $F_1(x)$ which differs from the distribution $F(x)$ of $X_2, X_3, \cdots$.

The distribution $F_1(x)$ has to reflect the initial condition of interruption. And if the interruption arises independently on the state of vehicle flow, $F_1(x)$ will be described by the limiting distribution of residual life time in renewal process $\{X_i\}$. This limiting distribution and it's mean are given in renewal theory by

$$F_1(x) = \int_0^x \frac{1-F(v)}{\mu} dv,$$

$$(3.15)\qquad E[X_1] = \int_0^\infty x\,dF_1(x) = \frac{\sigma^2+\mu^2}{2\mu}.$$

Now the expectation value of N conditional on X_1 is obtained applying equation *(3.10)* to the case $r \to r+\delta-x$ and adding unity to the result. And we have

$$E[N|X_1=x] = \frac{1}{\mu-\delta}\,(\,r-x+E[I_f|X_1=x]\,)\,.$$

For $x \geqq r$, the right hand side vanishes because $E[I_f|X_1=x]=x-r$ from *(3.2)* and *(3.9)*. Then the exact $E[N]$ is represented by

$$\text{(3.16)} \qquad E[N] = \frac{1}{\mu-\delta}\int_0^\infty (\,r-x+E[I_f|X_1=x]\,)\,dF_1(x)\,.$$

The lower bound for general inputs can be given from *(3.15)* and *(3.16)* by

$$E[N] \geqq \frac{1}{\mu-\delta}\int_0^\infty (\,r-x\,)\,dF_1(x) = \frac{r}{\mu-\delta}\left(1-\frac{\mu}{r}\cdot\frac{1+c_a^2}{2}\right),$$

where c_a is the coefficient of variation of interarrival time.

To obtain the upper bound, it needs some assumption on the tail distribution of X_i. Here we consider the γ-*MRLA* arrival flow defined by

$$\int_t^\infty \frac{1-F(v)}{1-F(t)}\,dv \leqq \gamma \qquad\qquad t \geqq 0\,,$$

as was done by Marshall [4] in the study of bounds on mean wait for usual *GI/G/1* queue. The γ-*MRLA*, or above inequality, means that the mean residual life time at an arbitrary time in renewal process $\{X_i\}$ is bounded above by γ. Then, $E[I_f|X_1=x, N=n]$ is also bounded above by γ, because I_f is the residual life time at the instant $r+n\delta$ in renewal process $\{X_i\}$. From this fact it follows that

$$E[I_f|X_1=x] = \sum_{n=0}^{\infty} E[I_f|X_1=x, N=n]\;Pr\{N=n\} \leqq \gamma,$$

and

$$E[I_f] = \int_0^\infty E[I_f|X_1=x]\,dF_1(x) \qquad \leqq \gamma.$$

Namely, the expectation value of I_f is bounded above by γ whether or not it is conditional on X_1 or N.

From *(3.16)* and above inequality, we have

$$E[N] \leqq \frac{1}{\mu-\delta}\left(r-\mu\cdot\frac{1+c_a^2}{2}+\gamma\right).$$

The upper bound with the γ-*MRLA* arrival flow was found. Especially for the

μ-*MRLA* arrival flow, we have

$$\frac{r}{\mu-\delta}\left(1-\frac{\mu}{r}\cdot\frac{1+c_a^2}{2}\right) \leqq \mathrm{E}[N] \leqq \frac{r}{\mu-\delta}\left(1+\frac{\mu}{r}\cdot\frac{1-c_a^2}{2}\right).$$

This shows that for a broad class of arrival distribution the approximated expression $\mathrm{E}[N]=r/(\mu-\delta)$ does not differ from the true value over μ/r in proportion, because $c_a \leqq 1$ for the μ-*MRLA* arrivals. When r tends to infinity, the relative error tends to zero. Therefore, it is possible to ignore the influence of initial condition or the first idle time insofar as r is considerably large compared with μ.

4. Delay

The expected delay caused by a single interruption will be given. Firstly, we define the random variable denoting the delay of the nth arriving vehicle by

$$(4.1) \qquad W_n = \begin{cases} T'_n - T_n = r - S_n & N \geqq n, \\ 0 & N < n. \end{cases}$$

Notice that W_n is the delay of the nth vehicle not in queue but in arriving stream and that it depends on N. From the definition, the difference $W_n - W_{n-1}$ is given by

$$W_n - W_{n-1} = \begin{cases} S_{n-1} - S_n = -Y_n & N \geqq n, \\ S_{n-1} - r & N = n-1, \\ 0 & N < n-1. \end{cases}$$

Taking the expectation value , we have

$$(4.2) \qquad \begin{aligned} \mathrm{E}[W_n]-\mathrm{E}[W_{n-1}] &= -\mathrm{E}[Y_n \mid N \geqq n]Pr\{N \geqq n\} - \mathrm{E}[r-S_{n-1} \mid N=n-1]Pr\{N=n-1\} \\ &= -\mathrm{E}[Y_n \mid N \geqq n-1]Pr\{N \geqq n-1\} - \mathrm{E}[r-S_{n-1}-Y_n \mid N=n-1]Pr\{N=n-1\}. \end{aligned}$$

Now the event $N \geqq n-1$ is decided by the values of $Y_1, Y_2, \cdots, Y_{n-1}$ from equation *(3.3)*; Y_n is independent on the condition $N \geqq n-1$. Therefore we find that $\mathrm{E}[Y_n \mid N \geqq n-1] = \mathrm{E}[Y_i] = \mu-\delta$. And from equation *(3.9)* it follows that $\mathrm{E}[r-S_{n-1}-Y_n \mid N=n-1] = \mathrm{E}[r-S_n \mid N=n-1] = \mathrm{E}[\delta - I_f \mid N=n-1]$. Using these two equations in *(4.2)* , we have for all $n \geqq 1$ with $W_0=r$

$$(4.3) \qquad \mathrm{E}[W_n] - \mathrm{E}[W_{n-1}] = -(\mu-\delta)\, Pr\{N \geqq n\} - (\mu - \mathrm{E}[I_f \mid N \geqq n-1])\, Pr\{N=n-1\}.$$

We can obtain the $E[W_m]$ adding above equation over $n=1, 2, 3, \cdots, m$ and using the relation

$$(\mu-\delta)\sum_{n=1}^{m} Pr\{N\geq n\} = (\mu-\delta)E[N] - (\mu-\delta)\sum_{n=m+1}^{\infty} Pr\{N\geq n\}$$
$$= r - (\mu-E[I_f]) - (\mu-\delta)\sum_{n=m+1}^{\infty} Pr\{N\geq n\}$$

which follows from *(3.10)* for $\mu > \delta$.
Then we have for $\mu > \delta$

$$E[W_m] = (\mu-\delta)\sum_{n=m+1}^{\infty} Pr\{N\geq n\} + (\mu-E[I_f|N\geq m])Pr\{N\geq m\} .$$

The expected total delay is given by

$$E[W] = \sum_{m=1}^{\infty} E[W_m]$$
$$= (\mu-\delta)\sum_{m=1}^{\infty} (m-1)Pr\{N\geq m\} + \sum_{m=1}^{\infty} (\mu-E[I_f|N\geq m])Pr\{N\geq m\}$$
$$= \frac{\mu-\delta}{2}(E[N^2]-E[N]) + \sum_{m=1}^{\infty} (\mu-E[I_f|N\geq m])Pr\{N\geq m\}.$$

Now, considering the relation $E[I_f|N\geq m]=E[I_f]=\mu$ in the above equation and using *(3.12)* and *(3.13)* for a Poisson input, or neglecting the second term and using *(3.12)* and *(3.13)* for other inputs, we obtain

$$(4.4) \qquad E[W] = \frac{1}{2}\left(\frac{r^2}{\mu-\delta} + r\left(\frac{\sigma^2}{(\mu-\delta)^2} -1\right) \right) \qquad \mu > \delta.$$

This expression for total delay is exact for a Poisson input and approximated for general inputs. Especially, the total delay for the μ-*MRLA* arrival flow is bounded by

$$\frac{\mu-\delta}{2}(E[N^2]-E[N]) \leq E[W] \leq \frac{\mu-\delta}{2}(E[N^2]- E[N]) + \mu E[N],$$

because $E[I_f|N\geq m] \leq \mu$ as described in section 3.4. This inequality shows that, if divided by $E[N]$, the expected delay per delayed vehicle is bounded within the mean interarrival time.

In the customary representation, the delay *(4.4)* is given by

$$E[W] = \frac{1}{2}\left(\frac{qr^2}{1-q/s} + r\left(\frac{I}{(1-q/s)^2}-1\right) \right) \qquad q < s.$$

The parameters q, s and I were defined in section 3.2.

It should be noticed that the approximated delay for general input is available when $q \gtrsim 1/r$ and that it does not vanish at $q=0$. Evidently, the exact expression for mean delay has to vanish at $q=0$. Hence, for practical use, it is appropriate to adopt a modified expression

$$\text{(4.6)} \qquad E[W] = \frac{1}{2}\left(\frac{qr^2}{1-q/s} + rI\left(\frac{1}{(1-q/s)^2}-1\right) \right) \qquad q<s.$$

In order to obtain this expression, we have reduced from (4.5) the residual value at $q=0$, that is $(I-1)r/2$. For a Poisson input the rigorousness is preserved yet, because $I=1$. And the modification affects the value of delay little for general inputs if $q \gtrsim 1/r$.

The first term of (4.5) or (4.6), which is equivalent to Clayton's expression for mean delay at a fixed-cycle traffic signal, is regarded as the contribution from the regularity of input flow. The second term may be considered to be the effect of fluctuation of input. The similar term is presented in Miller's and also in Newell's expression for mean delay, but in the latter without derivation.

5. Discussion

The method and results given in this paper are not sensitive to the details of model. Here, several aspects of the insensitivity will be discussed. Firstly, as was seen already, the influence of initial condition of blockage or of the first idle time was not essential insofar as r is large compared with μ. Next, the ending condition for queueing, equation (3.1), may be questioned because it allows the last delayed vehicle and the first undelayed one to close extremely. If necessary, it may be changed so as to keep the minimum headway δ between them by the replacement of T'_{n-1} with T'_n in (3.1). But the replacement causes mere modification of r to $r+\delta$ in all results. Furthermore, the minimum headway δ can be kept between any two vehicles if the interarrival distribution which vanishes below δ was adopted. In this case no matter arises, because the approximated expressions are described by moments of distribution. Finally, the precise regularity of interdeparture time is not an essential assumption required. With slight modofication we can extend our method to the case where the interdeparture time fluctuates. And it will be revealed that the similar approximated results are given, with δ denoting the mean interdeparture time and σ^2 denoting the sum of variances of interarrival time and interdeparture time. But in real traffic situation, the variance of interdeparture time will be very small.

Though the model analysed is rather simple, the results are applicable to the queue at a fixed-cycle traffic signal with light traffic. For, the overflow is known to be very small in wide range of traffic density. When the density increases and the degree of saturation exceeds *0.7* or *0.8*, the overflow will play the dominant role in delay or in queueing and must be taken into account.

An accidental interruption for a traffic stream due to a pedestrian-crossing or a vehicle-crossing at minor-major intersection is fully describrd by this model, as was stated in [2]. The results may be useful to measure the effect and relaxation time of such an interruption.

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