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A QUEUEING ANALYSIS OF CONVEYOR-SERVICED PRODUCTION STATION WITH GENERAL UNIT-ARRIVAL

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This paper analyses the conveyor-serviced production station that operates in conjunction with conveyor. As an operating policy, here we adopt the sequential range policy proposed by Beightler and Crisp, Jr. In-process storage is treated as a stationary imbedded Markov process with a general arrival of units. For an unloading station, the expected number of units in the reserve and the expected time of delay per unit produced are derived in the case where the capacity of the reserve is finite or infinite. An numerical example is given for the Erlangian arrival case. These results will be used for a loading station, as its analysis is identical to that for an unloading station if the variables are properly redefined.

1. Introduction

This paper develops a stochastic analysis of the conveyor-serviced production station with SRP, that is, the sequential range policy discussed in reference [1]. The purpose of adopting the SRP as an operating policy is to reduce the total amount of delay involved in obtaining units from the conveyor.

Here we limit to consider the unloading station that removes material from the conveyor, as the loading station that loads processed material to the conveyor, only reverses the unloading activity. Details of the physical system with the SRP are omitted here and see reference [1], except that the flow chart for the SRP is revised as Fig. 1.

The system is treated as a stationary imbedded Markov process and we take the range as $c(\geqq 0)$ units of time, whereas Beightler and Crisp were interested in studying the conveyor system to be treated properly as a stationary discrete Markov process in time and in space, and the range was taken as $c(=0,1,2,...)$ units of space. And the inter-arrival

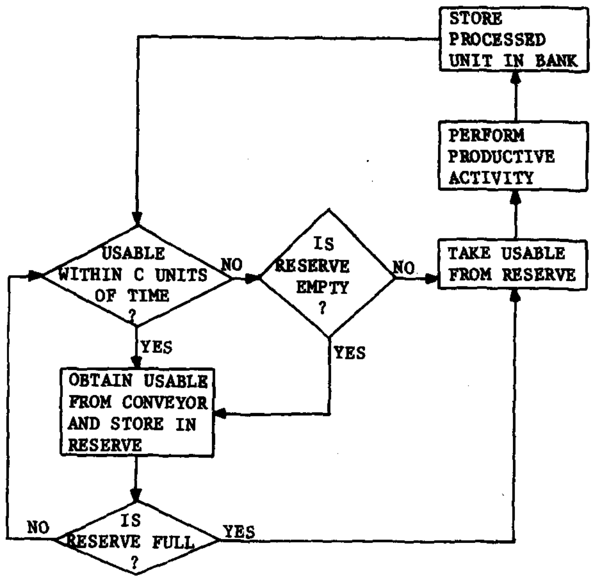
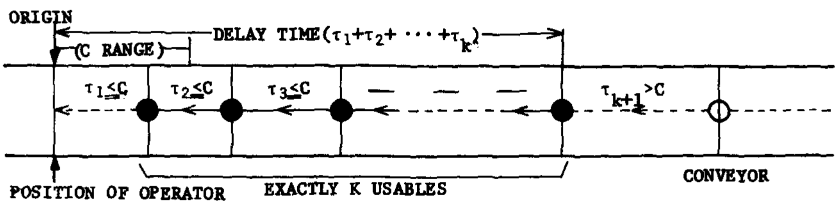
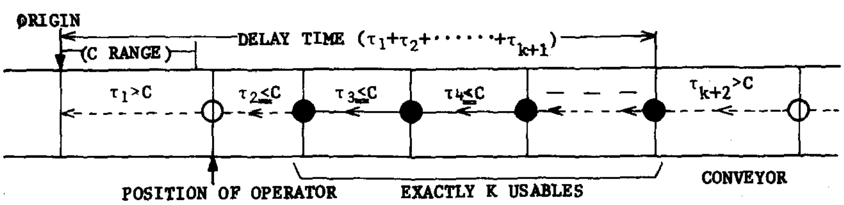


Fig. 1. Flow chart for sequential range policy (usables are units that arrive at an operator **within** the constant range of c units of time sequentially)



(i) Case when i=1



(ii) Case when i=2

Fig. 2. A sequence of exactly k usables that is indicated by a series of black circles

time of units on the conveyor is defined to distribute according to a general type, whereas in their paper one of the discrete types, Bernoulli distribution, was assumed.

In the following sections we consider the behavior of the system, and obtain the expected number of units in the reserve and the expected time of delay per unit produced in the case where the reserve capacity L is finite or infinite.

2. Several Definitions

Let the ordered arrival time of unit on the conveyor in the stationary state be denoted by t_i (i=1, 2, ...) such that t_1 is the arrival time of the first unit to arrive, measured from an arbitrarily chosen point in time, and t_{i+1} is that of the first unit to arrive after the arrival time t_i. Let the random variable (r.v.) τ_i represents the time between t_i and t_{i-1}, whereas the r.v. τ_1 represents the time between an arbitrarily chosen point in time and t_1.

We define the distribution function of the r.v. τ_i(i=2, 3, ...) as a general function A(t), which has a continuous derivative a(t)(=A'(t)), that is to say,

$$(1) \qquad \Pr(\tau_i \leqq t) = A(t),\ i = 2, 3, \ldots ,$$

and then the mean inter-arrival time, α^{-1} is given by

$$(2) \qquad \alpha^{-1} = \int_0^\infty \{1 - A(t)\}dt = \int_0^\infty ta(t)dt.$$

Using (1) and (2), the distribution function $A_o(t)$[1)] of the r.v. τ_1 is derived (see reference [2])and is as follows;

$$(3) \qquad \Pr(\tau_1 \leqq t) = A_o(t) = \alpha\int_0^t \{1 - A(u)\}du.$$

This is called the next arrival distribution function. The mean next arrival time, α_o^{-1} is given in the following

1) Here "o" is used for the meaning of "origin".

(4) $$\alpha_0^{-1} = \int_0^\infty \{1 - A_0(t)\}dt = \alpha\int_0^\infty t\{1 - A(t)\}dt = \frac{\alpha}{2}\int_0^\infty t^2 a(t)dt,$$

if exists.

In accordance with the SRP, a sequence of exactly k usables to be available from the conveyor is defined such that τ_i, τ_{i+1}, ... , $\tau_{i+k-1} \leqq c$, and $\tau_{i+k} > c$, where i = 1, 2, ... (see Fig. 2). When the probability that a sequence of exactly k usables is available is denoted by $P_0(k)$ or $P(k)$, according to the case when i = 1, or $i \geqq 2$ in the above definition, they are obtained as below;

(5) $$P_0(k) = \begin{cases} 1 - A_0(c) & , k = 0 \\ A_0(c)A^{k-1}(c)\{1 - A(c)\}, & k=1, 2, \ldots, \end{cases}$$

(6) $$P(k) = A^k(c)\{1 - A(c)\}, \ k = 0, 1, 2, \ldots .$$

In the subsequent development, we shall also have need for the probability that k or more usables are available from the conveyor. This probability corresponding to (5), or (6) respectively is obtained as follows;

(7) $$G_0(k) = \sum_{n=k}^{n=\infty} P_0(n) = \begin{cases} 1, & k = 0 \\ A_0(c)A^{k-1}(c), & k = 1, 2, \ldots , \end{cases}$$

(8) $$G(k) = \sum_{n=k}^{n=\infty} P(n) = A^k(c) , \quad k = 0, 1, 2, \ldots .$$

3. Markovian Analysis of the Reserve

Similar to the reference [1], the reserve may be treated as a Markov chain in which the state of the system is given by the number of holes or the number of units in the reserve, and the stages correspond to the points in time just after the storage of the processed unit in its bank.

Let P_n represent the steady-state probability in which there are n holes or L-n units in the reserve at the point in time immediately following the storage of the processed unit in the bank, and n runs from 1 to L.

Here we can easily obtain the simultaneous equations for the state probabilities P_n's and they are as follows;

$$(9)\begin{cases} P_1 = \sum_{i=1}^{i=L-1} G_o(i)P_i + G(L-1)P_L, \\ P_n = \sum_{i=0}^{i=L-n} P_o(i)P_{n+i-1} + P(L-n)P_L, \quad n = 2, 3, \ldots, L \end{cases}$$

Substituting (5)$\sim$(8) into (9), we can easily solve the simultaneous equations (9), and their solution is

$$(10) \qquad P_n = K[\{1 - A_o(c)\}/A(c)]^{n-1}, \qquad n = 1, 2, \ldots, L$$

where K is an arbitrary constant. To decide K in (10), we have to use the following normalization condition: $\sum_{n=1}^{n=L} P_n = 1$. And thus we have

$$(11) \qquad K = \begin{cases} (1 - A)/(1 - A^L), & \text{for } A \neq 1, \\ 1/L & \text{, for } A = 1 \end{cases}$$

where

$$A = \{1 - A_o(c)\}/A(c).$$

Using the above solution we can obtain the mean number of holes in the reserve, E(h), or the mean number of units, E(u), as follows;

$$
(12)\begin{cases}
E(h) = \sum_{n=1}^{n=L} nP_n \\
\quad = \begin{cases} 1/(1 - A) - LA^L/(1 - A^L), & \text{for } A \neq 1, \\ (1 + L)/2 & , \text{ for } A = 1 \end{cases} \\
E(u) = L - E(h) \\
\quad = \begin{cases} 1/(A - 1) - L/(A^L - 1), & \text{for } A \neq 1, \\ (L - 1)/2, & \text{for } A = 1. \end{cases}
\end{cases}
$$

Next we consider the particular case when $L = \infty$. From (10) and (11), we obtain

$$(13) \quad P_n = A^{n-1}(1 - A), \quad n = 1, 2, \dots ,$$

provided that the following condition is satisfied; $A < 1$, that is,

$$(14) \quad A(c) + A_o(c) > 1.$$

In this case we have, from (12)

$$(15) \quad E(h) = 1/(1 - A), \quad E(u) \to \infty.$$

In the case where $A(c) + A_o(c) < 1$, the steady state probability, q_n that there are n units in the reserve at the point in time just after the storage of the processed unit in the bank, is easily obtained using (10) and (11) and is as follows;

$$(16) \quad q_n = A^{-(n+1)}(A - 1), \quad n = 0, 1, \dots .$$

Using (16), we have

(17) $E(u) = \sum_{n=0}^{n=\infty} nq_n = 1/(A - 1), \quad E(h) \to \infty,$

where $A > 1$.

Needless to say, if $A = 1$, both $E(h)$ and $E(u)$ is not finite.

4. Distribution for the Time of Delay

Generally, cycle time at a station consists of the service time plus the delay (or idle) time. The time of delay per units produced, T is considered to be the time until the next procession of unit commences, measured from the point in time immediately following the storage of the procession of unit in the bank. The time spent to store the units in the reserve is neglected.

The pdf f(t) of r.v. T in the stationary state will be derived in this section. For this purpose three probability density functions are defined as below;

(18a) $$h(t) = \begin{cases} a(t)/A(c), & 0 \leqq t \leqq c \\ 0 & , \text{elsewhere} \end{cases}$$

(18b) $$h_o(t) = \begin{cases} a_o(t)/A_o(c), & 0 \leqq t \leqq c \\ 0 & , \text{elsewhere} \end{cases}$$

(18c) $$h_o^{(c)}(t) = \begin{cases} a_o(t)/\{1 - A_o(c)\}, & c \leqq t < \infty \\ 0 & , \text{elsewhere} \end{cases}$$

where

$$a_o(t) = A_o'(t) = \alpha\{1 - A(t)\}.$$

These represent the truncated distributions of the inter-arrival time of units, and using these notions, we can easily write down the explicit expression of the pdf f(t) as follows;

(19) $f(t) = f_1(t) + f_2(t) + f_3(t)$

in which

(20a) $f_1(t) = \delta(t)(1 - P_L)\{1 - A_o(c)\}$,

where $\delta(t)$ is the Dirac's δ-function,

(20b)
$$f_2(t) = \sum_{i=1}^{i=L} P_i \{\sum_{j=1}^{j=i} P_o(j) h_o * h^{*(j-1)}(t) + G_o(i+1) h_o * h^{*(i-1)}(t)\}, \quad t > 0$$

where notation (*) indicates the convolution of the concerned functions, and

(20c)
$$f_3(t) = P_L\{1 - A_o(c)\}\{\sum_{j=1}^{j=L} P(j-1) h_o^{(c)} * h^{*(j-1)}(t) + G(L) h_o^{(c)} * h^{*(L-1)}(t)\}, \qquad t \gtrless c.$$

Here the function $f_1(t)$ represents the term in the case of no-delay, the function $f_2(t)$ represents the term in the case where at least one unit is available in the first time-range, and the function $f_3(t)$ represents the term in the case where no unit is available in the first time-range.

Let denote the Laplace transform of a function L(.) by $\hat{L}(s)$ and from (19) and (20), $\hat{f}(s)$ is given as follows;

(21) $\hat{f}(s) = \hat{f}_1(s) + \hat{f}_2(s) + \hat{f}_3(s)$,

in which

$$\hat{f}_1(s) = (1 - P_L)\{1 - A_o(c)\}, \tag{22a}$$

$$\hat{f}_2(s) = A_o(c)\hat{h}_o(s) \sum_{i=1}^{i=L} P_i \hat{I}_i(s) \tag{22b}$$

where

$$\hat{I}_i(s) = \sum_{j=1}^{j=i} \{1 - A(c)\}\{A(c)\hat{h}(s)\}^{j-1} + A(c)\{A(c)\hat{h}(s)\}^{i-1}$$

$$= [1 - A(c) - \{A(c)\hat{h}(s)\}^i + A^i(c)\hat{h}^{i-1}(s)]/\{1 - A(c)\hat{h}(s)\}, \tag{23}$$

and

$$\hat{f}_3(s) = P_L \{1 - A_o(c)\}\hat{h}_o^{(c)}(s)\hat{I}_L(s) \tag{22c}$$

In the derivation of (22b) and (22c), (5) ∿ (8) have been used. After simple calculations and introducing another notations, $\hat{h}(s)$, $\hat{h}_o(s)$ and $\hat{h}_o^{(c)}(s)$ are expressed as below;

$$\hat{h}(s) = \int_0^c e^{-st}a(t)dt/A(c) \equiv \hat{a}(s, c)/A(c), \tag{24a}$$

$$\hat{h}_o(s) = \int_0^c e^{-st}a_o(t)dt/A_o(c) \equiv \hat{a}_o(s, c)/A_o(c), \tag{24b}$$

and

$$\hat{h}_o^{(c)}(s) = \int_c^\infty e^{-st}a_o(t)dt/\{1 - A_o(c)\}$$

$$= \{\hat{a}_o(s) - \hat{a}_o(s, c)\}/\{1 - A_o(c)\}, \tag{24c}$$

where $\hat{a}_o(s) = \hat{a}_o(s, \infty)$.

Substituting (24) into (22b) and (22c), we have

$$(25)\quad \begin{cases} \hat{f}_2(s) = \hat{a}_o(s,\ c)\ \sum_{i=1}^{i=L} P_i\hat{I}_i(s), \\ \hat{f}_3(s) = \{\hat{a}_o(s) - \hat{a}_o(s,\ c)\}\ P_L\hat{I}_L(s). \end{cases}$$

Therefore, substituting (22a) and (25) into (21), expression (21) becomes

$$\begin{aligned} \hat{f}(s) &= (1 - P_L)\{1 - A_o(c)\} \\ &\quad + \hat{a}_o(s,\ c)\ \sum_{i=1}^{i=L} P_i\hat{I}_i(s) + \{\hat{a}_o(s) - \hat{a}_o(s,\ c)\}P_L\hat{I}_L(s) \\ (26)\qquad &= (1 - P_L)\{1 - A_o(c)\} \\ &\quad + \hat{a}_o(s,\ c)\ \sum_{i=1}^{i=L-1} P_i\hat{I}_i(s) + \hat{a}_o(s)P_L\hat{I}_L(s), \end{aligned}$$

where $\hat{I}_i(s)$ is given in (23).

Here we are able to obtain the expression for the expected time of delay, $E(d)$, using (26). That is,

$$\begin{aligned} (27)\quad E(d) = -\hat{f}'(0) &= \alpha_o^{-1}(c) + \alpha^{-1}(c)A_o(c)/\{1 - A(c)\} \\ &\quad - (1 - A)[\alpha^{-1}(c)/\{1 - A(c)\} - A^{L-1}\{\alpha_o^{-1} - \alpha_o^{-1}(c) \\ &\quad + \alpha^{-1}(c)(1 - A_o(c))/(1 - A(c))\}]/(1 - A^L), \end{aligned}$$

where $A = \{1 - A_o(c)\}/A(c)$. (see Appendix for the above derivation)

In the case where $L = \infty$, $E(d)$ is given as below;

$$E(d)=\begin{cases} \alpha_o^{-1}(c) + \alpha^{-1}(c)\{1 - A_o(c)\}/\ A(c), & \text{for } A(c) + A_o(c) > 1 \\ [1 - A(c)/\{1 - A_o(c)\}]\alpha_o^{-1} + \alpha^{-1}(c) + \alpha_o^{-1}(c)A(c)/\{1 - A_o(c)\}, & \end{cases}$$

$$(28)\qquad\qquad \text{for } A(c) + A_o(c) < 1$$

Evaluation of min.E(d) is obtained by giving the adequate value of c for any given arrival time distribution A(t) and capacity L, using (27) or (28) and is given in §6 for some particular cases.

The behavior of the function E(d) may be clarified by studying its properties as c approaches its extreme limits:

$$(29)\quad E(d) = \alpha_o^{-1}, \qquad \text{for } c = 0$$

$$(30)\quad \lim_{c\to\infty} E(d) = \alpha_o^{-1}$$

5. The Case of Erlangian Inter-arrival Time Distribution

When the distribution of inter-arrival time is the Erlangian distribution with phase ℓ, A(t) and $A_o(t)$ is given as follows;

$$(31)\begin{cases} A(t) = 1 - e^{-\ell\lambda t}\sum_{i=0}^{i=\ell-1}(\ell\lambda t)^i/i!, \\ A_o(t) = 1 - e^{-\ell\lambda t}\sum_{i=0}^{i=\ell-1}(1 - i/\ell)(\ell\lambda t)^i/i!. \end{cases}$$

Differentiating the formulas (31), we have

$$(32)\begin{cases} a(t) = t^{\ell-1}e^{-\ell\lambda t}(\ell\lambda)^\ell/(\ell - 1)!, \\ a_o(t) = \lambda e^{-\ell\lambda t}\sum_{i=0}^{i=\ell-1}(\ell\lambda t)^i/i!. \end{cases}$$

From (31) and (32), various parameters to be used are obtained as below;

$$(33a)\quad \alpha^{-1}(c) = \Gamma_{\ell\lambda c}(\ell + 1)/(\lambda\ell!),$$

where $\Gamma_\alpha(\beta) = \int_0^\alpha e^{-x}x^{\beta-1}dx$,

$$(33b)\quad \alpha_o^{-1}(c) = 1/(\lambda\ell^2)\sum_{i=0}^{i=\ell-1}\Gamma_{\ell\lambda c}(i + 2)/i!$$

and

$$(33c)\quad \alpha_o^{-1} = (1 + \ell^{-1})/(2\lambda).$$

Using the formulas (31) and (33), the expressions of E(u) and E(d) for the Erlangian case will be, if necessary, obtained, but those for the case when $\ell=1$, that is, the negative exponential case, are written here:

$$E(u) = (1 - e^{-\lambda c})/(2e^{-\lambda c} - 1) + L/(1 - A_1^L),$$

where $A_1 = e^{-\lambda c}/(1 - e^{-\lambda c})$, and

$$E(d) = \lambda^{-1}\{1 - \lambda c A_1(1 - A_1^{L-1})/(1 - A_1^L)\}$$

and

$$E(d) = \lambda^{-1}, \quad \text{for } c = 0,$$

$$\lim_{c\to\infty} E(d) = \lambda^{-1}.$$

Especially for the case when $L=\infty$, we have

$$E(u) = \begin{cases} \infty, & \text{for } e^{-\lambda c} < 1/2 \ (\lambda c > 0.693 \ldots) \\ (1 - e^{-\lambda c})/(2e^{-\lambda c} - 1), & \text{for } e^{-\lambda c} > 1/2 \end{cases}$$

$$E(h) = \begin{cases} (1 - e^{-\lambda c})/(1 - 2e^{-\lambda c}), & \text{for } e^{-\lambda c} < 1/2 \\ \infty, & \text{for } e^{-\lambda c} > 1/2 \end{cases}$$

and

$$E(d) = \begin{cases} \lambda^{-1}\{1 - \lambda c e^{-\lambda c}/(1 - e^{-\lambda c})\}, & \text{for } e^{-\lambda c} < 1/2 \\ \lambda^{-1}(1 - \lambda c), & \text{for } e^{-\lambda c} > 1/2. \end{cases}$$

6. Numerical Example[2)]

As an numerical example, the behaviors of E(u) and E(d) are presented here for the Erlangian arrival case. In this case the values of E(u) are calculated from the formula (12) in the case where $L<\infty$, and (17) in the

2) This numerical calculation was done by TOSBAC 3400-41 in the Computing Center of Hiroshima University.

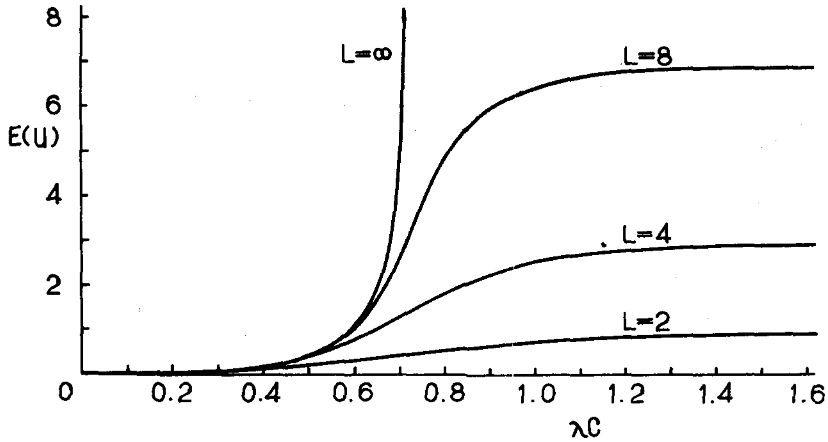


Fig. 3. Graph of mean reserve size, E(u), case when ℓ=4 (graph in case where L=1 concides with horizontal axis and is omitted)

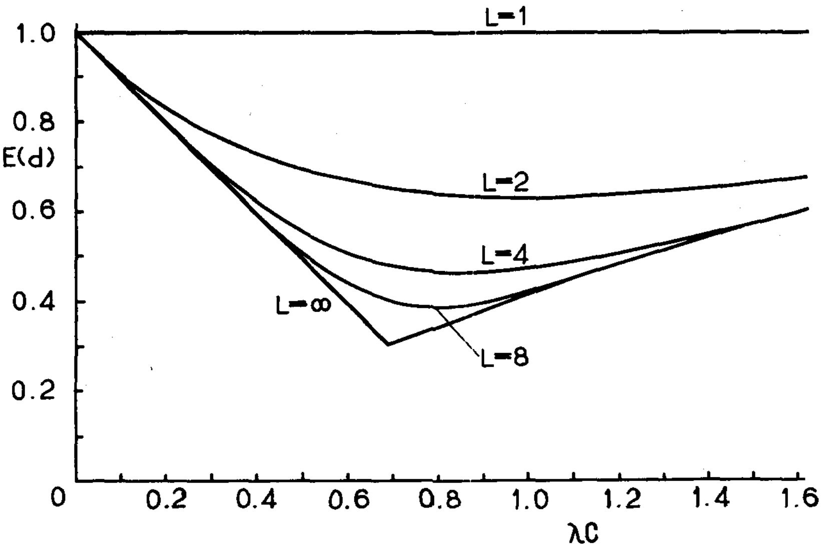


Fig. 4. Graph of λE(d), case when ℓ=1

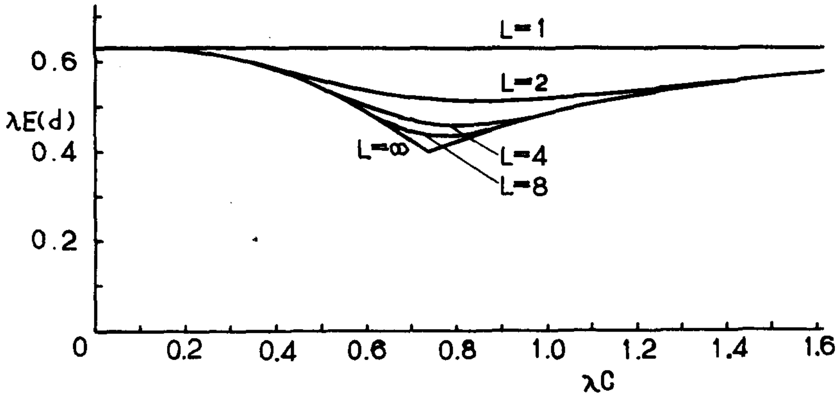


Fig. 5. Graph of λE(d), case when ℓ=4

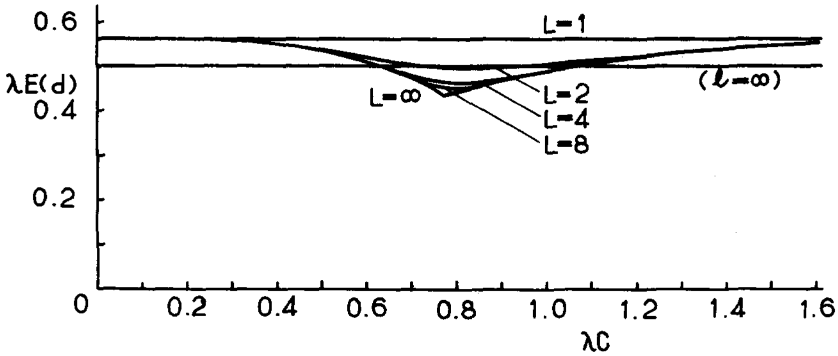


Fig. 6. Graph of λE(d), case when ℓ=8(∞)

Tab. 1. Table of λc_m and $\lambda E_m(d)$ (values are round number)

l	λc_m	$\lambda E_m(d)$
1	0.6931	0.3069
2	0.7086	0.3596
3	0.7231	0.3858
4	0.7352	0.4022
5	0.7453	0.4137
6	0.7539	0.4223
7	0.7614	0.4290
8	0.7680	0.4344
9	0.7738	0.4390
10	0.7791	0.4428
15	0.7994	0.4557
20	0.8137	0.4633
∞	1.0000	0.5000

case where $L=\infty$, substituting the values of $A(c)$ and $A_0(c)$ to be given by the formula (31). And also those of $E(d)$ are calculated from the formula (27) in the case where $L<\infty$, and (28) in the case where $L=\infty$, substituting the values of $\alpha^{-1}(c)$, $\alpha_0^{-1}(c)$ and α_0^{-1} which are obtained from the formula (33).

Fig.3 shows an example for the behavior of $E(u)$ in the case where the Erlangian phase $\ell=4$. Fig. 4, 5 and 6 show the behavior of $\lambda E(d)$ in the case where $\ell= 1$, 4 and 8 respectively.

Viewing Fig. 4, 5 and 6, we see that, needless to say, the minimum of the mean delay time diminishes with L, the capacity of the reserve, and for the case where L is not small it occurs in the neibourhood of λc_m, in which c_m is the solution of the following equation; $A(c) + A_o(c) = 1$. Its mean delay time, $E_m(d)$ is given as follows; $E_m(d) = \alpha_o^{-1}(c_m) + \alpha^{-1}(c_m)$.

Tab. 1 indicates the values of λc_m and$\lambda E_m(d)$ when $\ell=1(1)10$, 15, 20 and ∞.

According to the increase of the value of ℓ, the value $\lambda E_m(d)$ (or $E_m(d)$) increases slowly toward 1/2 (or $1/(2\lambda)$) when $L= \infty$, but the constant value $\lambda E(d)$ $(=(1 + \ell^{-1})/2)$ in the case where L=1 decreases monotonously from 1 to 1/2. The effect of the decrease of the mean delay by the increase of the capacity L diminishes gradually with the increase of the value ℓ, and is none in the case where $\ell=\infty$. It is noticed that the minimum $\lambda E(d)$, the minimum value of $\lambda E_m(d)$, occurs at the minimum point, $\lambda c_m = 0.6931 \ldots$ in the case where $\ell=1$ and $L=\infty$.

Acknowledgement

The authors are grateful to Mitsuru Kuroda of Gifu University to the valuable suggestions for the numerical calculations.

References

[1] C.S. Beightler and R.M. Crisp, Jr., "A Discrete-time Queuing Analysis of Conveyor-serviced Production Stations," Opns. Res., 16, 986-1001, (1968).

[2] J. Riordan, Stochastic Service Systems, Wiley, New York, 11-12, (1962).

Appendix

E(d) is derived as follows.

From (23), we have

$$\hat{I}_i(0) = 1,$$

$$\hat{I}_i'(0) = \hat{a}'(0,c)\{1 - A^{i-1}(c)\}/\{1- A(c)\}.$$

Also from (24), we have the following formulas;

$$\hat{a}(0, c) = A(c), \qquad \hat{a}_o(0, c) = A_o(c),$$

$$\hat{a}'(0, c) = - \int_0^c ta(t)dt \equiv - \alpha^{-1}(c),$$

$$\hat{a}_o'(0, c) = - \int_0^c ta_o(t)dt \equiv - \alpha_o^{-1}(c), \qquad \hat{a}_o'(0) = - \alpha_o^{-1}.$$

From (26), it follows that

$$\text{(A1)} \qquad E(d) = - \hat{f}'(0) = - \hat{a}_o'(0, c) - \hat{a}_o(0, c) \sum_{i=1}^{i=L} P_i\hat{I}_i'(0)$$

$$- P_L[\hat{a}_o'(0) - \hat{a}_o'(0, c) + \{1 - \hat{a}_o(0, c)\}\hat{I}_L'(0)],$$

and upon substituting of (1) and (2) into (A1), we have

$$\text{(A2)} \qquad E(d) = \alpha_o^{-1}(c) + A_o(c)\alpha^{-1}(c)[\sum_{i=1}^{i=L}P_i\{1 - A^{i-1}(c)\}]/\{1 - A(c)\}$$

$$+ P_L[\alpha_o^{-1} - \alpha_o^{-1}(c) +\alpha^{-1}(c)\{1 - A_o(c)\}\{1 - A^{L-1}(c)\}/\{1 - A(c)\}].$$

Substituting (10) and (11) into (A2) and after simple calculations, we obtain (27).