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OPTIMAL REPLACEMENT POLICY FOR MINIMAL REPAIR MODEL

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Abstract

The purpose of this paper is to generalize the 'minimal repair model' proposed by R.Barlow and L.Hunter, by introducing a breakdown cost. It is natural to consider that the replacement cost for failed system is larger than that of unfailed system. This additional cost is called 'breakdown cost'. For this system, the optimal maintenance policy is the following '(t,T)-policy'.

> "Replace a system when the first failure after t hours operating occurs or the total operating time reaches T ($0 \leq t \leq T$), whichever occurs first. For the intervening failures repair it."

A computational algorithm for the (t,T)-policy is obtained.

1. Introduction

For a complex system, it may be too expensive to replace or overhaul a system at any failure occasion. Naturally, we have to repair and use it again. But, in such a case, we may expect the mean life of a repaired system to be less than that of a new one. R.Barlow and L.Hunter [1] proposed a maintenance model called 'minimal repair model' which reflects the situation stated above. In this model, it is assumed that the system failure rate is not disturbed by any (minimal) repair of failures between successive replacements and the system regenerates completely after a replacement. In some practical situations, on the other hand, the average cost of maintenance made after failure exceeds that before failure. Taking into account of this fact, we shall consider a generalization of the above model by introducing a breakdown cost suffered for each failed system.

The purpose of this paper is to show that if the criterion is to minimize the expected total discounted cost, the optimal maintenance policy has the following form.

> "Replace a system when the first failure after t hours operating occurs or the total operating time reaches T, whichever occurs first, but for the intervening failures repair it on these occasions. ($0\leqq t\leqq T\leqq\infty$, $T>0$)"

This policy will be called '(t,T)-policy'. We also present a useful computational algorithm of the optimal (t,T)-policy under the average cost criterion.

In the next section, the problem and the assumptions are formulated precisely. In this paper, the word 'repair' means any action such that the failure rate remains unchanged and the word 'replacement' means the one such that the system renews completely, after performing respective actions. The word 'mainteneance' is a general name for these two actions.

2. Problem Formulation

Let the life distribution of a new system be F(x). Put for all $x\geqq 0$,

$$\bar{F}(x) = 1-F(x) \ , \quad q(x) = F'(x)/\bar{F}(x) \ , \quad Q(x) = \int_0^x q(y)dy \ , \tag{1}$$

where q(x) is the failure rate. It is well known that

$$\bar{F}(x) = e^{-Q(x)}. \tag{2}$$

We assume that

$$q(x) \text{ is differentiable and strictly increasing to infinity.} \tag{3}$$

Let $\Delta(y)$ denote the residual life of a system of which age is $y\geqq 0$. $\Delta(0)$ means the life of a new system. Let $F_y(x)$ be the distribution function of $\Delta(y)$ and put

$$\bar{F}_y(x) = 1-F_y(x) \ , \tag{4}$$

where $F_0(x)=F(x)$. Since the failure rate of $F_y(x)$ is given by $q(x+y)$, which is the assumption itself that the failure rate remains undisturbed, it follows that

$$\bar{F}_y(x) = e^{-[Q(x+y)-Q(y)]}. \tag{5}$$

It is to be noted that $\Delta(y)$ can be defined independently of the failure history of a system.

The following cost structure is imposed for our model. A cost R is suffered for each unfailed system which is replaced. A cost R' is suffered for each failed system which is replaced; this includes all costs resulting from failure and its replacement. We shall put $D=R'-R$, which will represent the additional cost caused by breakdown. We shall call this 'breakdown cost'. Since the failure rate remains unchanged by any repair, there is no advantage to be gained by repairing a good system prior to actual failure. Hence, we never repair a good system. Denote by C the cost suffered for each failed system which is repaired; this of course includes all costs of failure and its repair. It will be natural to suppose that

$$R' \geqq R > 0\ , \quad R' \geqq C > 0, \quad C \geqq D = R'-R \geqq 0\ . \tag{6}$$

If D=0, our model reduces to the original one seen in R.Barlow and F.Proschan [2;pp.96-98]. For convenience the maintenance time is not taken into account, but this assumption may be little restrictive by regarding maintenance cost as imputed cost including maintenance time. Further, we assume that failures are instantly detected and repaired or replaced.

We shall now characterize the class of possible maintenance policies. At any instant of time, the following alternative actions are to be considered;

A_1 ; Keep the present system.

A_2 ; Repair the present system.

A_3 ; Replace the present system.

A policy, to be denoted by p, is a prescription for taking actions at each point in time. Since, by virtue of (5), the future failure property of a system does not depend upon its history but only upon its present age, it suffices to consider policies that are independent of the history. Thus, a policy p is a {1, 2,3}-valued function p(x,i), where (x,i) denotes the 'state' of the system in current use, that is, $x \geq 0$ is its age and it is good or failed according as i=0 or 1. One interprets p(x,i)=j, j=1,2,3, as follows; at any time point if the state of a system is (x,i), then action A_j is taken. Any policy satisfies $p(x,0) \neq 2$ and $p(x,1) \neq 1$, since one never repairs a good system and never keeps a failed one. We assume for simplicity p(x,i) to be continuous to the right over all $x \geq 0$, for each i.

Using the notation defined above, the (t,T)-policy stated in the introduction can be expressed as

$$\begin{cases} p(x,0) = 1 \ , \quad x < T \ , \quad p(x,1) = 2 \ , \quad x < t \\ p(x,0) = 3 \ , \quad x \geq T \ , \quad p(x,1) = 3 \ , \quad x \geq t \ . \end{cases} \tag{7}$$

It is to be noted that if t=0, t=T, and $T=\infty$, then the (t,T)-type policies reduce to Policies I, II, and II', respectively, which were proposed and discussed by R.Barlow and L.Hunter [1] and H.Morimura [3].

3. Optimality of the (t,T)-Policy

In this section, the criterion for optimal policy is taken to minimize the expected total discounted maintenance cost over an infinite planning horizon, i.e., $E\int_0^\infty e^{-\alpha x} dM_p(x)$, where $M_p(x)$ denotes the accumulated maintenance cost up to time x when policy p is used, and $\alpha>0$ is the discount rate.

In what follows, we shall explore the property of an optimal policy by applying the usual technique of dynamic programming. Let g(x,i) denote the expected total discounted cost incurred when the initial state is (x,i) and an optimal policy is employed from time zero onward. If a system is good (i=0), we either keep or replace it, hence it must be true that

$$g(x,0) \leq R+g(0,0) \ , \tag{8}$$

where the right side is the consequence of immediate replacement. If, on the other hand, a system is failed (i=1), we either repair or replace it, so that

$$g(x,1) = \min [\ C+g(x,0)\ ,\ R'+g(0,0)\ ] \ . \qquad (9)$$

Hence, if we define

$$L = [\ x \mid g(x,0) < R+g(0,0)\ ]\ , \quad M = [\ x \mid C+g(x,0) < R'+g(0,0)\ ]\ , \qquad (10)$$

then for a good (failed) system, action $A_1(A_2)$ is better than action A_3 if its age x belongs to L(M), and otherwise action A_3 is as good as or better that action $A_1(A_2)$, where words and symbols in parentheses should be read together. But, as will later be shown,

$$g(x,0) \text{ is increasing and continuous in } x \geqq 0\ , \qquad (11)$$

from which we find that

$$L = [0,T) \quad \text{and} \quad M = [0,t) \qquad (12)$$

for some T and t, respectively, with $0 \leqq t \leqq T \leqq \infty$ and $T>0$, by virtue of (6). Thus, (7) is readily obtained. That is, an optimal policy is the (t,T)-policy.

Here we shall inquire into when we expect the extreme cases mentioned at the end of Section 2.

i) If C=R', then $M=\phi$, i.e., t=0, which is Policy I with parameter T.

ii) If C=D, then L=M, i.e., t=T, which is Policy II with Parameter T(=t).

iii) If D=0(R=R'), it may be intuitively obvious that a good system should not be replaced, since we can use it until the next failure without any additional cost. To prove this, suppose that one starts with a good state (x,0). If g(x) denotes the expected total discounted cost obtained by taking action A_1 up to the first failure and using an optimal policy thereafter, it follows that

$$g(x,0) \leqq g(x) = \int_0^\infty e^{-\alpha y} g(x+y,1)dF_x(y)\ . \qquad (13)$$

But, using (9) with R'=R, we have

$$g(x) \leqq [R+g(0,0)]\int_0^\infty e^{-\alpha y}dF_x(y) < R+g(0,0)\ . \qquad (14)$$

Therefore, $g(x,0)<R+g(0,0)$ for all $x\geqq 0$, which means $L=[0,\infty)$, i.e., $T=\infty$, which is Policy II' with parameter t. Notice that the optimality of Policy II' has been proved for the original minimal repair model with no breakdown cost.

It remains to prove (11). Before doing so, we shall state here the concept of nonhomogeneous Poisson processes ('NPP', for short) which is rather well known but will play an important role in our arguments.

> "A counting process $\{N(t);\ t\geqq 0\}$ is said to be an NPP with intensity function $\lambda(t)\geqq 0$ if i) $N(0)=0$, ii) $\{N(t);\ t\geqq 0\}$ has independent increments, iii) P[two or more events in $(t,t+h)]=o(h)$, and iv) P[exactly one event in $(t,t+h)]=\lambda(t)h+o(h)$. [4; p.24]

By the definition, it is easily verified that if $\{N'(t);\ t\geqq 0\}$ and $\{N''(t);\ t\geqq 0\}$ are two independent NPP's with intensity functions $\lambda'(t)$ and $\lambda''(t)$, respectively, the pooled process $\{N(t);\ t\geqq 0\}$, $N(t)=N'(t)+N''(t)$, is also an NPP with intensity function $\lambda(t)=\lambda'(t)+\lambda''(t)$.

With the above preparation, we now turn to the proof of (11). Suppose $x\leqq y$ and consider two systems $S(x)$ and $S(y)$ of which initial states are $(x,0)$ and $(y,0)$, respectively. Let $X(t)$ and $Y(t)$ denote the total numbers of failures which have occured up to time t, given that one starts respectively with $S(x)$ and $S(y)$ at time zero and always keeps them unless they fail, in which case only repair action is taken. By the assumption that the failure rate is not disturbed by repair action, $\{X(t);\ t\geqq 0\}$ and $\{Y(t);\ t\geqq 0\}$ are independent NPP's with intensity functions $q(x+t)$ and $q(y+t)$, respectively. We next introduce a supplementary NPP $\{W(t);\ t\geqq 0\}$, which is independent of the above two processes, by letting its intensity function be $q(y+t)-q(x+t)\geqq 0$. If we put

$$Z(t) = X(t) + W(t)\ , \quad t \geqq 0\ , \tag{15}$$

then $\{Z(t);\ t\geqq 0\}$ is also an NPP with intensity function $q(y+t)$. Hence, $\{Y(t);\ t\geqq 0\}$ and $\{Z(t);\ t\geqq 0\}$ are independent and identical processes. On the other hand, the state of $S(y)$ at time t can be written as $(y+t, Y(t)-Y(t-0))$, because if t is a failure epoch of $S(y)$, then $Y(t)-Y(t-0)=1$, and otherwise $=0$. Keeping this in mind and letting p be an optimal policy, define

$$\tau = \inf [\ t \mid p(y+t, Y(t)-Y(t-0))=3\ ] \tag{16}$$

and

$$\tau' = \inf [\ t \mid p(y+t, Z(t)-Z(t-0))=3\ ] , \tag{17}$$

where if $p(y+t,Y(t)-Y(t-0))\neq 3$ and $p(y+t,Z(t)-Z(t-0))\neq 3$ for all $t \geqq 0$, we put $\tau=\infty$ and $\tau'=\infty$, respectively. Under the optimal policy p, τ is the first replacement time of S(y) and the replacement cost at that time is given by R or R' according as $Y(\tau)-Y(\tau-0)=0$ or 1. Further, $[\{Y(t);\ t\geqq 0\},\tau]$ and $[\{Z(t);\ t\geqq 0\},\tau']$ are apparently independent and identically distributed, though the latter may not have a practical interpretation. Hence, it follows that

$$\begin{aligned} g(y,0) &= E\{\ C\int_0^{\tau-0} e^{-\alpha t}dY(t) + e^{-\alpha\tau}[D\cdot(Y(\tau)-Y(\tau-0))+R+g(0,0)]\ \} \\ &= E\{\ C\int_0^{\tau'-0} e^{-\alpha t}dZ(t) + e^{-\alpha\tau'}[D\cdot(Z(\tau')-Z(\tau'-0))+R+g(0,0)]\ \} . \end{aligned} \tag{18}$$

Since, by (15), $Z(t)-X(t)$ is increasing in $t\geqq 0$, we have

$$g(y,0) \geqq E\{\ C\int_0^{\tau'-0} e^{-\alpha t}dX(t) + e^{-\alpha\tau'}[D\cdot(X(\tau')-X(\tau'-0))+R+g(0,0)]\ \} . \tag{19}$$

But, the right side of (19), which represents the expected total discounted cost obtained by replacing S(x) at time τ' and using an optimal policy thereafter, is no smaller than g(x,0), the minimum that can be achieved starting from S(x). Therefore,

$$g(y,0) \geqq g(x,0) \quad \text{for} \quad y \geqq x , \tag{20}$$

which is the first assertion of (11). The proof of the second will proceed by the same technique as above, hence we shall mention the point briefly. Change the roles of S(y) and S(x) by S(x-0) and S(x+0), respectively. Then, noting the continuity of q(x), one will get $g(x-0,0)\geqq g(x+0,0)$, hence $g(x-0,0)=g(x+0,0)$, since g(x,0) is increasing in x, as shown above. Thus (11) is proven.

4. Computation of Optimal Policies

Once we have demonstrated the structure of the optimal policy, it remains to determine its parameter values. In this section, the criterion used is to minimize the expected total (undiscounted) maintenance cost per unit time, which we call the 'average cost', for short.

The first problem is to evaluate the average cost $A(t,T)$ of a given (t,T)-policy. Under this policy, the expected replacement interval and the expected total maintenance cost in this interval are given by

$$E \min [\ t+\Delta(t),\ T\ ] = t + \int_0^{T-t} \bar{F}_t(x)dx \tag{21}$$

and

$$CQ(t) + [\ F_t(T-t)R'+\bar{F}_t(T-t)R\ ] = CQ(t) + F_t(T-t)D + R\ , \tag{22}$$

respectively. Hence, by using the renewal reward theorem [4; pp.51-54], we have

$$A(t,T) = \frac{CQ(t) + F_t(T-t)D + R}{t + \delta(t,T)}\ , \quad (\ 0 \leqq t \leqq T \leqq \infty\ , \quad T > 0\ ) \tag{23}$$

where we put

$$\delta(t,T) = \int_0^{T-t} \bar{F}_t(x)dx\ . \tag{24}$$

In what follows, we shall attempt to minimize $A(t,T)$ with respect to (t,T). We assume for simplicity that

$$R' > C > D > 0\ , \quad (\ R' > R > 0 \quad \text{since} \quad R=R'-D\ )\ , \tag{25}$$

instead of (6). This assumption will avoid the extreme cases stated in the previous section. In fact, we can see that if $(\hat{t},\hat{T})$ is an optimal pair minimizing $A(t,T)$, then

$$0 < \hat{t} < \hat{T} < \infty\ , \tag{26}$$

of which proof will be presented at the end of this section.

Differentiating $A(t,T)$ with respect to t and T, respectively, we have

$$\frac{\partial A(t,T)}{\partial t} = \frac{q(t)}{[t+\delta(t,T)]^2}\, \Phi(t,T)\ , \tag{27}$$

where

$$\Phi(t,T) = [C-\overline{F}_t(T-t)D][t+\delta(t,T)] - [CQ(t)+F_t(T-t)D+R]\delta(t,T) \ , \tag{28}$$

and

$$\frac{\partial A(t,T)}{\partial T} = \frac{\overline{F}_t(T-t)}{[t+\delta(t,T)]^2} \Psi(t,T) \ , \tag{29}$$

where

$$\Psi(t,T) = Dq(T)[t+\delta(t,T)] - [CQ(t)+F_t(T-t)D+R] \ . \tag{30}$$

By virtue of (26), a necessary condition that a pair $(\hat{t},\hat{T})$ minimizes $A(t,T)$ is that it satisfies $\partial A(t,T)/\partial t = \partial A(t,T)/\partial T = 0$, or equivalently,

$$\Phi(t,T) = \Psi(t,T) = 0 \ , \tag{31}$$

from which it follows that

$$q(T)\delta(t,T) + \overline{F}_t(T-t) - C/D = 0 \tag{32}$$

and

$$q(T) - [\ CQ(t)+D+R-C\ ]/Dt = 0 \ , \tag{33}$$

and moreover, from $\Psi(t,T)=0$, we find

$$A(t,T) = Dq(T) \ . \tag{34}$$

If a $(\hat{t},\hat{T})$-policy is optimal, then $(\hat{t},\hat{T})$ is of course a solution of (32) and (33), and the resulting minimum value of $A(t,T)$ is given by $Dq(\hat{T})$. Therefore, $(\hat{t},\hat{T})$ must have the minimum T among all (t,T)'s that satisfy conditions (32) and (33), and consequently, $\hat{T}$ is unique. On the other hand, denoting the left-hand side of (32) by $B(t,T)$, it follows that

$$\begin{aligned} \partial B(t,T)/\partial t &= q(T)[\ q(t)\delta(t,T) - 1\ ] + q(t)\overline{F}_t(T-t) \\ &\leqq q(T)[\ \int_0^{T-t} q(x+t)\overline{F}_t(x)dx - 1\ ] + q(t)\overline{F}_t(T-t) \\ &= -[\ q(T) - q(t)\ ]\overline{F}_t(T-t) < 0 \quad \text{for} \quad 0 \leqq t < T \end{aligned} \tag{35}$$

and

$$\partial B(t,T)/\partial T = q'(T)\delta(t,T) > 0 \quad \text{for} \quad T > t \ . \tag{36}$$

Further, $B(t,t)=1-(C/D)<0$ and $B(t,\infty)=\infty$ for all $t\geqq 0$. Hence, if we let $T(t)$ be T that satisfies (32) for each t, then $T(t)$ is a strictly increasing function of t, by the implicit function theorem.

According to what precedes, one can conclude that the optimal $(\hat{t},\hat{T})$-policy is uniquely determined; $\hat{t}$ is the minimum zero of

$$b(t) = q(T)\delta(t,T) + \overline{F}_t(T-t) - C/D , \tag{37}$$

where

$$T = q^{-1}\{ [CQ(t)+D+R-C]/Dt \} , \quad (0 < t < T) , \tag{38}$$

and $\hat{T}$ is given by (38) with $t=\hat{t}$. Moreover, the resulting minimum average cost is

$$A(\hat{t},\hat{T}) = Dq(\hat{T}) . \tag{39}$$

Since $b(t)$ is a function of t only, its minimum zero will be relatively easy to compute. By (38), we have $T \to \infty$ as $t \to 0$, so that (37) implies that $b(t)$ is positive to the left of where $b(t)$ crosses zero for the first time.

As a numerical example, let

$$q(x) = x , \quad D = 4 , \quad C = 5 , \quad R = 6 , \quad (R' = 10) .$$

Then,

$$b(t) = T\int_t^T e^{-(x^2-t^2)/2}dx + e^{-(T^2-t^2)/2} - 5/4 \quad \text{where} \quad T = 5(t^2+2)/8t ,$$

from which we see that

$$\hat{t} = 1.032 , \quad \hat{T} = 1.856 , \quad A(\hat{t},\hat{T}) = 7.425 .$$

Figure 1 graphs $b(t)$. In this example, we have $T\leqq t$ for $t\geqq\sqrt{10/3}$, so that t exists in $(0,\sqrt{10/3})$.

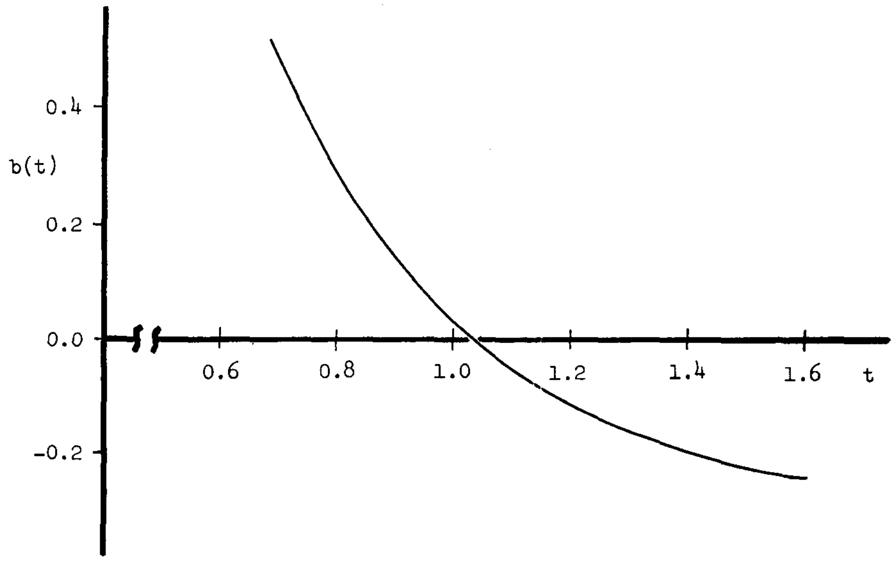

Fig. 1 Graph of b(t)

It remains to prove (26). Since

$$\Phi(0,T) = (C-D-R)\delta(0,T) < 0 \,, \quad T > 0 \,, \tag{40}$$

it follows from (27) that $\partial A(t,T)/\partial t<0$ for sufficiently small $t>0$. Hence, we have $\hat{t}>0$. We next prove $\hat{T}<\infty$. Since

$$\partial\Psi(t,T)/\partial T = Dq'(T)[t+\delta(t,T)] > 0 \tag{41}$$

and $\Psi(t,\infty)=\infty$, it follows from (29) that $\partial A(t,T)/\partial T>0$ for sufficiently large T, so that $\hat{T}<\infty$, if $\hat{t}<\infty$. But, using (23), one can easily see that

$$A(t,T) \geqq CQ(t)/[t+\delta(0,\infty)] \to \infty \quad \text{as} \quad t \to \infty \,, \tag{42}$$

which implies $\hat{t}<\infty$. Therefore, $\hat{T}<\infty$. Finally, if $\hat{t}=\hat{T}$, then necessarily

$$\Psi(\hat{t},\hat{T}) = Dq(\hat{T})\hat{T} - [CQ(\hat{T})+R] \geqq 0 \,. \tag{43}$$

But, it also follows that

$$Cq(\hat{T})\hat{T} - [CQ(\hat{T})+R] = 0 \,, \tag{44}$$

because $\hat{T}$ must minimize A(T,T), i.e.,

$$A(\hat{t},\hat{T}) = A(\hat{T},\hat{T}) \leq A(T,T) = [CQ(T)+R]/T \ . \tag{45}$$

Recalling D<C and comparing (43) and (44), we have a contradiction. Hence, $\hat{t}<\hat{T}$ must be true. The proof of (26) is complete.

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