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AN ALGORITHM FOR FINDING ALL EXTREMAL RAYS OF POLYHEDRAL CONVEX CONES WITH SOME COMPLEMENTARITY CONDITIONS

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Abstract

In this paper, we show a method for finding all extremal rays of polyhedral convex cones with some complementarity conditions. The polyhedral convex cone is defined as the intersection of half-spaces expressed by linear inequalities. By a complementarity extremal ray, we mean an extremal ray vector that satisfies some complementarity conditions among its elements. Our method is iterative in the sense that, knowing all sub-complementary extremal rays of the intersection of several half-spaces, we add repeatedly a new half-space to the half-spaces on the foregoing stage and determine all sub-complementary extremal rays of the new polyhedral convex cone thus formed, until all half-spaces are taken into consideration. Since, in the process of computation, we deal only with sub-complementary extremal rays, we could avoid the exceeding growth of the number of extremal rays. And it is of interest to note that the more complementarities there are, the less amount of computations we need. In the latter part, we apply this method to the general linear complementarity problem, to the non-convex quadratic programming and to a mathematical programming with control variables.

1. Problem

We define a polyhedral convex cone P_m by a set of nonnegative vector x (εR^n) in the intersection of m half-spaces. That is, (1.1) $P_m=\{x|\ x\geq 0,\ a_1'x\geq 0,\ldots,\ a_m'\geq 0\}$,

where x, $a_1,\ldots,a_m$ are vectors in R^n and the symbol ' denotes the transposition of matrices or vectors. We introduce the slack variables $\lambda=(\lambda_1,\ldots,\lambda_m)'$ to transfer the inequalities in (1.1) into equalities. Thus, we have

(1.2) $P_m=\{x|\ x\geq 0,\ \lambda\geq 0,\ a_1'x-\lambda_1=0,\ldots,\ a_m'x-\lambda_m=0\}$.

Now, we put some complementarity conditions on the elements of x and λ. For example, $x_1x_2=0$, $x_1\lambda_3=0$, $\lambda_1\lambda_4=0$, $x_3\lambda_2\lambda_5=0$ etc.. We call them the complementarity conditions.

The problem is to find out all extremal rays of P_m that satisfy these conditions. We call an extremal ray in such conditions a complementary extremal ray and a vertex of a polyhedral set in such conditions a complementary vertex.

Our method is iterative in the sense that knowing all complementary extremal rays of the polyhedral convex cone

(1.3) $P_{k-1}=\{x|\ x\geq 0,\ a_1'x\geq 0,\ldots,\ a_{k-1}'x\geq 0\}$,

we add a constraint $a_k'x\geq 0$ to it to determine all complementary extremal rays of the polyhedral convex cone

(1.4) $P_k=\{x|\ x\varepsilon P_{k-1},\ a_k'x\geq 0\}$.

Here, when we mention of the complementary extremal rays of P_k, we only consider the complementarity conditions among $x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_k$. We take no account of the complementarity conditions related to $\lambda_{k+1},\ldots,\lambda_m$. The latter conditions are taken into consideration step by step as we proceed our algorithm and when k attains m, the complementarity conditions among all variables $x_1,\ldots,x_n$, $\lambda_1,\ldots,\lambda_m$ will be taken into consideration. As, at step k of the algorithm, we only consider a subset of the given complementarity conditions, we call it the sub-complementarity conditions. Similarly, we mean by a sub-complementary extremal ray or a sub-complementary vertex that satisfies these sub-complementarity conditions among its elements and corresponding slack variables. In regard to P_k , let

(1.5) $C_k=\{x|\ x\varepsilon P_k,\ 1'x=1\}$,

where $1'=(1,\ldots,1)$. C_k is a convex polyhedron. As is well known, there is a one to one correspondence between the vertices of C_k and the extremal rays of P_k. Indeed, the correspondence between $x\varepsilon P_k$ $(x\neq 0)$ and $y=x/(1'x)\varepsilon C_k$ is one to one and the vertices of C_k and the extremal rays of P_k correspond with each other. Also, by this correspondence, the sub-complementary extremal rays of P_k correspond to the sub-complementary vertices of C_k and vice versa. So, we hereafter deal with the set of sub-complementary vertices of C_k which we denote V_k.

For our problem, if we could find all vertices of C_m, we could choose among them the complementary extremal rays. M.L. Balinski [1] showed an algorithm for finding all vertices of convex polyhedral sets by means of the simplex method. But we wish to find only vertices in the complementarity conditions. As far as such a purpose is aimed, Balinski's method will not be said to be very effective. On the other hand, Motzkin, Raiffa, Thompson and Thrall [3] presented " the Double Description Method" for linear programming problems, in which they tried to find out all extremal rays of a polyhedral convex cone. But, this method will not be so effective as the simplex method for linear programs, because the number of extremal rays grows exceedingly as the number of variables and constraints increases.

Our method is basically along Motzkin's method to which we add care of degeneracy and complementarity. And strong complementarity conditions will avoid the exceeding growth of the number of extremal rays.

2. Algorithm

Step 1. Initialization.

For

(2.1) $C_o=\{x|x\geq 0,\ 1'x=1\}$,

the sub-complementary vertex set is

(2.2) $V_o=\{e_i|e_i$: the i-th unit vector, $i=1,\ldots,n\}$.

Repeat the following steps for $k=1,\ldots,m$.

Step 2. Adding a constraint.

Suppose all sub-complementary vertices of C_{k-1} are known. Let it be

(2.3) $V_{k-1}=\{v_i\}$,

and let

(2.4) $\lambda_{ik}=a_k'v_i$.

Step 2.1. If $\lambda_{ik}\geq 0$ for all $v_i\varepsilon V_{k-1}$, then the adding constraint $a_k'x\geq 0$ is not binding. That is

$$C_k=C_{k-1}.$$

Let

$$\bar{V}_k=V_{k-1}.$$

(Go to step 2.5.)

Step 2.2. If $\lambda_{ik}<0$ for all $v_i\varepsilon V_{k-1}$, then C_k is null.

(The end.)

Step 2.3. If $\lambda_{ik}\leq 0$ for all i, then let

$$V_k=\{v_i|v_i\varepsilon V_{k-1},\ \lambda_{ik}=0\}.$$

(Go to the beginning of step 2. Increase k by one.)

Step 2.4. If, for some i and some j, $\lambda_{ik}>0$ and $\lambda_{jk}<0$, then try the following [Common Zero Test] for v_i and v_j. If they pass the test, then compose a vector w_{ij} by

(2.5) $w_{ij}=-\{\lambda_{jk}/(\lambda_{ik}-\lambda_{jk})\}v_i+\{\lambda_{ik}/(\lambda_{ik}-\lambda_{jk})\}v_j$.

The w_{ij} is on the line segment joining v_i and v_j and on the hyperplane H_k : $a_k'x=0$. Try this process for all pairs of v_i (with $\lambda_{ik}>0$) and v_j (with $\lambda_{jk}<0$).

Then let

$$\bar{V}_k=\{v_i, w_{ij} \mid v_i \varepsilon V_{k-1},\ a_k'v_i \geq 0\ ;\ w_{ij} \text{ by } (2.5)\}.$$

(Go to step 2.5.)

<u>Step 2.5.</u> Try the following [Sub-Complementarity Test] to the elements of $\bar{V}_k$ to remove all non sub-complementary vertices of C_k and let the remaining set be $\bar{V}_k$.

(Go to step 2.6.)

<u>Step 2.6.</u> Try the following [Degeneracy Test] to $\bar{V}_k$ to remove all non-vertex points of C_k from $\bar{V}_k$ and let the remaining set be V_k.

(Go to the beginning of step 2. Increase k by one.)

[Common Zero Test]

For v_i and v_j, let

(2.6) $\qquad \lambda_{is}=a_s'v_i \qquad (s=1,\dots,k-1),$

(2.7) $\qquad \lambda_{js}=a_s'v_j \qquad (s=1,\dots,k-1)$

and let the extended vectors v_i^o and v_j^o of v_i and v_j be

(2.8) $\qquad v_i^o=(v_{i1},\dots,v_{in},\lambda_{i1},\dots,\lambda_{ik-1},\lambda_{ik})',$

(2.9) $\qquad v_j^o=(v_{j1},\dots,v_{jn},\lambda_{j1},\dots,\lambda_{jk-1},\lambda_{jk})'$

respectively. They are (n+k)-vectors. If v_i^o and v_j^o have no less than (n-2) common zeros in their corresponding elements, then they pass the test. Otherwise, they fail.

[Sub-Complementarity Test]

For each v_i of $\bar{V}_k$, check the sub-complementarity among the elements of its extended vector v_i^o. If it does not satisfy the conditions, then remove v_i from $\bar{V}_k$.

[Degeneracy Test]

$\bar{V}_k$ consists of $v_i \varepsilon V_{k-1}$ and w_{ij} composed by (2.5). Let $\hat{V}_k$ be the subset of $\bar{V}_k$ composed of the points on the hyperplane H_k : $a_k'x=0$. Of course $w_{ij} \varepsilon \hat{V}_k$. If w_{ij} can be expressed by a convex combination of other points of $\hat{V}_k$, w_{ij} is not a vertex of C_k. To see this test the following.

If there exist $w_{ij} \varepsilon \hat{V}_k$ and $y_t \varepsilon \hat{V}_k$ whose extended vectors we denote w^o_{ij} and y^o_t respectively, such that for every positive elements of y^o_t , the corresponding elements of w^o_{ij} are also positive and there is at least one positive element of w^o_{ij} whose corresponding element of y^o_t is zero, then w_{ij} is not a vertex of C_k. And we remove it from $\bar{V}_k$.

3. Validity of the method

[Lemma 1] Let W_{k-1} be the set of all vertex of C_{k-1} and let W_k be the set of points obtained from W_{k-1} by applying step 2 of the preceding section to W_{k-1} instead of V_{k-1} except [Sub-Complementarity Test]. Then W_k is the vertex set of C_k.

Proof : If all vertices of C_k are non-degenerate, the lemma is true even if W_k is obtained from W_{k-1} by applying step 2 of the preceding section except [Degeneracy Test] , (see [3]). But when some vertices of C_k are degenerate, it may be happen that some non-vertex points of C_k are contained in $\bar{W}_k$ (corresponding to $\bar{V}_k$ in step 2.4.). [Degeneracy Test] prevents such troubles. We need to try the test only to the newcomers on the hyperplane H_k : $a_k'x=0$. Let w_i and $w_j \varepsilon W_{k-1}$ be positioned on the opposite sides of H_k and

(3.1) $$\lambda_{ik}=a_k'w_i>0,$$

(3.2) $$\lambda_{jk}=a_k'w_j<0.$$

If they pass [Common Zero Test], we define a newcomer w_{ij} by

(3.3) $$w_{ij}=-\{\lambda_{jk}/(\lambda_{ik}-\lambda_{jk})\}w_i+\{\lambda_{ik}/(\lambda_{ik}-\lambda_{jk})\}w_j.$$

Now, let $\hat{W}_k$ be the subset of $\bar{W}_k$ composed of the points on the hyperplane H_k. If, for w_{ij}, there exists $y_t \varepsilon \hat{W}_k$ such that for every positive elements of y_t^o (the extended vector of y_t), the corresponding elements of w_{ij}^o are also positive and there is at least one positive element of w_{ij}^o for which the corresponding element of y_t^o is zero, then w_{ij} is not a vertex of C_k. This can be seen as follows.

Let

(3.4) $$\xi_1=(w_{ij}+\varepsilon y_t)/(1+\varepsilon)$$

and

(3.5) $$\xi_2=(w_{ij}-\varepsilon y_t)/(1+\varepsilon),$$

where ε is a sufficiently small positive number. Then $\xi_1,\xi_2 \varepsilon C_k \cap H_k$ and $\xi_1 \neq \xi_2$. And we have,

(3.6) $$w_{ij}=\{(1+\varepsilon)/2\}\xi_1+\{(1-\varepsilon)/2\}\xi_2.$$

Thus, w_{ij} is not a vertex of C_k.

Conversely, as $\hat{W}_k$ contains all vertices of C_k on H_k and every non-vertex point of C_k on H_k can be expressed by a convex combination of vertices on H_k, all non-vertex point of W_k can be removed by [Degeneracy Test]. This can be seen as follows. First, let w_i and w_j be any two different vertices of C_k on H_k, then their extended vectors w_i^o and w_j^o have their positive elements at, at least, one different position. For, if they have their positive elements wholly at common positions, let ξ_1 and ξ_2 be the vectors defined by (3.4) and (3.5) respectively, after replacing w_{ij} by w_i and y_t by w_j on the right hand sides. Then, $\xi_1,\xi_2 \varepsilon C_k \cap H_k$ and $\xi_1 \neq \xi_2$. And we have,

(3.7) $$w_i=\{(1+\varepsilon)/2\}\xi_1+ \{(1-\varepsilon)/2\}\xi_2.$$

Thus, w_i is not a vertex of C_k on H_k and also w_j. Using this fact and using the representation of a non-vertex point of C_k on H_k by at least two different vertices of C_k on H_k, we can conclude that all non-vertex points of W_k can be removed by [Degeneracy Test]. (Q.E.D.)

[Lemma 2] Let V_k be the sub-complementary vertex of C_k. Then we can get V_k from V_{k-1} by step 2.

Proof : Assume the vertex set of C_{k-1} is known, which has the sub-complementary vertex set V_{k-1} and the non-sub-complementary vertex set U_{k-1}. Similarly, the vertex set W_k of C_k consists of V_k and non-sub-complementary U_k. By lemma 1, W_k is composed of some vertices of C_{k-1} and some of the newcomers defined by (3.3). The former we denote $\{w_i\}$, the latter $\{w_{ij}\}$. Then we have,

(1) If $w_i \varepsilon U_{k-1}$, then also $w_i \varepsilon U_k$.

(2) If $w_i \varepsilon V_{k-1}$, then w_i may belong to U_k or to V_k.

(3) If w_{ij} is defined by (3.3) and

(3a) if $w_i \varepsilon V_{k-1}$ and $w_j \varepsilon V_{k-1}$, then w_{ij} may belong to U_k or to V_k,

(3b) otherwise, w_{ij} belongs to U_k.

Thus, we have shown that V_k can only be obtained from V_{k-1}. (Q.E.D.)

As V_o has such property, we demonstrated the validity of the method.

Now, we have the following theorem:

[Theorem 1] V_k defined in the section 2 is the sub-complementary vertex set of C_k.

[Corollary] Any sub-complementary point of C_k can be expressed by a convex combination of vectors in V_k.

Remark : We could also replace the step 2 of the preceding algorithm by the dual simplex method, because the added constraint is a cutting plane.

4. Example and computational remark

Example. Solve the system,

(4.1a) $\lambda_1 = x_1 + x_2 + x_3 + x_4 - x_5 + 2x_6 \geq 0$

(4.1b) $\lambda_2 = x_1 - x_2 + x_3 - x_4 + x_5 - 3x_6 \geq 0$

(4.1c) $\lambda_3 = -x_1 - x_2 + 2x_6 \geq 0$

(4.1d) $\lambda_4 = -x_1 + x_2 + x_6 \geq 0$

(4.1e) $\lambda_5 = x_1 - x_2 + x_6 \geq 0$

(4.1f) $x_1, x_2, x_3, x_4, x_5, x_6, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0$

with the complementarity conditions

(4.1g) $x_1\lambda_1 = x_2\lambda_2 = x_3\lambda_3 = x_4\lambda_4 = x_5\lambda_5 = 0.$

We show the process of solution in Table 4.1. To avoid decimal numbers, we were not restricted to the condition $1'x=1$. In this table, a row means a point and its extended form. The points V1 to V6 are unit vectors corresponding to C_o. And Vk=V(i,j) means that the point Vk is obtained from Vi and Vj by the formula (2.5). _ means non-complementary elements. Final solutions are V3, V9, V13 and V18.

		x_1	x_2	x_3	x_4	x_5	x_6	λ_1	λ_2	λ_3	λ_4	λ_5
	V1	$\underline{1}$						$\underline{1}$	*	*	*	*
	V2		1					1	-1	*	*	*
C_o	V3			1				1	1	0	0	0
	V4				1			1	-1	*	*	*
	V5					1		-1	*	*	*	*
↑	V6						1	2	-3	*	*	*
	V7=V(1,5)	1				1		0	2	-1	*	*
	V8=V(2,5)		1			1		0	0	-1	*	*
C_1	V9=V(3,5)			1		1		0	2	0	0	0
	V10=V(4,5)				1	1		0	0	0	0	0
↑	V11=V(5,6)					2	1	0	-1	*	*	*
	V12=V(2,3)		1	1				2	0	-1	*	*
	V13=V(3,4)			1	1			2	0	0	0	0
C_2	V14=V(3,6)			$\underline{3}$			1	5	0	$\underline{2}$	*	*
	V15=V(7,11)	1				$\underline{5}$	2	0	0	$\underline{3}$	1	3
↑	V16=V(9,11)			$\underline{1}$		5	2	0	0	$\underline{4}$	*	*
C_3	V17=V(7,15)	2				4	1	0	3	0	-1	*
C_4	V18=V(8,15)	1	3			8	2	0	0	0	4	0
C_5	V19=V(8,16)		4	1		9	2	0	0	0	6	-2
↑	V20=V(12,14)		2	5			1	9	0	0	3	-1

Table 4.1.

Remark : Degeneracy may happen rarely. And we need not to try [Degeneracy Test] at every step. We may try it at the final stage to final candidates.

5. Applications

Recently, many papers have published on the linear complementarity problem [2]. But we can apply so-called principal pivoting method or complementary pivot method only for matrices with special structures. There would be many other complementarity problems whose matrices are not of such structures, for example, the Kuhn-Tucker conditions for nonconvex quadratic programmings.

In this section, we apply our method to the general linear complementarity problem, to the nonconvex quadratic programming and to a mathematical programming with control variables.

(a) The general linear complementarity problem

We define a generalized linear complementarity problem as the problem to find $x\varepsilon R^n$ which satisfies the following system :

(5.1) $Ax=b,$

(5.2) $x\geq 0$

and

(5.3) the given complemntarity conditions among the elements of x,

where A is an (m,n) matrix, $m\leq n$, rank(A)=m, and b is an m-vector.

As rank(A) is m, there is a regular submatrix M of order m of A. By multiplying M^{-1} to (5.1) from the left and by rearranging the result, we have the canonical form :

(5.4) $\lambda=By+d,$

where $\lambda\varepsilon R^m$ is the vector of the basic variables corresponding to M, $y\varepsilon R^{n-m}$ is the vector of the nonbasic variables, $d=M^{-1}b\varepsilon R^m$ and B is an (m, n-m) matrix.

Then we have the following theorem :

[Theorem 2] The general linear complementarity problem which satisfies (5.1) to (5.3), can be reduced to the complementary extremal ray problem of the polyhedral convex cone F defined by

(5.5) $$F=\{(y,t)\mid \lambda=By+dt\geq 0,\ y\geq 0,\ t\geq 0,\ t\in R^1\}.$$

And when $d\neq 0$, any complementary ray of F with a positive t can be normalized so as to become a solution of the original problem and any solution x of the original problem can be represented as the sum of the convex linear combination x_1 of the normalized complementary extremal rays of F and of the nonnegative combination x_2 of the complementary extremal rays of F with $t=0$.

Proof : The relationship between the solutions of the inhomogeneous system (5.1) to (5.2) and the homogeneous system $\lambda=By+dt$ is well known, (see, for example, [4]). We can get the theorem by adding the complementarity conditions to the relationship. (Q.E.D.)

(b) The nonconvex quadratic programming

It is well known that the Kuhn-Tucker conditions for a quadratic programming is a linear complementarity problem. When the coefficient matrix of the quadratic form in the objective function to be minimized is positive semidefinite (i.e. the convex quadratic programming), we have an efficient algorithm to solve it, due to P. Wolfe [6]. For nonconvex case, we have not such a good algorithm. But as the minimizing point satisfies the Kuhn-Tucker conditions, we can solve the corresponding linear complementarity problem by our method to choose the global optimum point among the solutions. Indeed, the example in the preceding section is the Kuhn-Tucker conditions for the following nonconvex quadratic programming.

Minimize $2x_1-3x_2+(x_1^2+2x_1x_2-x_2^2)/2$,

subject to $x_1+x_2\leq 2$,

$$x_1-x_2\leq 1,$$
$$-x_1+x_2\leq 1,$$
$$x_1,x_2\geq 0.$$

Thus, we have the global optimum point ($x_1=1/2$, $x_2=3/2$). As to the detail of the algorithm which includes several devices to reduce the amount of computations, see [5].

(c) A mathematical programming with control variables

We consider the following two linear programming problems including control variables λ.

[Problem I]

(5.6) Maximize $(c+K\lambda)'x$,

(5.7) subject to $Ax\leq b+F\lambda$, $x\geq 0$, $\lambda\geq 0$.

[Problem II]

(5.8) Maximize $(d+L\lambda)'x$,

subject to the same constraints with [Problem I], where x, a, $c\varepsilon R^n$, $\lambda\varepsilon R^k$, $b\varepsilon R^m$ and matrices A,F,K and L are of order (m,n), (m,k), (n,k) and (n,k), respectively.

For a given λ, there may be two optimum points of the two problems. Our problem is how to determine the control variables λ to let the two optimum points coincide with each other. By applying the Kuhn-Tucker conditions for this problem, we get the following theorem :

[Theorem 3] In order that the optimum points of the two problems coincide with each other, it is necessary and sufficient to find $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{\lambda}$, $\bar{\xi}$, $\bar{\eta}$ and $\bar{\zeta}$ which satisfy

(5.9a) $\xi=-Ax+F\lambda+b$,

(5.9b) $\eta=A'y-K\lambda-c$,

(5.9c) $\zeta = A'z - L\lambda - d,$

(5.9d) $\eta'x = \zeta'x = \xi'y = \xi'z = 0,$

where $\xi \in R^m$, η and $\zeta \in R^n$ and all variables must be nonnegative.
And the $\bar{x}$ is the common optimum point and the $\bar{\lambda}$ is the corresponding value of the control vector.

Proof : Obvious. (Q.E.D.)

Because (5.9) is a generalized linear complementarity problem, we can apply our method to get the solutions.

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