AN INVENTORY MODEL WITH DEPENDENT INTERVALS

OF DEMAND

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Abstract

The inventory problem for continuous time is studied in which the demand process is composed of two different processes, one is a compound Poisson process and the other is a generalized semi-Markov chain or a process whose intervals of occurrence are constant.

We assume that holding and shortage costs are convex.   And a set-up charge for ordering are considered.

The form and bounds of an optimal policy are determined and some numerical results in the special case are added.

Finally, we consider some combined policy at the view point of practical use and compare with a simple (s, S) policy.

1. Introduction

It is usually assumed that demand are independently and identically distributed in different periods, and in the continuous case, assumed that intervals of demand are subject to an exponential distribution. The case of arbitrary interval distribution has been analyzed in [1] and [2], where independent intervals of demand are assumed.

We consider, for example, the situation where a part of demand is required by fixed customers and other part, by a

floating purchasing power. In such a case, intervals of demand are no longer independent.

We treat such a case and develop the continuous inventory model with the following two types of demand (type 1 and type 2).

Those are superposed of a compound Poisson process and the other process: type 1 has a constant interval of occurrence with an independent and identical distribution of demand size, and type 2 is a generalized semi Markov chain. Type 2 is an extension of type 1.

Section 2 gives the structure of the model.

Section 3 presents a proof of the optimality of an (s(t), S(t)) policy where t denotes a parameter defined in §3.

In section 4, we discuss about a special demand process of type 2 that is a superposition of two compound Poisson processes.

Section 5 gives upper and lower bounds of the optimal critical numbers for N truncated decision periods.

In section 6, numerical examples are given, observing the effect of parameter t.

Since optimal policy (s(t), S(t)) are very complicated both in calculation and practical use, section 7 presents a relatively simple policy which is a combined policy of (s, S) policy and not always optimal, but in some cases, is better than simple (s, S) policy.

2. Notations and the Model

We formulate the model more precisely.

Following notations are used.

x: inventory before ordering

y: inventory after ordering

$\Phi_2(\xi)$: the probability distribution function of demand

size of compound Poisson process, one component of

superposition of two processes.

$\Phi_1(\xi)$: that of the other component of superposition

$\bar{\tau}$:   deterministic delivery lag

$g(y)$:   an expected holding and shortage cost function

λ: parameter of interval of compound Poisson process

Following assumptions are made.

1. Decisions concerning whether to order stock replacements and how much to order may be immediately after a demand has arisen.

2. Delivery of an order is assumed to require a fixed time, $\bar{\tau}$.

3. No restriction is placed on the size or number of orders that may be outstanding at any time.

4. There is a fixed cost of ordering, K, which is incurred when the order is made.

5. A holding and shortage penalty cost is charged against inventory level after $\bar{\tau}$ time later after ordering.

$g(y)$ is a non negative convex function in y with $g(y) \to \infty$ as $|y| \to \infty$.

6. Complete backlogging is postulated.

7. Future costs are discounted by a discount rate $e^{-\alpha}$ per unit time.

8. Since no disposal is permitted, $y \geq x$ at the same point of time.

9. Infinite planning horizon is assumed.

Since interest cost is included in holding cost, the unavoidable outlay of the purchasing price per item need not be considered.

The decision objective is to minimize the expected value of present and discounted future avoidable cost.

3. Inventory Equations and Form of an Optimal Policy

First, we shall consider type 1. If we call compound Poisson process C type and the other process with constant interval (T) D type, type 1 is a superposition of C type and D type.

Let $f_D(y)$ denote the expected value of discounted avoidable cost at a time immediately after a demand of D type, having present stock level y, if an optimal policy is followed.

Also, $f_c(t, y)$, for $0 < t < T$, denotes the expected value of discounted avoidable cost at a time immediately after a demand of C type which occurs at time T-t after D type occurred, having present stock level y, if an optimal policy is followed.

Now, we define $f_c(T, y) \equiv f_D(y)$ expediently, and moreover, abbreviate index c, then $f(t, y) \equiv f_c(t, y)$ for $0 < t \leq T$ without confusion.

Here, we will introduce the functional equation that $f(t, x)$ satisfies. Consider the system immediately after a demand of type D has occurred. We pay attention to time when first C type demand occurs after the origin, occurrence time of D type. Let it be $(t, t+dt)$. When $k\bar{\tau} < T < (k+1)\bar{\tau}$ for $k=2, 3, \ldots$, we classify the following six cases. (1) For $0 < t < \bar{\tau}$, only C type demand occurs during an interval $(\bar{\tau}, \bar{\tau}+t)$ (2) For $\bar{\tau} < t < T-\bar{\tau}$, nothing occurs during an interval $(\bar{\tau}, t)$ and only C type demand occurs during an interval $(t, \bar{\tau}+t)$ (3) For $T-\bar{\tau} < t < k\bar{\tau}$, nothing occurs during an interval $(\bar{\tau}, t)$, and only C type during (t, T), D type occurs at T and only C type occurs during $(T, \bar{\tau}+t)$ (4) For $k\bar{\tau} < t < T$, nothing occurs during $(\bar{\tau}, t)$, only C type during (t, T), D type at T, only C type during $(T, \bar{\tau}+t)$ (5) For $T < t < T+\bar{\tau}$, noting occurs during $(\bar{\tau}, T)$, D type at T, nothing during (T, t), only C type during $(t, \bar{\tau}+T)$ (6) For $T+\bar{\tau} < t$, nothing occurs during $(\bar{\tau}, T)$, D type at T, nothing during $(T, \bar{\tau}+T)$.

The cost of storage or shortage during an interval of length t if $t < T$ and length T if $t \geq T$, but placed $\bar{\tau}$ units of time later, is considered. We denote the discounted cost of storage and shortage expected with the demand size distribution during a time interval (t_1, t_2) with stock level y, by $L(y, \overline{t_1 t_2})$.

Based on the above classification, we obtain the following functional equation. We have[1]

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1): $\delta(z) \equiv 0$ for $z=0$ and 1 for $z>0$

$$f(T,x) = \inf_{y \geq x}[K \cdot \delta(y-x) + \int_0^{\bar{\tau}} \lambda e^{-\lambda t}\{L(y,\overline{\tau,\tau+t}) + e^{-\alpha t}\int_{\xi=0}^{\infty} f(T-t,y-\xi)d\Phi_2(\xi)\}dt$$

$$+\int_{\tau}^{t-\bar{\tau}} \lambda e^{-\lambda t}\{L(y,\overline{\tau,\tau+t}) + e^{-\alpha t}\int_{\xi=0}^{\infty} f(T-t,y-\xi)d\Phi_2(\xi)\}dt$$

$$+ \int_{T-\tau}^{k\bar{\tau}} \lambda \cdot e^{-\lambda t}\{L(y,\overline{\tau,\tau+t}) + e^{-\alpha t}\int_{\xi=0}^{\infty} f(T-t,y-\xi)d\Phi_2(\xi)\}dt$$

$$+ \int_{k\tau}^{T} \lambda \cdot e^{-\lambda t}\{L(y,\overline{\tau,\tau+t}) + e^{-\alpha t}\int_{\xi=0}^{\infty} f(T-t,y-\xi)d\Phi_2(\xi)\}dt$$

$$+ \int_{T}^{\bar{\tau}+T} \lambda \cdot e^{-\lambda t}\{L(y,\overline{\tau,\tau+T}) + e^{-\alpha T}\int_{\xi=0}^{\infty} f(t,y-\xi)d\Phi_1(\xi)\}dt$$

$$+ \int_{\bar{\tau}+T}^{\infty} \lambda \cdot e^{-\lambda t}\{L(y,\overline{\tau,\tau+T}) + e^{-\alpha T}\int_{\xi=0}^{\infty} f(T,y-\xi)d\Phi_1(\xi)\}dt]$$

Calculating each $L(y, \overline{t_1 t_2})$ for six intervals where demand occurrences are given above, and integrating by t and summing up, we denote it by $L_T(y)$.

Then, $f(T, x)$ satisfies the functional equation

$$(3.1) \quad f(T,x) = \inf_{y \geq x}\{K \cdot \delta(y-x) + L_T(y) + \int_{t=0}^{T}\int_{\xi=0}^{\infty} \lambda \cdot e^{-(\lambda+\alpha)t} f(T-t,y-\xi)d\Phi_2(\xi)dt$$

$$+ e^{-(\lambda+\alpha)T}\int_{\xi=0}^{\infty} f(T,y-\xi)d\Phi_1(\xi)\}$$

And for $T > 2\bar{\tau}$,

$$(3.2) \quad L_T(y) = \frac{1}{\lambda+\alpha}[(e^{-(\lambda+\alpha)\bar{\tau}} - e^{-(\lambda+\alpha)T}) \cdot \{\sum_{j=1}^{\infty} \frac{(\lambda\bar{\tau})^{j+1}}{(j+1)!} G_{0,j+1}(y) + g(y)\}$$

$$+ (e^{-(\lambda+\alpha)T} - e^{-(\lambda+\alpha)(T+\bar{\tau})})\{\sum_{j=1}^{\infty} \frac{(\lambda\bar{\tau})^{j+1}}{(j+1)!} G_{1,j+1}(y) + G_{1,0}(y)\}],$$

where[2)]

$$(3.3) \quad G_{i,j}(y) = \int_0^\infty d\Phi_1^{(i)} * \Phi_2^{(j)}(\xi) \cdot g(y-\xi)$$

When $T \le 2\bar{\tau}$, the same functional eq. (3.1) holds, but for $\bar{\tau} \le T \le 2\bar{\tau}$,

$$(3.4) \quad L_T(y) = \frac{1}{\lambda+\alpha}[(e^{-(\lambda+\alpha)\bar{\tau}} - e^{-(\lambda+\alpha)T}) \sum_{j=1}^\infty \frac{(\lambda\bar{\tau})^{j+1}}{(j+1)!} \cdot G_{0,j+1}(y)$$

$$+ (e^{-(\lambda+\alpha)T} - e^{-(\lambda+\alpha)(T+\bar{\tau})})\{ \sum_{j=1}^\infty \frac{(\lambda\bar{\tau})^{j+1}}{(j+1)!} G_{1,j+1}(y) + G_{1,0}(y)\}$$

$$+ e^{-(\lambda+\alpha)T} \sum_{j=1}^\infty (\frac{\lambda}{\lambda+\alpha})^{j+1} \cdot G_{0,j+1}(y)]$$

$$+ \frac{1}{\alpha} e^{-\lambda T}(e^{-\alpha\bar{\tau}} - e^{-\alpha T})g(y) - e^{-(\lambda+\alpha)\bar{\tau}} \sum_{j=1}^\infty \lambda^{j+1} G_{0,j+1}(y) \sum_{\ell=0}^{j+1} \frac{(\lambda+\alpha)^{\ell-j-2}(\bar{\tau}-T)^\ell}{\ell!}$$

For $\frac{\bar{\tau}}{k+1} \le T < \frac{\bar{\tau}}{k}$, $k = 1, 2, \ldots$

$$(3.5) \quad L_T(y) = \frac{1}{\lambda+\alpha}\{ \sum_{j=1}^\infty \frac{(\lambda\bar{\tau})^{j+1}}{(j+1)!}\{G_{k,j+1}(y) \cdot (e^{-(\lambda+\alpha)\bar{\tau}} - e^{-(\lambda+\alpha)(k+1)T})$$

$$+ G_{k+1,j+1}(y) \cdot (e^{-(\lambda+\alpha)(k+1)T} - e^{-(\lambda+\alpha)(T+\bar{\tau})}) - \frac{e^{-(\lambda+\alpha)(k+1)T}}{(\bar{\tau})^{j+1}} \sum_{r=0}^{j+1} (kT-\bar{\tau})^{j-r+1}$$

$$\times \frac{(j+1)!}{(j-r+1)!} \frac{1}{(\lambda+\alpha)^r} + \frac{j!}{(\lambda+\alpha)^{j+1}} e^{-(\lambda+\alpha)(\bar{\tau}+T)})\}] + \frac{1}{\alpha}(e^{-\alpha\bar{\tau}} - e^{-\alpha(k+1)T})$$

$$\times e^{-\lambda(k+1)T} G_{k,0}(y) + \frac{1}{\alpha}(\frac{\alpha}{\lambda+\alpha} \cdot e^{-(\lambda+\alpha)\bar{\tau}} - e^{-\lambda(k+1)T-\alpha\bar{\tau}} + \frac{\lambda}{\lambda+\alpha} e^{-(\lambda+\alpha)(k+1)T})$$

$$\times G_{k1}(y) + \frac{1}{\alpha}(e^{-\alpha(k+1)T-\lambda(\bar{\tau}+T)} - e^{-(\lambda+\alpha)(\bar{\tau}+T)}) G_{k+1,0}(y)$$

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2): $\Phi^{(i)}$ is the i-fold convolution of Φ with itself. * denotes convolution mark.

Especially for $\bar{\tau} = 0$,

(3.6) $\quad L_T(y) = \frac{1}{\lambda+\alpha}\{1 - e^{-(\lambda+\alpha)T}\}g(y)$

In the same way, $f(t, x)$ for $0 < t < T$, satisfies the functional equation

(3.7) $\quad f(t,x) = \inf_{y \geq x}\{K \cdot \delta(y-x) + L_t(y) + \int_{\tau=0}^{t}\int_{\xi=0}^{\infty}\lambda \cdot e^{-(\lambda+\alpha)\tau}f(t-\tau,y-\xi)d\tau d\Phi_2(\xi)$

$$+ e^{-(\lambda+\alpha)t}\int_{\xi=0}^{\infty}f(T,y-\xi)d\Phi_1(\xi)\},$$

where $L_t(y)$ denotes the discounted expected cost of storage or shortage cost during an interval between the origin t and next first occurrence of demand, but placed $\bar{\tau}$ units of time later.

For $\bar{\tau} = 0$,

(3.8) $\quad L_t(y) = \frac{1}{\lambda+\alpha}(1 - e^{-(\lambda+\alpha)t})g(y)$

From the above functional eqs. (3.1) and (3.7), we will derive the form of an optimal policy. So we introduce the following lemma which can be deriven by slitely rewritting Theorem 2.1 in Kalymon [3].

Lemma. Let $\widetilde{E}$ be any subspace of N dimensional Euclid space. For arbitrary $U \in \widetilde{E}$ and any real number x, the following function $f(u, x)$ is defined, where $L_u(y)$ is a convex function of y and $P(v|u)$ is a function satisfying $dP(v|u) \geq 0$ and $\int_E dP(v|u) \leq 1$.

(1) $\quad f(u,x) = \inf_{y \geq x}\{K \cdot \delta(y-x) + L_u(y) + \int_{v \in \widetilde{E}}\int_{\xi=0}^{\infty}dP(v|u)f(v,y-\xi)d\Phi_v(\xi)\}$

Then, $f(u,x)$ has an optimal policy of the $(s(u), S(u))$ form.

The above lemma shows the following.

Theorem 3.1. For demand process of type 1, there exists an optimal policy of the $(s(t), S(t))$ form $(0 < t \leq T)$.

Proof. It suffices to show that eqs. (3.1), (3.7) are included in the special case of eq. (1). If we put $P(u|t)$ for $0 < t \leq T$ as follows

$$P(T|t) = e^{-(\lambda+\alpha)t}$$

$$dP(t-\tau|t) = \lambda \cdot e^{-(\lambda+\alpha)\tau} d\tau \qquad \text{for } 0 < \tau < t,$$

then eqs. (3.1) and (3.7) are rewritten

$$(3.9) \quad f(t,x) = \inf_{y \geq x} \{K \cdot \delta(y-x) + L_t(y) + \int_{u \in \widetilde{E}} \int_{\xi=0}^{\infty} dP(u|t) f(u, y-\xi) d\phi_u(\xi)\},$$

where $\widetilde{E} = (0, t) \cup T$ and $\phi_u(\xi) = \phi_2(\xi)$ for $u \neq T$ and otherwise $\phi_u(\xi) = \phi_1(\xi)$. $L_t(y)$ is convex function of y from assumptions in section 2, so the results is deriven from the above lemma.

In the same way, we will give the functional equation of an optimal cost function of type 2. In type 2, state i of a generalized semi Markov chain implies that the distribution of the size of demand is Ψ_i. We call A_i type for state i. The joint distribution function of state j after v or less v time later given that the state is i, is denoted by $F(v, j|i)$.

$f_{A_i}(y)$: expected value of discounted avoidable cost at a time immediately after a demand of A_i type, conditional on inventory y and an optimal policy.

$f_c(t, i, y)$: expected value of discounted avoidable cost at a time immediately after a demand of C type which occurs at time t after A_i type occurred, conditional on inventory y and an optimal policy.

Then, $f_{A_i}(y)$ and $f_c(t, i, y)$ satisfy the functional

equations

$$(3.10) \quad f_{A_i}(x) = \inf_{y \geq x} \{ K\delta(y-x) + L_{A_i}(y) + \sum_j \int_{v=0}^{\infty} \int_{t=0}^{v} \lambda e^{-(\lambda+\alpha)t} d_v F(v, j|i) dt $$

$$\times \int_{\xi=0}^{\infty} f_c(t, i, y-\xi) d\Phi_2(\xi)$$

$$+ \sum_j \int_{v=0}^{\infty} \int_{t=v}^{\infty} \lambda e^{-\lambda t} e^{-\alpha v} d_v F(v, j|i) dt \int_{\xi=0}^{\infty} f_{A_j}(y-\xi) d\Psi_j(\xi) \}$$

$$= \inf_{y \geq x} \{ K \cdot \delta(y-x) + L_{A_i}(y) + \int_{t=0}^{\infty} \int_{\xi=0}^{\infty} \lambda e^{-(\lambda+\alpha)t} \bar{F}(t|i) f_c(t, i, y-\xi) d\Phi_2(\xi) dt$$

$$+ \sum_j \widetilde{F}(\lambda+\alpha, j|i) \int_{\xi=0}^{\infty} f_{A_j}(y-\xi) d\Psi_j(\xi) \},$$

where $\quad dF(v|i) \equiv \sum_j dF(v, j|i), \quad \bar{F}(t|i) \equiv \int_{v=t}^{\infty} dF(v|i),$

$$\widetilde{F}(\beta, j|i) \equiv \int_{v=0}^{\infty} e^{-\beta v} dF(v, j|i)$$

$$(3.11) \quad f_c(t, i, x) = \inf_{y \geq x} \{ K \cdot \delta(y-x) + L_{(t,i)}(y)$$

$$+ \int_{\tau=0}^{\infty} \int_{\xi=0}^{\infty} \lambda e^{-(\lambda+\alpha)\tau} \bar{K}(i, t, \tau) f_c(t+\tau, i, y-\xi)$$

$$\times d\Phi_2(\xi) d\tau + \sum_j \widetilde{K}(i, t, j, \lambda+\alpha) \int_{\xi=0}^{\infty} f_{A_j}(y-\xi) d\Psi_j(\xi) \},$$

where $d_v K(i, t, v, j)$ denotes the joint distribution of

residual time v and next state j at time t after state i, in

semi Markov chain. $d_v K(i,t,v) \equiv \sum_j d_v K(i,t,v,j), \quad \bar{K}(i,t,\tau)$

$$\equiv \int_{v=\tau}^{\infty} d_v K(i,t,v), \quad \widetilde{K}(i,t,j,\beta) \equiv \int_{v=0}^{\infty} e^{-\beta v} d_v K(i,t,v,j)$$

And $L_{A_i}(y)$ and $L_{(t,i)}(y)$ denote analogous meanings of $L_t(y)$ in type 1.

In the case of $\bar{\tau} = 0$,

$$L_{A_i}(y) = \frac{1}{\lambda+\alpha}\{1-\widetilde{F}(\lambda+\alpha\,|\,i)\}g(y)$$

$$L_{(t,i)}(y) = \frac{1}{\lambda+\alpha}\{1-\widetilde{K}(i,t,\lambda+\alpha)\}g(y)$$

Thus, we obtain the following

Theorem 3.2   For demand process of type 2, there exists an optimal policy of the $(s(u), S(u))$ form, where $U \in \widetilde{E}$,

$\widetilde{E} = \{A_i, (t,i), i=1,2,\ldots, 0 < t < \infty\}$

Proof   In the same manner of Theorem 3.1, we put

$$P(S_j\,|\,S_i) = \widehat{F}(\lambda+\alpha, j\,|\,i) \quad i, j = 1,2,\ldots$$

$$dP((t,i)\,|\,S_i) = \lambda e^{-(\lambda+\alpha)t} \cdot \bar{F}(t\,|\,i) \text{ for } 0 < t < \infty, \; i=1,2,\ldots$$

$$P(S_j\,|\,(t,i)) = \widetilde{K}(i,t,j,\lambda+\alpha) \qquad \text{for } 0 < t < \infty, \; i=1,2,\ldots$$

$$dP((t+\tau, i)\,|\,(t,i)) = \lambda e^{-(\lambda+\alpha)\tau} \cdot \bar{K}(i,t,\tau)$$
$$\text{for } 0 < t < \infty, \; i=1,2,\ldots$$

Then, eqs. (3.10) and (3.11) are reduced to the same type as eq. (1) of lemma and the results are obtained.

4. Special case of type 2: in the case of a compound
    Poisson demand process

In the demand process of type 2, we take specially a
compound Poisson process of parameter λ_2 as a semi Markov
demand process, so superposition of two process is also a
compound Poisson process. In this case, we will try two
different methods of analysis and show that those results
are consistent.

For a compound Poisson process with a parameter $\lambda = \lambda_1$
$+ \lambda_2$ and the probability distribution function of demand
size, $\frac{1}{\lambda}(\lambda_1\Phi_2 + \lambda_2\Phi_1)$, the optimal cost function satisfies
the following functional equation.

For simplicity, we assume $\bar{\tau} = 0$.

(4.1.) $\displaystyle f(x) = \inf_{y \geq x} \{ K \cdot \delta(y-x) + \frac{1}{\lambda+\alpha} \cdot g(y)$

$\displaystyle \qquad\qquad + \frac{1}{\lambda+\alpha} \int_0^\infty f(y-\xi) \cdot (\lambda_1 d\Phi_2 + \lambda_2 d\Phi_1)(\xi) \}$

While, according to the argument in section 3, eqs. (3.10)
and (3.11), for a superposition of two compound Poisson
processes with parameter λ_1 and λ_2 and the probability dis-
tribution of demand size, Φ_1 and Φ_2, respectively, we show
that the optimal cost function satisfies the same functional
equation as (4.1).

Since in eqs. (3.10) and (3.11)

$$ L_{A_i}(y) = \frac{1}{\lambda+\alpha} \cdot g(y), \quad \widetilde{K}(t,\beta) = \frac{\lambda_2}{\lambda_2+\beta} , $$

$$L_{(t,i)}(y) = \frac{1}{\lambda+\alpha} \cdot g(y),$$

the optimal cost functions $f_{A_i}(x)$ and $f_c(t,x)$ satisfy the functional eqs.

(4.2) $f_{A_i}(x) = \inf_{y \geq x} \{K \cdot \delta(y-x) + \frac{1}{\lambda+\alpha} g(y)$

$$+ \int_{t=0}^{\infty} \int_{\xi=0}^{\infty} e^{-(\lambda+\alpha)t} f_c(t,y-\xi) d(\lambda_1 \Phi_2(\xi)) dt$$

$$+ \frac{1}{\lambda+\alpha} \int_{\xi=0}^{\infty} f_{A_i}(y-\xi) d(\lambda_2 \Phi_1(\xi)) \}$$

(4.3) $f_c(t,x) = \inf_{y \geq x} \{K \cdot \delta(y-x) + \frac{1}{\lambda+\alpha} \cdot g(y) + \int_{\tau=0}^{\infty} \int_{\xi=0}^{\infty} e^{-(\lambda+\alpha)\tau} f_c(t+\tau, y-\xi)$

$$\times d(\lambda_1 \Phi_2(\xi)) d\tau + \frac{1}{\lambda+\alpha} \int_{\xi=0}^{\infty} f_{A_i}(y-\xi) d(\lambda_2 \Phi_1(\xi)) \}$$

Now, we take $f_c(t,x) \equiv h(x)$ independent of t, eqs. (4.2) and (4.3) coinside with each other, and $f_{A_i}(x) = f_p(t,x) = h(x)$. Moreover, this satisfies eq. (4.1). Since eqs. (4.1), (4.2), (4.3) have unique solution (See [3]), we have $f(x) = h(x)$.

5.   Bounds on s(u), S(u)

We define $f^n (n \geq 1)$ by the following equation.

(5.1) $f^n(u,x) = \inf_{y \geq x} \{K \cdot \delta(y-x) + L_u(y) + \int_{v \in \widehat{E}} \int_{\xi=0}^{\infty} dP(v|u) f^{n-1}(v,y-\xi)$

$$\times d\Phi_v(\xi) \}$$

$$f^0(u,x) = 0,$$

where u, $\widetilde{E}$, $L_u(y)$, $P(v|u)$ are defined in eq. (1)

Remark: We call f^n n truncated decision period case. And under appropriate conditions, $f^n(u, x)$ converges $f(u, x)$ in eq. (1) as $n \to \infty$ monotonically and uniformly in any finite interval. (See [3])

We denote critical numbers in eq. (5.1) as $s_n(u)$ and $S_n(u)$. Then, specially for demand process of type 1 and type 2 bounds on $s_n(u)$, $S_n(u)$ are given in the following theorem.

Theorem 5.1   For demand processes of type 1 and type 2,

$$(5.1) \quad \underline{s}(u) \leq s_n(u) \leq \bar{s}(u) \leq \underline{S}(u) \leq S_n(u) \leq \bar{S}(u)$$

where $\underline{s}(u)$, $\bar{s}(u)$, $\underline{S}(u)$, $\bar{S}(u)$ are defined as following.

$$(5.2) \quad \min_y L_u(y) = L_u(\underline{S}(u))$$

Let $\underline{s}(u)$ be the smallest number such that

$$(5.3) \quad L_u(\underline{s}(u)) \leq L_u(\underline{S}(u)) + K$$

Define $\bar{s}(u)$ as the smallest number such that

$$(5.4) \quad L_u(\bar{s}(u)) \leq L_u(\underline{S}(u)) + (1-D(u)) \cdot K,$$

where

$$(5.5) \quad D(u) = \frac{1}{\lambda+\alpha} \{\lambda + \alpha \cdot e^{-(\lambda+\alpha)u}\}, \quad 0 < u \leq T, \text{ for a demand}$$

process of type 1,

(5.6) $D(u) = \int_0^\infty d_v F(v|j) \frac{1}{\lambda+\alpha}\{\lambda + \alpha \cdot e^{-(\lambda+\alpha)v}\}$ for $u = A_j$,

and

(5.7) $D(u) = \int_0^\infty d_v K(j,t,v) \frac{1}{\lambda+\alpha}\{\lambda + \alpha \cdot e^{-(\lambda+\alpha)v}\}$

for $u = (t,j)$, in a demand process of type 2.

Define $\bar{S}(u)$ as the smallest number greater than $\underline{S}(u)$ for which

(5.8) $L_u(\bar{S}(u)) \gneq L_u(\underline{S}(u)) + K \cdot D(u)$

Proof: The method of proof is followed as same line as Veinott & Wagner [7], B. A. Kalymon [3]. So we will show only the outline.

(1) The Proof of $\underline{S}(u) \leq S_n(u)$ for all u, n.

Let Y_n be an optimal policy with $s_n(u)$, $S_n(u)$ structure with $S_n(u) \leqq \underline{S}(u)$ for some n, u.

Define the policy $Y_n{}'$ by

$$Y_n{}' = \begin{cases} \underline{S}(u) & \text{for} \quad x_n < s_n(u), \quad u_n = u \\ x_n & \text{for} \quad x_n \geqq s_n(u) \quad u_n = u \\ y_n & \text{for} \qquad\qquad\qquad u_n \neq u \end{cases}$$

and for $i = n-1, n-2, \ldots, 1$, $y_i{}' = \max\{x_i{}', y_i\}$, where $x_i{}'$ represent the inventory in period i when following $y_i{}'$.

For $i < n$, $K \cdot [\delta(y_i - x_i) - \delta(y_i{}' - x_i{}')] \geqq 0$

Also, either $y_i' = y_i$, in which case $L_{u_i}(y_i') = L_i(y_i)$ or $y_i < y_i' \leq \underline{S}(u)$, and theorefore, by convexity of $L_u(y)$ and the definition of $\underline{S}(u)$

$$L_{u_i}(y_i') \leq L_{u_i}(y_i)$$

Then for $x_n = x < s_n(u)$ and $u_n = u$, we have[3]

$$f^n(u,x|Y_n) - f^n(u,x|Y_n') \geq K \cdot E\{\delta(y_n-x_n) - \delta(y_n'-x_n)\}$$

$$+ L_u(y_n) - L_u(y_n') = L_u(S_n(u)) - L_u(\underline{S}(u)) > 0$$

Since $f^n(u,x|Y_n) = f^n(u,x|Y_n')$ for all other u_n, and for $u_n = u$ with $x_n \geq s_n(u)$, we have shown that Y_n is not an optimal policy, which is contradiction.

(2) The proof of $\underline{s}(u) \leq s_n(u)$

Suppose for an optimal policy Y_n, $s_n(u) < \underline{s}(u)$ for some n, u. By (1), $\underline{S}(u) \leq S_n(u)$. Let Y_n' be an alternative policy such that

$$Y_n' = \begin{cases} S_n(u) & \text{for } x_n < s_n(u), & u_n = u \\ \underline{S}(u) & \text{for } s_n(u) \leq x_n < \underline{s}(u), & u_n = u \\ x_n & \text{for } x_n \geq \underline{s}(u), & u_n = u \\ y_n & \text{for } u_n \neq u \end{cases}$$

and for $i = n-1, n-2, \ldots, 1$, $y_i' = \max\{x_i', y_i\}$.

Then, as in similar argument

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3): E denotes expectation mark.

$$f^n(u,x|Y_n) - f^n(u,x|Y_n') \geq [L_u(x) - (L_u(\underline{S}(u)) + K)] > 0$$

for $u_n = u$, with $s_n(u) \leqq x_n = x < \underline{s}(u)$. This is a contra-
diction.

(3) The proof of $s_n(u) \leqq \bar{s}(u)$.

Let Y_n be an optimal policy with $\bar{s}(u) < s_n(u)$.

Define Y_n' by

$$Y_n' = \begin{cases} S_n(u) & \text{for } x_n < \bar{s}(u), \quad u_n = u \\[2ex] x_n & \text{for } x_n \geq \bar{s}(u), \quad u_n = u \\[2ex] y_n & \text{for } u_n \neq u \end{cases}$$

and for $i = n-1, n-2, \ldots, 1, y_n' = y_n$.
For $u_n = u$ and $\bar{s}(u) \leqq x_n = x < s_n(u)$, at first consider type 1,
then

$$f^n(u,x|Y_n) - f^n(u,x|Y_n') = K + L_u(S_n(u))$$

$$- L_u(x) + E \int_0^u \lambda e^{-\lambda t} e^{-\alpha t} dt [K \cdot \delta(y_{n-1} - x_{n-1})$$

$$- K \cdot \delta(y_{n-1} - x_{n-1}')] + E \int_u^\infty \lambda \cdot e^{-\lambda t} e^{-\alpha u} dt [K \cdot \delta(y_{n-1} - x_{n-1})$$

$$- K \cdot \delta(y_{n-1} - x_{n-1}')]$$

Here, the notation of E is a expectation mark with respect
to demand size. Remark that x_{n-1} (also x_{n-1}') is a different
random variable according to the occurence time of demand,
$t < u$ or $t \geqq u$. But in any way, $\delta(y_{n-1} - x_{n-1}) - \delta(y_{n-1} - x_{n-1}') \geqq -1$.

Then, we obtain

$$f^n(u,x|Y_n) - f^n(u,x|Y_n') \geqq (1-D(u)) \cdot K + L_u(\underline{S}(u)) - L_u(\bar{s}(u)) \geqq 0,$$

where $\quad D(u) = \frac{1}{\lambda+\alpha} \{\lambda + \alpha \cdot e^{-(\lambda+\alpha)u}\} \quad$ for type 1.

In an analogous manner, for $u_n = u$ and $\bar{s}(u) \leqq x_n = x < s_n(u)$ and for type 2,

$$f^n(u,x|Y_n) - f^n(u,x|Y_n') = K + L_u(S_n(u)) - L_u(x)$$

$$+ E \{\sum_k \int_0^\infty d_v F(v,k|j) \int_0^v \lambda e^{-\lambda t} e^{-\alpha t} dt [K \cdot \delta(y_{n-1} - x_{n-1})$$

$$- K \cdot \delta(y_{n-1} - x_{n-1}')\} + E \{\sum_k \int_0^\infty d_v F(v,k|j) \int_v^\infty \lambda e^{-\lambda t} e^{-\alpha v} dt [K \cdot \delta(y_{n-1}$$

$$- x_{n-1}) - K \cdot \delta(y_{n-1} - x_{n-1}')] \qquad \text{for } u = A_j$$

and

$$f^n(u,x|Y_n) - f^n(u,x|Y_n') = K + L_u(S_n(u)) - L_u(x)$$

$$+ E \int_0^\infty d_v K(j,t,v) \int_0^v \lambda e^{-\lambda t} e^{-\alpha t} dt [K \cdot \delta(y_{n-1} - x_{n-1})$$

$$- K \cdot \delta(y_{n-1} - x_{n-1}')]$$

$$+ E \int_0^\infty d_v K(j,t,v) \int_v^\infty \lambda e^{-\lambda t} e^{-\alpha v} dt [K \cdot \delta(y_{n-1} - x_{n-1}) - K \cdot \delta(y_{n-1} - x_{n-1}')]$$

$$\text{for } u = (t,j)$$

We put $\quad D(u) = \int_0^\infty d_v F(v|j) \frac{1}{\lambda+\alpha} \{\lambda + \alpha \cdot e^{-(\lambda+\alpha)v}\} \quad$ for $u = A_j$

and $\quad D(u) = \int_0^\infty d_v K(j,t,v) \frac{1}{\lambda+\alpha} \{\lambda + \alpha \cdot e^{-(\lambda+\alpha)v}\}$ for $u = (t,j)$

Then, the same reason for type 1, we obtain

$$f^n(u,x|Y_n) - f^n(u,x|Y_n') \geq 0$$

So, Y_n' is also an optimal policy. Repeating the procedure for every u for which $\bar{s}(u) < s_n(u)$, we arrive at an optimal policy for which $s_n(u) \leq \bar{s}(u)$ for all u.

(4)   The proof of $S_n(u) \leq \bar{S}(u)$.

Let Y_n be an optimal policy with $S_n(u) > \bar{S}(u)$ and define Y_n' by

$$Y_n' = \begin{cases} \underline{S}(u) & \text{for} \quad x_n < s_n(u), \quad u_n = u, \\ x_n & \text{for} \quad x_n \geq s_n(u), \quad u_n = u, \\ Y_n & \text{for} \quad u_n \neq u \end{cases}$$

and for $i = n-1, n-2, \ldots, 1, y_i' = y_i$.

Then, in the same manner as (3), for $x_n = x < s_n(u)$ and $u_n = u$,

$$f^n(u,x|Y_n) - f^n(u,x|Y_n') \geq L_u(S_n(u)) - L_u(\underline{S}(u)) - K \cdot D(u)$$

$$\geq L_u(\bar{S}(u)) - L_u(\underline{S}(u)) - K \cdot D(u) \geq 0$$

Thus Y_n' is also an optimal policy. Repeating the procedure for every u for which $\bar{S}(u) < S_n(u)$, we arrive at an optimal policy for which $S_n(u) \leq \bar{S}(u)$ for all u, as required.

6. Some Examples

We calculate optimal policies for a special case of type 1 whose component processes have both deterministic demand size, for compound Poisson process demand size is 1 and for other process it is M. We consider all variables discrete, then consider that intervals of Poisson process followed geometric distribution, $p(X = k) = p \cdot q^{k-1}$, $k = 1, 2, \ldots$

For $\bar{\tau} = 0$ and $g(y) = \begin{cases} h \cdot y & y \geq 0 \\ -r \cdot y & y < 0 \end{cases}$, bounds of optimal critical numbers are as follows.

$$(6.1) \quad -[\frac{K}{r\beta}] \lessdot s_n(t) \lessdot -[\frac{1-\beta}{r \cdot \beta} \cdot K] \lessdot 0 \leq S_n(t) \leq [\frac{K}{h}],$$

where β denotes discount rate.

We calculate in n truncated decision case, and apply it for infinite time horizon. (in our examples, convergences are attained in $n \leq 20$).

We take the cases where T = 30, M = 30, h = 1.0, q = 0.1 β = 0.9 and

	r	k
1st case	9.0	2
2nd case	99.0	2
3rd case	9.0	8
4th case	99.0	8
5th case	999.0	8
6th case	999.0	30
7th case	99.0	32

From (6.1), $s_n(t) = 0$ for all cases and we calculate only $\dot{S}_n(t)$.

Fig. 1 $\sim$ 4 illustrates how $S(t)$ varies with t. Since time t denotes that C type demand occurs at time T-t after D type occurred, we conjecture that when t is near T(=30), $S(t)$ gets small, because next occurrence of D type is far and C type demand is low.

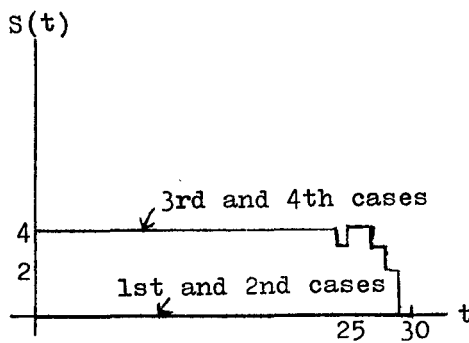Thus, we would estimate that $S(t)$ is first constant in some interval and next increases up some levels and then decreases linearly to zero.
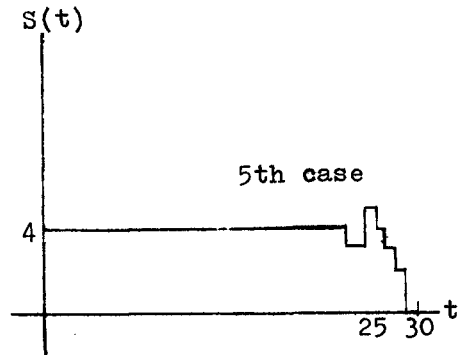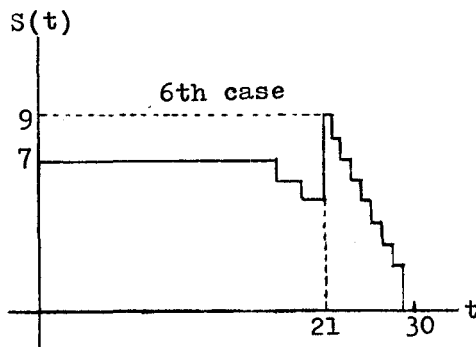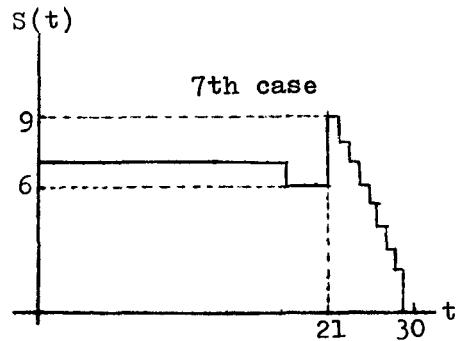

Fig.1

Fig.2

Fig.3

Fig.4

7. A Further Consideration

We consider same demand process as type 1 in §6. An optimal policy (s(t), S(t)) is so complicated both in calculation and in practical use that, observing the results of examples in §6 we present a combination of the simple strategies that is not always optimal but will be better than the simple (s, S) policy in some cases.

In [4], Popp discussed the simple strategies as above, but, from his assumption of independent intervals between demands, any combined policy can not be better than simple (s, S) policy and his results are erroneous.

Add following assumptions to those in §2.

1. $\bar{\tau} = 0$

2. The holding costs are given per unit of time and unit of quantity by h.

3. A discount can be neglected.

4. No delay is allowed.

5. M is relatively large and an inventory level can not exceed M.

6. A stocking level Q is discrete variable.

The criterion of optimization are minimal costs per unit of time. From above Assumption 1 and 4, optimal s(t) = 0.

Now, we define three strategies, Policy 1, Policy 2 and Policy 3.

Policy 1: a simple (s, S) policy

Policy 2: a directly ordered policy, that is a policy without stock on hand

Policy 3: a combined policy where decision of a simple (s,S) policy is followed for some interval length t_1 ($t_1 > 0$) and after the time t_1, directly ordered policy is followed.

Let $C_1(Q)$ and $C_3(Q, t_1)$ be respectively the costs per unit of time and $C_1(Q, t)$ and $C_3(Q, t_1, t)$ the costs over time interval $(0,t)$, following policy 1 and 3 with stock on hand Q. From the above assumption 5, $Q < M$. Then, a time when a demand with constant intervals occurs, is always an ordering point and a renewal point under this inventory process.

As well known results of renewal theory,

$$(7.1) \quad C_1(Q) = \frac{C_1(Q,T)}{T} = \frac{Q}{2}h + \frac{K}{T}(1 + \sum_{k=1}^{\infty} G^{*kQ}(T)) \quad \text{for } M > Q,$$

where $G(t)$ denotes the exponential distribution with parameter λ and $G^{*1}(t)$ the l th convolution of $G(t)$, that is l-Erlang distribution.

As for policy 3, we use following notations.

X_i: interval of ordering when Policy 1 is followed.

They have an independent and identical Q-Erlang (G^{*Q}) distribution.

$N(t)$: the number of ordering until time t.

Then, for $x \geq t_1 > 0$,

$$P(N(t_1) = k, \ x \leq X_1 + \ldots + X_k \leq x+dx)$$

$$= \int_0^{t_1} g^{*(k-1)Q}(y) \cdot g^{*Q}(x-y) \, dy \, dx \quad \text{for } k \geq 2$$

$$P(N(t_1) = 1, \; x \le X_1 \le x+dx) = g^{*Q}(x)\,dx,$$

where $\quad g^{*Q}(x) = \dfrac{dG^{*Q}(x)}{dx}$

Using the above,

(7.2) $\displaystyle C_3(Q,t_1,T) = \sum_{k=1}^{\infty} \int_0^{\infty} C_3(Q,t_1,T \,|\, N(t_1)=k, \; X_1+ \ldots +X_k=x)\,dP(N(t_1) = k,$

$X_1+\ldots+X_k=x)$

$\displaystyle = \sum_{k=1}^{\infty} \int_{x=t_1}^{T} \{K \cdot (k+\lambda(T-x)) + h \cdot \frac{Qx}{2}\}\,dP(N(t_1)=k, X_1+\ldots+X_k=x)$

$\displaystyle + \sum_{k=1}^{\infty} \int_{x=T}^{\infty} \{K \cdot k + h \cdot \frac{QT}{2}\}\,dP(N(t_1)=k, X_1+\ldots+X_k=x)$

$\displaystyle = K + \frac{hQT}{2} + (K\lambda T - \frac{hQT}{2}) \cdot G^{*Q}(T) + (K-K\lambda T + \frac{hQT}{2}) \cdot M_Q(t_1)$

$\displaystyle + (K\lambda T - \frac{hQT}{2}) \int_0^{t_1} m_Q(y) G^{*Q}(T-y)\,dy + (\frac{hQ}{2} - K\lambda) \int_{x=t_1}^{T} \int_{y=0}^{t_1} x \cdot g^{*Q}(x-y) m_Q(y)\,dx\,dy$

$\displaystyle + (\frac{hQ}{2} - K\lambda) \int_{x=t_1}^{T} x \cdot g^{*Q}(x)\,dx,$

where

$$M_Q(t) = \sum_{k=1}^{\infty} G^{*kQ}(t), \quad m_Q(t) = \frac{\alpha M_Q(t)}{dt} = \sum_{k=1}^{\infty} g^{*kQ}(t)$$

Of course, from (7.1) and (7.2),

$$C_3(Q, T, T) = C_1(Q, T)$$

Now, we calculate an optimal t_1.

(7.3) $\dfrac{\partial C_2(Q, t_1, T)}{\partial t_1} = (K - K\lambda T + \dfrac{hQ}{2}T)m_Q(t_1) + (K\lambda T - \dfrac{hQ}{2}T)$

$\times m_Q(t_1) \cdot G^{*Q}(T-t_1) - (\dfrac{hQ}{2} - K\lambda) \cdot t_1 \cdot M_Q(t_1) + (\dfrac{hQ}{2} - K\lambda) \cdot t_1 \cdot G^{*Q}(t_1)$

$+ (\dfrac{hQ}{2} - K\lambda)m_Q(t_1) \cdot t_1 \cdot G^{*Q}(T-t_1) + (\dfrac{hQ}{2} - K\lambda) \cdot m_Q(t_1) \cdot \dfrac{Q}{\lambda} \cdot G^{*(Q+1)}(T-t_1)$

$- (\dfrac{hQ}{2} - K\lambda) \cdot t_1 \cdot g^{*Q}(t_1)$

(7.4) $\dfrac{\partial C_3(Q, t_1, T)}{\partial t_1}\Bigg|_{t_1=T} = (K - K\lambda T + \dfrac{hQ}{2}T) \cdot m_Q(T) + (K\lambda - \dfrac{hQ}{2}) \cdot T \cdot g^{*Q}(T)$

$+ (K\lambda - \dfrac{hQ}{2}) \cdot T \cdot (M_Q(T) - G_Q(T))$

Now, for policy 2,

(7.5) $C_2 = \dfrac{K}{T} + K\lambda$

As in [4], the comparison of inventory policies uses the policies with optimized decision variables, and the sign > is used for a preference, then

(7.6) P2 < P1 for $k\lambda > 2h$

From (7.4), if $k\lambda > \dfrac{hM}{2}$ and $\lambda T < 1$ for example,

$\dfrac{\partial C_3(Q, t_1, T)}{\partial t_1}\Bigg|_{t_1=T} > 0$

Then, if $k\lambda > \dfrac{hM}{2}$ and $\lambda T < 1$

P3 > P1 > P2

Acknowledgement

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