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ON APPROXIMATION IN TIME IN OPTIMAL INVENTORY PROCESSES

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Abstract

In the inventory processes, it may be better to count the member of items only when the supply is low, instead of keeping records every periods. In this paper, the problem that we want to study is that of determining when to examine the number of items remaining in stock. We first show that the fundamental equation of Δ-processes can be reduced to one involving only a single variable. Secondary, we shall obtain an inequality connecting the returns of the Δ and $2\ \Delta$-processes. Thirdly, we present an estimation of error for this type of the approximation in time using the above inequality. Finally, we carry through a simulation of our inventory model.

1. Introduction

In introducing the basic optimal inventory equation, explicit use was made of the assumption that observations and orders are made at each period [1]. The assumption however, is a questionable one. Instead of keeping records every period, it may be better to count the number of items only when the supply is low, and even to pay a penalty charge for getting items quickly when the supply is very low. The problem that we want to study is that of determining when to examine the number of items remaining in stock. In [2], we showed that at a certain point the cost of accuracy in quantity of keeping record is greater than the gain that is obtained by using the records. In this paper, we shall discuss the analytic and computational studies of the approximation in time in the optimal inventory processes.

We first show that the functional equation of Δ processes can be reduced to one involving only a single variable, and therefore that all of the theory developed for the special case of Δ-processes is applicable in general. Secondly, we shall obtain an inequality connecting the returns of the Δ and $2\ \Delta$-processes. Thirdly, we present an estimation of error for this type of the "approximation in time," using the above-mentioned inequality. Finally, we carry through a simulation of this inventory model.

2. Mathematical Formulation of the Approximation in Time

An inventory model with back-log is adopted. The inventory periods $I_0, I_1, \ldots, I_R, \ldots, I_{2R}, \ldots$, are numbered from left to right. An inventory model involving observation only at periods $I_0, I_R, I_{2R}, \ldots,$ is considered. At the beginning of the period nR, (n=0,1,...,N-1), we assume an observation to be taken, and a regular order made. Assuming that we are given the ordering cost, the penalty cost, the holding cost, the observation cost, and the distribution of demand at any stage, we wish to determine ordering policies which minimize the total expected cost.

Assumptions

(2.1) x_{nR} is the stock level at the nR-th stage, prior to the delivery of the quantity ordered at the nR-th stage and the demand at the nR-th stage (n=0,1,...,N-1). The inventory stock levels at other times are estimated.

(2.2) y_{nR} is the quantity ordered at the beginning of the nR-th stage (n=0,1,...,N-1).

(2.3) The demands z_n are identically, independently distributed nonnegative random variables. The continuous density function will be denoted by $\varphi(\xi)$.

(2.4) The holding cost function is given by h(z). We assume that h(0)=0 and that h is a continuous, convex, increasing function of z.

(2.5) The penalty cost functions are given by p(z). We assume that p(0)=0 and that p(z) is a continuous, convex, increasing function of z.

(2.6) The ordering cost function c(z) is given by c(z)=cz (z>0), = 0(z<0).

(2.7) The observation cost function $d(z)$ is given by $d(z) = dz\ (z>0),\ = 0\ (z<0)$.

(2.8) There is a discount factor α such that $0<\alpha<1$.

(2.9) $h'(0)<0,\quad p'(0)>0$.

We let

(2.10) $$L(y,\varphi)=\int_0^y h(y-\xi)\varphi(\xi)d\xi+\int_y^\infty p(\xi-y)\varphi(\xi)d\xi.$$

Then $L(y,\varphi^i)$ represents the expected inventory cost and the expected penalty cost and φ^i is the i-fold convolution of φ.

Let $f_{nR}(x)$ represent the total expected discounted cost for nR-periods if an optimal policy is follows. Then for $n=0,1,\ldots,N-1$,

(2.11) $$f_{nR}(x)=\min_{y\geq 0}[d(x)+c(y)+L(x+y,\varphi^1)+\alpha L(x+y,\varphi^2)+\ldots$$
$$+\alpha^{R-1}L(x+y,\varphi^R)+\alpha\int_0^\infty f_{(n-1)R}(x,y-\xi)\varphi^{R+1}(\xi)d\xi],$$
$$f_R(x)=\min_{y\geq 0}[d(x)+c(y)+L(x+y,\varphi^1)+L(x+y,\varphi^2)+\ldots$$
$$+\alpha^{R-1}L(x+y,\varphi^R)],$$
$$f_0(x)=0.$$

We let, for $n=0,1,\ldots,N-1$,

(2.12) $$F_n(x)=f_{nR}(x),$$
$$g_R(x,y)=d(x)+c(y)+L(x+y,\varphi^1)+\ldots+\alpha^{R-1}L(x+y,\varphi^R)$$

Then we have from (11) for $n=0,1,\ldots,N-1$,

$$(2.13)\quad F_n(x)=\min_{y\geqq 0}[g_R(x,y)+\alpha\int_0^{\infty}F_{n-1}(x+y-\xi)\varphi^{R+1}(\xi)d\xi]$$
$$=\min_{w\geqq x}[(d-c)x+G_n(w)],$$

$$F_1(x)=f_R(x),$$

$$F_0(x)=0,$$

where for $x+y=w$

$$G_n(w)=c(w)+L(w,\varphi^1)+\alpha L(w,\varphi^2)+\ldots+\alpha^{R-1}L(w,\varphi^R)$$
$$+\alpha\int_0^{\infty}F_{n-1}(w-\xi)\varphi^{R+1}(\xi)d\xi.$$

Equation (2.13) is identical with the usual optimal inventory equation. The analog of this equation for an infinite period model is

$$(2.14)\quad F(x)=\min_{w\geqq x}[g_R(x,w-x)+\alpha\int_0^{\infty}F(w-\xi)\varphi^{R+1}(\xi)d\xi],$$

where $\quad F(x)=\lim_{n\to\infty}F_n(x).$

Then we have the following theorem.

Theorem 1. For an infinite period model, if x is the effective inventory on hand at the beginning of the first period, and

$$L'(0,\varphi^1)+\alpha L'(0,\varphi^2)+\ldots+\alpha^{R-1}L'(0,\varphi^R)+c<0,$$

then (a) the optimal ordering policy in the first period is to order $\max[0,\bar{x}-x]$ units of stock, where the critical level $\bar{x}$ is the unique root of the equation

$$(2.15)\quad \alpha d+c(1-\alpha)+L'(w,\varphi^1)+\alpha L'(w,\varphi^2)+\ldots+\alpha^{R-1}L'(w,\varphi^R)=0.$$

(b) $F(x)$ is convex and twice continuous differentiable, except possibly at $(\bar{x}=x)$. In addition,

$$(2.16)\quad F'(x)=d-c, \qquad (x\leqq\bar{x}),$$

$$=d+L'(x,\varphi^1)+\alpha L'(x,\varphi^2)+\ldots+\alpha^{R-1}L'(x,\varphi^R)$$

$$+\alpha\int_0^\infty F'(x-\xi)\varphi^{R+1}(\xi)d\xi$$

$$(x>\bar{x}).$$

We show that this theorem holds for the n-period model $(n\geqq R+1)$, where we replace $\bar{x}$ by $\bar{x}_n$ and $F(x)$ by $F_n(x)$.

Proof. The proof proceeds by induction on the number of total periods in the inventory program. Specifically, we shall truncate the model to n periods, and, subsequently, let $n\to\infty$. For n=1, we have

$$(2.17)\quad F_1(x)=\min_{w\geqq x}[(d-c)x+G_1(w)].$$

Let $\bar{x}_1$ be defined as the smallest value of w for which

$$(2.18)\quad G_1(\bar{x}_1)=\min_{w\geqq x}[G_1(w)].$$

By assumption, we have $G_1'(0)<0$ and since $G_1(w)\to\infty$ as $w\to\infty$, we infer that $0<\bar{x}<\infty$.

Clearly, the value of w that minimizes $G_1(w)$ also minimizes $(d-c)x+G_1(w)$ with respect to w.

If we can show that $G_1(w)$ is convex, then $\bar{x}_1$ will be the root of the equation

$$(2.19)\quad G_1'(w)=0.$$

Differentiating $G_1(w)$ twice with respect to w yields

$$
\begin{aligned}
G_1'(w)=&c+\int_0^w h'(w-\xi)\varphi^1(\xi)d\xi-\int_w^\infty p'(\xi-w)\varphi^1(\xi)d\xi\\
&+\alpha\int_0^w h'(w-\xi)\varphi^2(\xi)d\xi-\alpha\int_w^\infty p'(\xi-w)\varphi^2(\xi)d\xi\\
&+\cdots\\
&+\alpha^{R-1}\int_0^\infty h'(w-\xi)\varphi^R(\xi)d\xi-\alpha^{R-1}\int_w^\infty p'(\xi-w)\varphi^R(\xi)d\xi,\\
&+[h(0)+p(0)][\varphi^1(w)+\alpha\varphi^2(w)+\ldots+\alpha^{R-1}\varphi^R(w)].
\end{aligned}
\tag{2.20}
$$

Since h(z) and p(z) are each continuous, convex and increasing functions, obviously G(w) is convex and $\bar{x}_1$ is the smallest root of (2.19). If follows that where $x\leqq\bar{x}_1$, the optimal policy calls for ordering to the level $\bar{x}_1$ and where $x>\bar{x}_1$, the optimal policy calls for non-ordering.

Then we have

$$
F_1(x)=\begin{cases}-cx+dx+G_1(\bar{x}_1), & (x\leqq\bar{x}_1),\\ -cx+dx+G_1(x), & (x>\bar{x}_1),\end{cases}
\tag{2.21}
$$

and from differentiation with respect to x we have

$$
F_1'(x)=\begin{cases}-c+d, & (x\leqq\bar{x}_1),\\ -c+d+G_1'(x), & (x>\bar{x}_1),\end{cases}
\tag{2.22}
$$

and

$$
F_1'(x)\geqq 0 \quad \text{except at } x=\bar{x}_1.
\tag{2.23}
$$

These considerations prove the theorem for the one period inventory model.

Assuming now that the theorem has been proved for the (n-1) period model, we shall show that it holds for an n-period model. Differentiating $G_n(w)$ twice with respect to w gives

$$
\begin{aligned}
G_n''(w) = {} & [h'(0)+p'(0)][\varphi^1(w)+\alpha\varphi^2(w)+\ldots+\alpha^{R-1}\varphi^R(w)] \\
& +\int_0^w h''(w-\xi)\varphi^1(\xi)d\xi+\alpha\int_w^\infty p''(\xi-w)\varphi^2(\xi)d\xi \\
& + \ldots \\
& +\alpha^{R-1}\int_0^w h''(w-\xi)\varphi^R(\xi)d\xi+\alpha^{R-1}\int_w^\infty p''(w-\xi)\varphi^R(\xi)d\xi \\
& +\alpha\int_0^\infty F''_{n-1}(w-\xi)\varphi^{R+1}(\xi)d\xi,
\end{aligned}
$$

In view of the assumptions, it is now clear that $G_n(w)$ is convex. The argument hereafter is identical to the one used for the one period model. It thus follows the theorem holds for the n-period model.

By applying a standard limiting argument, we can show that $F_n(x)$ converges to $F(x)$ and similarly that the critical number $\bar{x}_n$ of the truncated model converges to $\bar{x}$ which is the critical number of the full dynamic model. The proof of the theorem is complete.

3. Inequality

Let us consider the maximization of a functional of the form

$$J(q)=\int_0^T \underset{v}{E}(h(p(t),q(t),v(t)))dt,$$

subject to a relation of the form

(a) $\frac{dp}{dt}=Q(p,q,v)$, $p(0)=p$,

(b) $R(p,q) \leqq 0$.

Here p and q are N-dimensional vectors, while R is an M-dimensional vector function. The maximization is over q and E is the expectation over v.

Since a solution of a problem of this type is only in rare instance obtainable in explicit form recourse must be had to some type of approximate solution if we are interested in numerical results. A method going back to Euler consists of approximating to the integral in J(q) by a sum of the form

$$J_n(q)=\sum_{i=0}^{n} \underset{v}{E}(h(p(i),q(i),v(i))),$$

and

(a)' $p(i+1)=p(i)+\Delta Q(p(i),q(i))$, $p(0)=p$, $(i=0,1,\ldots,N)$,

(b)' $R_j(p(i),q(i))\leqq 0$, $(j=1,\ldots,M)$,

where

$$\Delta=T/n,\quad p(i)=p(iT/n),\quad q(i)=g(iT/n).$$

Let us keep Δ fixed, equal to T/n, and define for $k=0,1,\ldots,n$, the sequence of functions

$$f_k(p)=\max_q\{J_n(q)\}.$$

It is easy to derive the recurrence relations

$$f_k(p)=\max_{q}{}_{(0)}[\Delta g(p,q)+\underset{v}{E}(f_{k+1}(p+\Delta Q(p,q,v)))],$$

$$f_0(p)=\max_{q}{}_{(0)}[\Delta g(p,q)],$$

where

$$g(p,q)=\underset{v}{E}(h(p(0),q(0),v(0))),$$

$$R_j(p(0),q(0))\leq 0, \quad j=1,2,\ldots,M.$$

Problems of this sort were discussed in the deterministic case, cf. [2]. In this paper, we shall consider the stochastic case.

3.1. Assumption.

Let us keep Δ fixed, equal to T/n, and consider the following maximization problem in n-stage decision process, with the state variables p and i,

$$(3.1) \quad f_i(p)=\max_q[\Delta g_i(p,q)+\sum_{j=1}^{n}\int f_j(p+\Delta Q(p,q,v))G_{ij}(v)dv]. \qquad (i=1,2,3,\ldots,M)$$

Introduce the following vectors and matrices to simplify our notation:

$$f(p)=\begin{pmatrix} f_1(p) \\ \vdots \\ f_n(p) \end{pmatrix}, \quad g(p,q)=\begin{pmatrix} g_1(p,q) \\ \vdots \\ g_n(p,q) \end{pmatrix}, \quad G(p,q,v)=\begin{pmatrix} G_{11}\cdots G_{1n} \\ \vdots \\ G_{n1}\cdots G_{nn} \end{pmatrix},$$

$$Q(p,q,v)=\begin{pmatrix} Q_1(p,q,v) \\ \vdots \\ Q_n(p,q,v) \end{pmatrix}$$

We take

$$(3.2) \quad f(p)=\max_q[\Delta g(p,q)+\int G(v)f(p+\Delta Q(p,q,v))dv],$$

where it is understood that the maximum is taken element by element.

We shall assume that $g(p,q)$ and $Q(p,q,v)$ are the continuous functions with some conditions on the modulus of continuity of the functions and $Q(p,q,v)$. The simplest is the uniform Lipschitz condition of the type

(3.3) $\|g(p_1,q)-g(p_2,q)\| \leqq K\|p_1-p_2\|\alpha,$

for some $K \geqq 0$ and α satisfying $0 \leqq \alpha \leqq 1$ for all q, satisfying qs and p_1,p_2 lying in some fixed interval $[-c,c]$, where $\|g(p_1,q) -g(p_2,q)\|=\max_i|g_i(p_1,q)-g_i(p_2,q)|$ and $\|p_1-p_2\|=\max_i|p_{1i}-p_{2i}|$.

Let us now consider the successive approximations determined by

$$f_o(p)=0.$$

(3.4) $f_{n+1}(p)=\max_q[\Delta g(p,q)+\int G(v)f_n(p+\Delta Q(p,q,v))dv].$

Assume that

(a) $g(p,q)$ and $Q(p,q,v)$ are jointly continuous in p,q, and v is the region of the $p\varepsilon D$, $q\varepsilon s$ and $v\varepsilon R$ satisfying the restriction $\|p\| \leqq c_1$, where $\|p\|=\max_i|p_i|$, and $\|q\| \leqq c_2$ where $\|q\|=\max|q_i|$, and $\|p\|<\infty$ where $\|v\|=\max_i|v_i|$.

Also, to satisfy (3.4) in this region, for α satisfying $\alpha^2+\alpha \geqq 1$.

(b) $\|Q(p,q,v)\| \leqq \alpha\ \|P\|+h$, where $\|Q(p,q,v)\|=\max_i|Q_i(p,q,v)|$.

(3.5)

(c) $G(v)$ is integrable over an infinite interval.

3.2. Lemma.

We shall repeatedly use the following results. The following lemma is well known [2].

Lemma 1.

(3.6) $$T_1(p)=\max_q[g(p,q)+\int G(v)f(p,q,v)dv],$$

$$T_2(p)=\max_q[h(p,q)+\int G(v)F(p,q,v)dv],$$

then

(3.7) $$\|T_1(p)-T_2(p)\| \leqq \max_q[\|g(p,q)-h(p,q)\|+\int\|G(v)\| \|f(p,q,v)-F(p,q,v)\|dv].$$

Let $h=\max\|Q(p,q,v)\|$ and let us call the interval $[-c_1hk\Delta,\ c_1+hk\Delta]$, the k-th interval. We choose T and Δ so that with c in the initial interval, the n-th interval is contained in $[-c_2,c_2]$. In this way, we preserve uniform bounds.

It is essential for our proof to establish a uniform Lipschitz condition for the member of the sequence.

Lemma 2. Consider the sequence $\{f_k(p)\}$ as defined by (3.4) under the condition of the Theorem. For $k=0,1,\ldots n$, we have

(3.8) $$\|f_k(p_1)-f_k(p_2)\| \leqq m\|p_1-p_2\|^\alpha,$$

for p_1 and p_2 in the $(s-k+1)$th interval, where m is independent of p_1,p_2 and is depend of k, or Δ.

Proof. The proof will proceed by an induction on k.

We have

$$f_1(u)=\max_q \Delta g(u,q),$$

(3.9)

$$f_1(w)=\max_{q'} \Delta g(w,q'),$$

where q and q' satisfy the constraints $q\varepsilon S$, $q'\varepsilon S$.

Applying Lemma 1, we obtain the inequality

(3.10) $$\|f_1(u)-f_1(w)\| \leqq \max_q \Delta \|g(u,q)-g(w,q)\| \leqq K\Delta \|u-w\|^{\alpha}.$$

Assume that we have demonstrated that

(3.11) $$\|f(u)-f(w)\| \leqq K_1\Delta \|u-w\|^{\alpha},$$

for k=0,1,2,...,n, for u and w in the (n-k-1)-th interval. Turning to the recurrence relation and applying Lemma 1, we obtain the relation

(3.12) $$\|f_{k+1}(u)-f_{k+1}(w)\| \leqq \max_q [\Delta \|g(u,q)-g(w,q)\| + \int \|G(v)\| \; \|f_k(u+\Delta Q(u,q,v))-f_k(w+\Delta Q(w,q,v))\| dv].$$

If u and w lie in the (n-k-2)-th interval, the points $u+\Delta Q(u,q,v)$, $w+\Delta Q(w,q,v)$ will certainly be included in the (n-k-1)-th interval.

We have

(3.13) $$\begin{aligned}\|f_{k+1}(u)-f_{k+1}(w)\| &\leqq \Delta K\|u-w\|^{\alpha}+\int \|G(v)\| \{K_k\Delta \|u+\Delta Q(u,q,v) \\ &\qquad -w-\Delta Q(w,q,v)\|^{\alpha}\}dv \\ &\leqq (K_k+A_1K)\Delta\|u-w\|^{\alpha}+A_1K_k\Delta^2\|u-w\|^{\alpha} \\ &= (K_k+A_1K-A_1K_k\Delta)\Delta\|u-w\|^{\alpha}\end{aligned}$$

for a fixed constant A_1.

This shows that we can take $K_k=A_2kK$, for some constant $A_2\geqq 1$. Since $K\Delta\leqq n = T$, we see that we have a uniform Lipschitz condition.

We now wish to demonstrate a result concerning the stability of the sequence $\{f_k(p)\}$ under perturbations of the function $\Delta g(p,q)$.

Lemma 3. Consider the two sequences.

$$(3.14)\quad \begin{aligned} f_{k+1}(p)&=\max_q[\Delta g(p,q)+\int G(v)f_k(p+\Delta Q(p,q,v))dv], \\ F_{k+1}(p)&=\max_q[\Delta\bar{g}(p,q)+\int G(v)f_k(p+\Delta Q(p,q,v))dv], \end{aligned}$$

with

$$(3.15)\quad \begin{aligned} f_0(p)&=\max_q\Delta g(p,q), \\ F_0(p)&=\max_q\Delta\bar{g}(p,q). \end{aligned}$$

We have, under the hypothesis of the theorem,

$$(3.16)\quad \| f_k(p)-F_k(p)\| \leqq K_k\Delta\max_q \max_p \| \Delta g(p,q)-\bar{g}(p,q)\|,$$

for k-0,1,...n. The notation $\max_q$ and $\max_p$ signifies that the maximum is taken over k-th interval as defined above.

3.3 Inequality.

Let us now prove some inequality.

We shall obtain an inequality connecting the returns of the Δ and 2Δ-processes.

To define the Δ-process, the interval $[0,T]$ is divided into equal interval of length Δ. Choices of y are made at the points 0, Δ, 2Δ and so on. The 2Δ-process is defined, with intervals of length 2Δ. Let $\{f_k(p)\}$ denote the sequence of returns from the Δ-process, as defined by the recurrence relations of (3.4) and let $\{S_k(p)\}$ denote the sequence of return from the 2Δ-process.

Let us now define the following intermediate process. The interval length is Δ, but the policies are restricted to those which employ the same q-value at the point $2k\Delta$ and $(2k+1)\Delta$. Let $\{h_{2k}(p)\}$ denote the sequence of return obtained in this way.

Then

$$
\begin{aligned}
&h_0(p)=0,\\
&h_2(p)=\max_q[\Delta g(p,q)+\Delta\int G(v)g(p+\Delta Q(p,q,v),q)dv],\\
(3.17)\quad &\vdots\\
&h_{2k+2}(p)=\max_q[\Delta g(p,q)+\Delta\int G(v)g(p+\Delta Q(p,q,v),q)dv\\
&\qquad +\int G(v)dv\int G(v_2)h_{2k}(p+\Delta Q(p,q,v)+\Delta Q(p,q,v_1))dv]
\end{aligned}
$$

Here q is subject to the constraint $q\varepsilon S$.

It is clear that

$$(3.18)\quad h_{2k}(p)\leqq f_{2k}(p),\ (k=0,1,2,\ldots).$$

Let us now compare $h_{2k}(p)$ and $f_{2k}(p)$.

It is easy to show that the sequence $\{h_{2k}(p)\}$ satisfies the same type of uniform Lipschitz condition as the one we derived for $\{f_k(p)\}$.

Hence we may write

(3.19) $$h_{2k+2}(p)=\max_q[2\Delta g(p,q)+\int G(v)h_{2k}(p+2\Delta Q(p,q,v))dv+E_k(p,q)],$$

where

(3.20) $$\| E_k(p,q)\| \leqq a_2\Delta^{\alpha(1+\alpha)},$$

since

(3.21) $$\Delta\int G(v)g(p+\Delta Q(p,q,v),q)dv=\Delta g(p,q)+a_1\Delta^{1+\alpha},$$

and

(3.22) $$\int G(v)dv\int G(v_1)h_{2k}(p+\Delta Q(p,q,v)+\Delta Q(p+\Delta Q(p,q,v_1)))dv_1$$
$$=\int G(v)h_{2k}(c+2\Delta Q(p,q,v)+0(\Delta^{1+\alpha}))dv$$
$$=\int G(v)h_{2k}(c+2\Delta Q(p,q,v))dv+0(\Delta^{\alpha(1+\alpha)}).$$

Applying Lemma 3, we see that

(3.23) $$\| h_{2k}(p)-S_k(p)\| \leqq a_2(n\Delta)\Delta^{\alpha(1+\alpha)-1}\leqq a_2T\Delta^b,$$

where $b=\alpha(1+\alpha)-1>0$.

Combining (3.18) and (3.23), we obtain

(3.24) $$f_{2k}(p)-S_k(p)\geqq -a_2T\Delta^b.$$

4. Approximation

Let us now consider the inventory control process described above. To define the Δ-process, the interval $[0,T]$ is divided with intervals of length Δ. Let $f_k(x)$ denote the sequence of return from the Δ-process, as defined by the recurrence relation.

$$(4.1)\quad f_k(x)=\min_{w\geq x}[\Delta(d(x)+c(w-x)+L(w,\varphi^1))+\int_0^\infty f_{k-1}(w-\xi)\varphi^1(\xi)d\xi].$$

To define the 2Δ-process, the interval $[0,T]$ is divided into equal intervals of length 2Δ. Choices of y are made at the points 0, 2Δ, 4Δ and so on. Let $\{S_k(x)\}$ denote the sequence of return from the 2Δ-process

$$(4.2)\quad S_k(x)=\min_{w\geq x}[\Delta(d(x)+c(w-x)+2L(w,\varphi^2))+\int_0^\infty S_{k-2}(w-\xi)\varphi^2(\xi)d\xi].$$

Let us now define the following intermediate process. The interval length is Δ, but the policies are restricted to those which employ the same y-value at the points $2k\Delta$ and $(2k+1)\Delta$. Let $\{h_{2k}(x)\}$ denote the sequence of returns obtained in this way. Then

$$(4.3)\quad \begin{aligned} &h_0(x_1)=0\\ &h_2(x_1)=\min_{w\geq x}[\Delta(d(x_1)+c(w-x_1))+\Delta L(w,\varphi^1)+\Delta(d(x_2)+(w-x_2))\\ &\qquad\qquad +\Delta L(w+(w-x_2),\varphi^2)],\\ &\vdots\\ &h_{2k+2}(x_1)=\min_{w\geq x_i}[\Delta(d(x_1)+c(w-x_1))+\Delta L(w,\varphi^1)+\Delta(d(x_2)+c(w-x_2))\\ &\qquad\qquad +\Delta L(w+(w-x_2),\varphi^2)\\ &\qquad\qquad +\int h_{2k}(w+(w-x_2)-\xi)\varphi^2(\xi)d\xi].\\ &\qquad\qquad (i=1,2) \end{aligned}$$

Let M be an arbitrary nonnegative integer. Consider the dynamic inventory process in which for $n>M$ periods remaining, the ordering is based on the single critical number $\bar{x}_1$, for

$f_k(x)$ and $\bar{x}_2$, for $g_k(x)$, and the ordering decision is based on the optimal critical number.

Let us put M=0 without loss of generality. We see that

$$f_n(x)=h_n(x)$$

$$S_n(x_1)=\begin{cases}\Delta(d(x_1)+c(\bar{x}_{(2)}-x_1)+2L(\bar{x}_{(2)},\varphi^2))+\int_0^\infty S_{n-2}(\bar{x}_{(2)}-\xi)\varphi^2(\xi)d\xi, & (x_1\leqq\bar{x}_{(2)}),\\ \Delta(d(x_1)+2L(x_1,\varphi^2))+\int_0^\infty S_{n-2}(x_1-\xi)\varphi^2(\xi)d\xi, & (x_1>\bar{x}_{(2)}),\end{cases} \tag{4.4}$$

$$h_{2n}(x_1)=\begin{cases}\Delta(d(x_1)+c(x_1,-x_1)+L(x_{(1)},\varphi^1)+d(x_2)+c(x_{(1)}-x_2) \\ \quad +L(\bar{x}_{(1)}+\bar{x}_{(1)}-x_2,\varphi^2))+\int_0^\infty h_{2n-2}(\bar{x}_{(1)}+\bar{x}_{(1)}-x_2-\xi)\varphi^2(\xi)d\xi, & (x_1\leqq\bar{x}_{(1)},x_2\leqq\bar{x}_{(1)}),\\ \Delta(d(x_1)+c(\bar{x}_{(1)}-x_1)+L(\bar{x}_{(1)},\varphi^1)+d(x_2)+L(\bar{x}_{(1)},\varphi^2) \\ \quad +\int_0^\infty h_{2n-2}(\bar{x}_{(1)}-\xi)\varphi^2(\xi)d\xi, & (x_1\leqq\bar{x}_{(1)},x_2>\bar{x}_{(1)}),\\ \Delta(d(x_1)+L(x_1,\varphi^1)+d(x_2)+c(\bar{x}_{(2)}-x_2)+L(x_1+\bar{x}_{(1)}-x_2,\varphi^1)) \\ \quad +\int_0^\infty h_{2n-2}(x_1+\bar{x}_{(1)}-x_2-\xi)\varphi^2(\xi)d\xi. & (x_1>\bar{x}_{(1)},x_2\leqq\bar{x}_{(1)}),\\ \Delta(d(x_1)+d(x_2)+L(x,\varphi^1)+L(x,\varphi^2)) \\ \quad +\int_0^\infty h_{2n-2}(x_1-\xi)\varphi^2(\xi)d\xi, & (x_1>\bar{x}_{(1)},x_2>x_{(2)}).\end{cases}$$

Let us now compare $h_{2n}(x)$ and $g_n(x)$. There are the following possibilities:

I. $x_1 \leqq \bar{x}_{(2)}$, $x_1 \leqq \bar{x}_{(1)}$, $x_2 \leqq \bar{x}_{(1)}$,

II. $x_1 \leqq \bar{x}_{(2)}$, $x_1 \leqq \bar{x}_{(1)}$, $x_2 > \bar{x}_{(1)}$,

III. $x_1 \leqq \bar{x}_{(2)}$, $x_1 > \bar{x}_{(1)}$, $x_2 > \bar{x}_{(1)}$,

IV. $x_1 \leqq \bar{x}_{(2)}$, $x_1 > \bar{x}_{(1)}$, $x_2 > \bar{x}_{(1)}$,

V. $x_1 < \bar{x}_{(2)}$, $x_1 > \bar{x}_{(1)}$, $x_2 \leqq \bar{x}_{(1)}$,

VI. $x_1 > \bar{x}_{(2)}$, $x_1 > \bar{x}_{(1)}$, $x_2 > \bar{x}_{(1)}$.

Let us set

(4.5) $\quad \| h_2(x) - S_1(x) \| \leqq \Delta \| E_1 \|$. (i=I, II, III, IV, V, VI)

I. Applying Lemma 2 and Lemma 3, we see that

$$(4.6) \quad \| h_{2n}(x) - S_n(x) \| \leqq \Delta \| E_1 \| + \| \int_0^\infty \{ h_{2n-2}(\bar{x}_{(1)} + \bar{x}_{(1)} - x_2 - \xi) - h_{2n-2}(\bar{x}_{(2)} - \xi) \} \varphi^2(\xi) d\xi + \int_0^\infty \{ h_{2n-2}(\bar{x}_{(2)} - \xi) - S_{n-1}(\bar{x}_{(2)} - \xi) \} \varphi^2(\xi) d\xi$$

$$\leqq \Delta \| E_1 \| + K_k \| \bar{x}_{(1)} + \bar{x}_{(1)} - x_2 - \bar{x}_{(2)} \|^{\alpha} + (a_2 T \Delta h - 2) \Delta \| E_1 \|$$

$$\leqq \Delta ((2n-1) \| E_1 \| + K_k \| \bar{x}_{(1)} + \bar{x}_{(1)} - x_2 - \bar{x}_{(2)} \|),$$

where

$$E_1=d(x_2)+c(2\bar{x}_{(1)}-\bar{x}_{(2)}-x_2)+(p+h)(\int_{\bar{x}_{(2)}}^{\bar{x}_{(1)}+\bar{x}_{(1)}-x_2}\Phi^2(\xi)d\xi$$

$$-\int_0^{\bar{x}_{(2)}}(\Phi^2(\xi)-\Phi^1(\xi))d\xi-\int_{\bar{x}_{(1)}}^{\bar{x}(2)}\Phi^1(\xi)d\xi)$$

$$=(d-c)x_2+c(2\bar{x}_{(1)}-\bar{x}_{(2)})$$

$$+(p+h)(\int_{\bar{x}_{(2)}}^{\bar{x}_{(1)}+\bar{x}_{(1)}-x_2}\Phi^2(\xi)d\xi-\int_0^{\bar{x}(2)}\Phi^2(\xi)d\xi-\int_{\bar{x}_{(1)}}^{\bar{x}_{(2)}}\Phi^2(\xi)d\xi).$$

Similarly, we have the results for II, III, IV, V, VI.

5. Simulation and Summary of Results

Let us put

$$(5.1)\qquad \varphi^1(\xi)=\begin{cases}0, & \xi<0,\\ \lambda e^{-\lambda\xi}, & \xi\geqq 0,\end{cases}$$

Then we have

$$\Phi^1(\xi)=\int_0^\xi\varphi^1(y)dy=\begin{cases}0, & \xi<0,\\ 1-e^{-\lambda\xi}-\lambda\xi e^{-\lambda\xi}, & \xi\geqq 0,\end{cases}$$

$$\Phi^2(\xi)=\int_0^\xi\Phi^1(\xi-y)\varphi^1(y)dy=\begin{cases}0, & \xi<0,\\ 1-e^{-\lambda\xi}-\lambda\xi e^{-\lambda\xi}, & \xi\geqq 0,\end{cases}$$

$$(5.2)\qquad \Phi^R(\xi)=\int_0^\xi\Phi^{R-1}(\xi-y)\varphi^{R-1}(y)dy=\begin{cases}0 & \xi<0,\\ 1-\sum_{i=0}^{R-1}\frac{\lambda^i\xi^i e^{-\lambda\xi}}{i!}, & \xi\geqq 0,\end{cases}$$

and

$$\bar{x}_{(1)}\,|\,\Phi^1(\xi)=\frac{p-c(1-\alpha)-\alpha d}{p+h},$$

(5.3) $$\bar{x}_{(2)} \mid \Phi^1(\xi)+\alpha\Phi^2(\xi)=\frac{p+\alpha p-c(1-\alpha)-\alpha d}{p+h},$$

$$\cdots$$

$$\bar{x}_{(R)} \mid \sum_{i=1}^{R} \Phi^i(\xi)=\frac{(p+\sum_{i=0}^{R} \alpha^i)-c(1-\alpha)-\alpha d}{p+h}.$$

Several inventory systems were simulated, in order to observe the effect of the delay of observation and ordering on the average inventory carried, the average shortage, the average replenishment per week and the total expected cost.

The results of the simulations are given in Table 1 and Fig. 1. As we might have expected, the shortage and the total expected cost increase with increasing variability. Thus we can determine the time of observation and control that balance the cost of emergency, the cost of observation, and the expected total cost that is obtained by using approximate times.

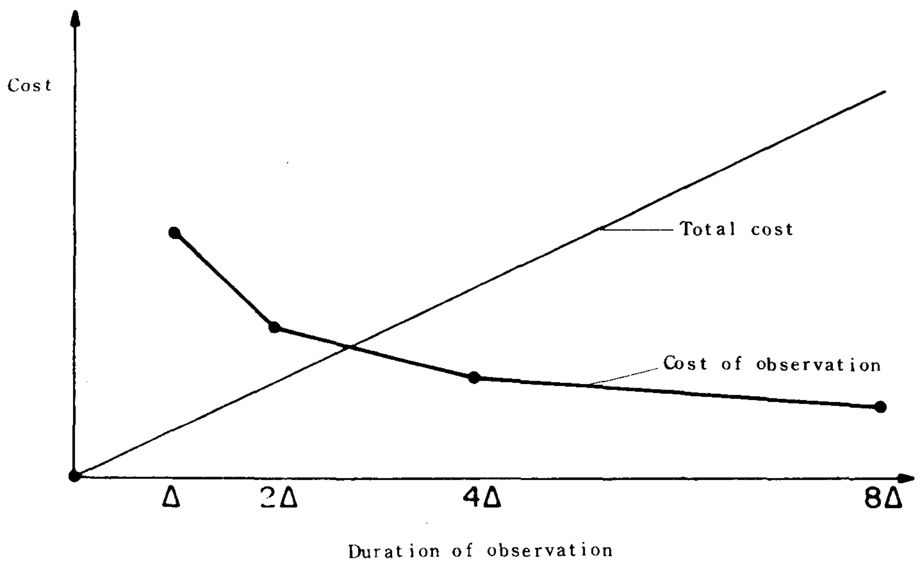

Fig. 1 Total cost and cost of observation

Table 1

Values of I_1, I_2, I_3 and average total cost in a 100-week simulation when Δ, 2Δ, 3Δ, 4Δ.

Δ	2Δ	3Δ	4Δ	
107.1	140.3	171.1	203.2	I_1
14.1	12.9	13.3	22.82	I_2
101.0	106.7	102.7	99.4	I_3
193.7	219.8	252.9	342.0	T

Table 2

Values of I_1, I_2, I_3 and average total cost in 100-week simulation when L=0, L=1, L=2.

L=0	L=1	L=2	
105.1	148.9	182.8	I_1
17.3	20.0	24.0	I_2
106.1	106.6	107.0	I_3
210.1	271.0	328.9	T

6. Discussion

Essentially our conclusion is the following. If we demand complete knowledge of the system at any stage, an appreciable time is required to accomplish this. During this time, the system is uncontrolled. In either case, the cost is increased. If, however, we use incomplete knowledge of the system to make a decision quickly, there is a nonnegligible probability that a non-optimal control action will be taken. We cannot have complete accuracy in both information and control.

Following this idea, we have examined the optimal inventory equation with a delay in delivery. If we use the system with a delay in delivery, there is an increased cost in total expected cost. In either case, we need the emergency cost.

The optimal inventory equation with delay in delivery is the following. Specifically, let us assume that an order be placed at the beginning of the period after λ periods from now. It is possible to write a functional equation as before -- the difficulty is that the functions involve λ variables, current stock x and orders $y_j (j=1,2,\ldots,\lambda,\ldots)$ due in the subsequent $\lambda-1$ periods.

If excess demands are backlogged, $f_n(x,y_1,\ldots,y_{\lambda-1})$ represents the expected discounted cost for an n-period problem when an optimal policy is followed. Then

$$(6.1)\quad f_n(x,y_1y_2,\ldots,y_{\lambda-1})=L(x)+\int_0^\infty L(x+y_1-s)\varphi(s)ds+\ldots$$
$$+\alpha^{\lambda-1}\int_0^\infty L(x+\sum_{i=0}^{\lambda-1}y_i-s)\varphi(s)ds+g_n(x+\sum_{i=1}^{\lambda-1}y_i),$$

where $g_n(x)$ satisfies the functional equation

$$(6.2)\quad g_n(x)=\min_{y\geq x}\{d(x)+c(y-x)+\alpha^{\lambda}\int_0^{\infty}L(y-s)\varphi_{\lambda}(s)ds$$
$$+\alpha\int_0^{\infty}g_{n-1}(y-s)\varphi(s)ds\}.$$

Other than replacing L(y) by $\alpha^{\lambda}\int_0^{\infty}L(y-s)\varphi_{\lambda}(s)ds$, the equation is identical with the zero lag time equation.

The effects of the delay in delivery can be seen by examining Table 2. In order to make meaningful comparisons the results L=0 and $\bar{x}_{(1)}$ should be compared with those for L=1,$\bar{x}$ and L=2,$\bar{x}_{(3)}$. It appears from the table that the delay tends to increase the expected total cost. Thus we can determine the delay in delivery that balances the cost of emergency and the total expected cost.

References

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