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RELIABILITY ANALYSIS OF A COMPLEX SYSTEM WITH DISSIMILAR UNITS UNDER PREEMPTIVE REPEAT REPAIR DISCIPLINE

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Abstract

This paper considers a system composed of two subsystems connected in series. One consists of an active and a warm standby unit where they are possibly dissimilar, while the other consists of several different units connected in tandem. Each unit is repaired upon failure by a single server and the failure rate is assumed to be constant, while the repair rate need not be constant. The system fails if both units in the standby group are simultaneously in a failed state or if any failure in the tandem group occurs. The system availability and the time to system failure are discussed. Finally, it is shown that our results include several earlier results as special cases.

1. Introduction

As is well-known, standby redundancy is one of the basic methods to increase reliability and depending upon the state of the redundant units, the redundancy of a unit falls into one of the following three types : cold, hot and warm. The last one is the most general type of standby redundancy since the first two are to be derived as special cases. In this paper, we discuss a system composed of two series subsystems. One consists of an active and a warm standby unit where they are possibly dissimilar, while the other consists of several different units connected in tandem. A two-unit redundant system has been studied by many authors. For instance, Gnedenko et al. [2] investigated a two-unit warm standby redundant system and Gaver [1] investigated a hot redundant system of two dissimilar units. Our model considered here is a generalized one of their models and we show that our results include their results as special cases. In Section 3, we analyze the system availability and in Section 4, the time to system failure are discussed and from their results the long-run availability and the mean time to system failure (MT SF) are derived. In the final Section, a few remarks concerning about a generalization of our model are stated.

2. Definition of the model

In this paper, we discuss a complex system composed of two series subsystems, U and V. Subsystem U consists of two possibly dissimilar units A_1 and A_2, one of them is in an active state and the other is in a warm standby state, while subsystem V consists of n different units A_{01}, ..., A_{on} connected in tandem. The failure rate of each unit is assumed to be constant and the failed unit is repaired by a single service channel. Denote by $\lambda_i > 0$ the failure rate of unit A_i when it is in an active state and by λ_i^* when in a warm standby state $(i = 1,2)$. Obviously, for $\lambda_i^* = \lambda_i$, we obtain the case of a hot standby, and for $\lambda_i^* = 0$, we obtain the case of a cold standby. Analogously, λ_{oj} denotes the failure rate of unit A_{oj} $(j = 1,\dots,n)$ and let $\lambda_o = \sum_{j=1}^{n} \lambda_{oj}$. Moreover, we assume that the repair time distribution function of unit A_i has the probability density function

$$g_i(x) = \mu_i(x)\exp\left[-\int_0^x \mu_i(u)du\right] \quad , \quad (i = 1,2)$$

where the function $\mu_i(x)$ is equivalent to age-specific failure rate in renewal theory and the operation of each unit is fully restored upon repair. Similarly, let

$$g_{oj}(x) = \mu_{oj}(x)\exp\left[-\int_0^x \mu_{oj}(u)du\right] \quad , \quad (j = 1,\dots,n)$$

be the probability density function of the repair time of unit A_{oj}. We denote by $\bar{g}_i(s)$ the Laplace transform of $g_i(x)$ with the first moment α_i and by $\bar{g}_{oj}(s)$ the Laplace transform of $g_{oj}(x)$ with the first moment α_{oj}. When an active unit in U fails, the standby unit takes its place instantaneously. The active unit is repaired, and then put in standby. The system fails if both units in standby group U are simultaneously in a failed state or if any failure in tandem group V occurs. When the system is failed, the failure rate of each good unit is zero. To shorten the sojourn time in system failure, in this paper, we adopt the following repair policy. If a unit in V fails when a unit in U is being repaired, then the failed V-unit has a right of replacing the U-unit from the service channel and the preempted U-unit obeys the "repeat" rule so that upon re-entry its repair is to be started from the begining. The reliability of detecting and switching is one and time to switch over is neglected.

3. System availability

3.1 Differential equations

First, we introduce the following notation.

$A_i^a(t)$ denotes that A_i is in an active state at time t,

$A_i^s(t)$ denotes that A_i is in a warm standby state at time t,

$A_i^r(t)$ denotes that A_i is undergoing emergency repair at time t,

$A_i^w(t)$ denotes that A_i is failed and waiting for the service at time t.

The notation $A_{oj}^a(t)$ etc. are analogous to $A_j^a(t)$ etc..

Next, we define the possible states of the system as follows:

$$E_1(t) = A_1^a(t) \cap A_2^s(t) \cap A_o^a(t)$$

$$E_2(t) = A_1^s(t) \cap A_2^a(t) \cap A_o^a(t) \qquad F_{1j}(t) = \underline{A_1^a(t)} \cap \underline{A_2^s(t)} \cap A_{oj}^r(t)$$

$$E_3(t) = A_1^a(t) \cap A_2^r(t) \cap A_o^a(t) \qquad F_{2j}(t) = \underline{A_1^s(t)} \cap \underline{A_2^a(t)} \cap A_{oj}^r(t)$$

$$E_4(t) = A_1^r(t) \cap A_2^a(t) \cap A_o^a(t) \qquad F_{3j}(t) = \underline{A_1^a(t)} \cap A_2^w(t) \cap A_{oj}^r(t)$$

$$F_1(t) = A_1^r(t) \cap A_2^w(t) \cap \underline{A_o^a(t)} \qquad F_{4j}(t) = A_1^w(t) \cap \underline{A_2^a(t)} \cap A_{oj}^r(t)$$

$$F_2(t) = A_1^w(t) \cap A_2^r(t) \cap \underline{A_o^a(t)}$$

where $A_o^a(t) = \bigcap_{j=1}^n A_{oj}^a(t)$ and the underlined unit is stopping its operation for the system failure. Thus it can be noted that $E_.(t)$ means that the system is operating and $F_.(t)$ means that the system is failed at time t.

Finally, let $\varepsilon(t)$ be the elapsed repair time of the unit currently in service, if any, at time t and define the following state probabilities:

$$p_i(t) = P\left\{E_i(t) \mid E_1(0)\right\}, \quad (i = 1,\dots,4)$$

$$p_i(t,x)dx + o(dx) = P\left\{E_i(t) \wedge [x < \varepsilon(t) < x+dx] \mid E_1(0)\right\}, \quad (i = 3, 4)$$

$$q_i(t,x)dx + o(dx) = P\left\{F_i(t) \wedge [x < \varepsilon(t) < x+dx] \mid E_1(0)\right\}, \quad (i = 1, 2)$$

$$q_{ij}(t,x)dx + o(dx) = P\left\{F_{ij}(t) \wedge [x < \varepsilon(t) < x+dx] \mid E_1(0)\right\}.$$

$$(i = 1,\dots,4\ ;\ j = 1,\dots,n)$$

By including the parameter x, after the method of supplementary variables, the process becomes Markovian and the differential equations for the above probabilities can be derived in a usual way:

(1) $\left\{\frac{d}{dt} + (\lambda_o + \lambda_1 + \lambda_2^*)\right\} p_1(t) = \int_o^t p_3(t,x)\mu_2(x)dx + \sum_{j=1}^n \int_o^t q_{1j}(t,x)\mu_{oj}(x)dx,$

(2) $\left\{\frac{d}{dt} + (\lambda_o + \lambda_1^* + \lambda_2)\right\} p_2(t) = \int_o^t p_4(t,x)\mu_1(x)dx + \sum_{j=1}^n \int_o^t q_{2j}(t,x)\mu_{oj}(x)dx,$

(3) $\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + [\lambda_o + \lambda_1 + \mu_2(x)]\right\} p_3(t,x) = 0,$

(4) $\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + [\lambda_o + \lambda_2 + \mu_1(x)]\right\} p_4(t,x) = 0,$

(5) $\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu_1(x)\right\} q_1(t,x) = \lambda_2 p_4(t,x),$

(6) $\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu_2(x)\right\} q_2(t,x) = \lambda_1 p_3(t,x),$

(7) $\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu_{oj}(x)\right\} q_{ij}(t,x) = 0. \quad (i = 1,\dots,4;\ j = 1,\dots,n)$

Each of these equations is to be solved under the following boundary conditions (8)-(11) and the initial condition (12).

(8) $p_3(t,o) = \lambda_2^* p_1(t) + \lambda_2 p_2(t) + \int_o^t q_1(t,x)\mu_1(x)dx + \sum_{j=1}^n \int_o^t q_{3j}(t,x)\mu_{oj}(x)dx,$

(9) $p_4(t,0) = \lambda_1 p_1(t) + \lambda_1^* p_2(t) + \int_o^t q_2(t,x)\mu_2(x)dx + \sum_{j=1}^n \int_o^t q_{4j}(t,x)\mu_{oj}(x)dx,$

(10) $q_i(t,0) = 0, \ (i = 1, 2)$ (11) $q_{ij}(t,0) = \lambda_{oj} p_i(t),$

$(i = 1,\dots,4;\ j = 1,\dots,n)$

(12) $p_1(0) = 1.$

3.2 Solution of the equations

Denote by $\bar{p}_i(s)$ etc. the Laplace transform of $p_i(t)$ etc.. Substituting the solutions of (3) and (7) into (1) and applying the Laplace transform to it, we obtain

$$(s+\lambda_o+\lambda_1+\lambda_2^*)\ \bar{p}_1(s) = 1 + \bar{p}_3(s,0)\bar{g}_2(s+\lambda_o+\lambda_1) + \bar{p}_1(s)\ \sum_{j=1}^{n}\lambda_{oj}\bar{g}_{oj}(s)\ ,$$

which gives

$$\bar{p}_1(s) = [1 + \bar{p}_3(s,0)\bar{g}_2(s+\lambda_o+\lambda_1)]/\alpha_1(s)\ , \tag{13}$$

where $\alpha_1(s) = s+\lambda_1+\lambda_2^*+\lambda_o[1-\bar{g}_o(s)]$ and $\bar{g}_o(s) = \frac{1}{\lambda}\sum_{j=1}^{n}\lambda_{oj}\bar{g}_{oj}(s)$.

By the similar argument to the case of $\bar{p}_1(s)$, it is shown that

$$\bar{p}_2(s) = \bar{p}_4(s,0)\bar{g}_1(s+\lambda_o+\lambda_2)/\alpha_2(s)\ , \tag{14}$$

where $\alpha_2(s) = s+\lambda_1^*+\lambda_2+\lambda_o[1-\bar{g}_o(s)]$ ..

Since

$$p_i(t) = \int_0^t p_i(t,x)dx, \quad (i = 3,\ 4)$$

it is easily shown that

$$\bar{p}_3(s) = \bar{p}_3(s,0)[1 - \bar{g}_2(s+\lambda_o+\lambda_1)]/(s+\lambda_o+\lambda_1)\ , \tag{15}$$

$$\bar{p}_4(s) = \bar{p}_4(s,0)[1 - \bar{g}_1(s+\lambda_o+\lambda_2)]/(s+\lambda_o+\lambda_2)\ . \tag{16}$$

On the other hand, applying the Laplace transform to (8) and substituting the solution of (5) into it, we obtain

$$\bar{p}_3(s,0) = \lambda_2^*\bar{p}_1(s) + \lambda_2\bar{p}_2(s) + \frac{\lambda_o}{\lambda_o+\lambda_2}\bar{p}_4(s,0)\ \times \tag{17}$$
$$\times\ [\bar{g}_1(s) - \bar{g}_1(s+\lambda_o+\lambda_2)] + \lambda_o\bar{g}_o(s)\bar{p}_3(s)\ .$$

Substituting (13)-(15) into (17) and rearranging with respect to $\bar{p}_3(s,0)$ and $\bar{p}_4(s,0)$ yields

$$\left\{1 - \frac{\lambda_2^*}{\alpha_1(s)}\bar{g}_2(\cdot\cdot) - \frac{\lambda_o\bar{g}_o(s)}{s+\lambda_o+\lambda_1}[1-\bar{g}_2(\cdot\cdot)]\right\}\bar{p}_3(s,0)$$

$$-\left\{\frac{\lambda_2}{\alpha_2(s)}\,\bar{g}_1(\cdot)+\frac{\lambda_2}{\lambda_o+\lambda_2}\,[\bar{g}_1(s)-\bar{g}_1(\cdot)]\right\}\bar{p}_4(s,0)=\frac{\lambda_2^*}{\alpha_1(s)}\,, \tag{18}$$

where $\bar{g}_1(\cdot)=\bar{g}_1(s+\lambda_o+\lambda_2)$ and $\bar{g}_2(\cdot\cdot)=\bar{g}_2(s+\lambda_o+\lambda_1)$.

Analogously, from (6) and (9), we obtain

$$-\left\{\frac{\lambda_1}{\alpha_1(s)}\,\bar{g}_2(\cdot\cdot)+\frac{\lambda_1}{\lambda_o+\lambda_2}\,[\bar{g}_2(s)-\bar{g}_2(\cdot\cdot)]\right\}\bar{p}_3(s,0)+\left\{1-\frac{\lambda_1^*}{\alpha_2(s)}\,\bar{g}_1(\cdot)-\frac{\lambda_o\bar{g}_o(s)}{s+\lambda_o+\lambda_2}\,[1-\bar{g}_1(\cdot)]\right\}\bar{p}_4(s,0)=\frac{\lambda_1}{\alpha_1(s)} \tag{19}$$

The solution to (18) and (19) is

$$\bar{p}_3(s,0)=\frac{\Delta_1(s)}{\Delta(s)}\,,\qquad \bar{p}_4(s,0)=\frac{\Delta_2(s)}{\Delta(s)}\,, \tag{20}$$

where

$$\Delta(s)= \tag{21}$$

$$\begin{vmatrix} \dfrac{\lambda_2^*-\alpha_1(s)}{s+\lambda_o+\lambda_1}\left\{\alpha_1(s)+[\lambda_o\bar{g}_o(s)-\lambda_2^*]\bar{g}_2(\cdot\cdot)\right\}, & \dfrac{\lambda_2}{\lambda_o+\lambda_2}\left\{\alpha_2(s)\bar{g}_1(s)+[\lambda_o\bar{g}_o(s)-s-\lambda_1^*]\bar{g}_1(\cdot)\right\} \\[2ex] \dfrac{\lambda_1}{\lambda_o+\lambda_1}\left\{\alpha_1(s)\bar{g}_2(s)+[\lambda_o\bar{g}_o(s)-s-\lambda_2^*]\bar{g}_2(\cdot\cdot)\right\}, & \dfrac{\lambda_1^*-\alpha_2(s)}{s+\lambda_o+\lambda_2}\left\{\alpha_2(s)+[\lambda_o\bar{g}_o(s)-\lambda_1^*]\bar{g}_1(\cdot)\right\} \end{vmatrix}$$

$$\Delta_1(s)=\begin{vmatrix} \lambda_2^*\,, & -\dfrac{\lambda_2}{\lambda_o+\lambda_2}\left\{\alpha_2(s)\bar{g}_1(s)+[\lambda_o\bar{g}_o(s)-s-\lambda_1^*]\bar{g}_1(\cdot)\right\} \\[2ex] \lambda_1\,, & \dfrac{\alpha_2(s)-\lambda_1^*}{s+\lambda_o+\lambda_2}\left\{\alpha_2(s)+[\lambda_o\bar{g}_o(s)-\lambda_1^*]\bar{g}_1(\cdot)\right\} \end{vmatrix}, \tag{22}$$

$$(23)\quad \Delta_2(s) = \frac{\alpha_2(s)}{\alpha_1(s)} \begin{vmatrix} \frac{\alpha_1(s)-\lambda_2^*}{s+\lambda_o+\lambda_1}\left\{\alpha_1(s)+[\lambda_o\bar{g}_o(s)-\lambda_2^*]\bar{g}_2(\cdot\cdot)\right\}, & \lambda_2^* \\ -\frac{\lambda_1}{\lambda_o+\lambda_1}\left\{\alpha_1(s)\bar{g}_2(s)+[\lambda_o\bar{g}_o(s)-s-\lambda_2^*]\bar{g}_2(\cdot\cdot)\right\}, & \lambda_1 \end{vmatrix}$$

Now, we define the probability

$$(24)\quad p_A(t) = P\left\{\text{at time } t\text{, the system is functioning} \mid E_1(0)\right\} = \sum_{i=1}^{4} p_i(t)\ .$$

From (13)-(16), the system availability $p_A(t)$ is determined by the Laplace transform

$$(25)\quad \begin{aligned}\bar{p}_A(s) &= \frac{1}{\alpha_1(s)} + \left\{\frac{\bar{g}_2(s+\lambda_o+\lambda_1)}{\alpha_1(s)} + \frac{1-\bar{g}_2(s+\lambda_o+\lambda_1)}{s+\lambda_o+\lambda_1}\right\}\bar{p}_3(s,0) \\ &+\left\{\frac{\bar{g}_1(s+\lambda_o+\lambda_2)}{\alpha_2(s)} + \frac{1-\bar{g}_1(s+\lambda_o+\lambda_2)}{s+\lambda_o+\lambda_2}\right\}\bar{p}_4(s,0)\end{aligned}$$

where $\bar{p}_3(s,0)$ and $\bar{p}_4(s,0)$ are given by (20) with (21)-(23).

3.3 Long-run availability

By applying an Abelian theorem to (25), it follows that

$$(26)\quad \begin{aligned}\lim_{t\to\infty} p_A(t) &= \lim_{s\to 0} s\sum_{i=1}^{4}\bar{p}_i(s) \\ &= \frac{A}{(\lambda_o+\lambda_1)(\lambda_1+\lambda_2^*)}\lim_{s\to 0} s\bar{p}_3(s,0) + \frac{B}{(\lambda_o+\lambda_2)(\lambda_1^*+\lambda_2)}\lim_{s\to 0} s\bar{p}_4(s,0),\end{aligned}$$

where $A = (\lambda_1+\lambda_2^*)+(\lambda_o-\lambda_2^*)\bar{g}_2(\lambda_o+\lambda_1)$,

$B = (\lambda_1^*+\lambda_2)+(\lambda_o-\lambda_1^*)\bar{g}_1(\lambda_o+\lambda_2)$.

Clearly, by (20) and (21), it holds that

$$(27)\quad \lim_{s\to 0} s\bar{p}_3(s,0) = \frac{\Delta_1(0)}{\Delta'(0)}\ ,\qquad \lim_{s\to 0} s\bar{p}_4(s,0) = \frac{\Delta_2(0)}{\Delta'(0)}$$

and, by (21)-(23), after some manipulation it is shown that

(28) $\Delta'(0) =$

$$\frac{1}{(\lambda_o+\lambda_1)(\lambda_o+\lambda_2)}\Big\{\lambda_1\lambda_2[(\lambda_1^*+\lambda_2)\alpha_1+\bar{g}_1(\lambda_o+\lambda_2)]A + \lambda_1\lambda_2[(\lambda_1+\lambda_2^*)\alpha_2+\bar{g}_2(\lambda_o+\lambda_1)]B$$

$$+ \lambda_o\Big[\frac{1}{\lambda_o+\lambda_2} + \frac{2}{\lambda_o+\lambda_1} + (\lambda_1+\lambda_2)\alpha_o\Big]AB\Big\},$$

(29) $$\Delta_1(0) = \frac{\lambda_2(\lambda_1+\lambda_2^*)}{\lambda_o+\lambda_2}B,$$

(30) $$\Delta_2(0) = \frac{\lambda_1(\lambda_1^*+\lambda_2)}{\lambda_o+\lambda_1}A,$$

where $\alpha_o = \frac{1}{\lambda_o}\sum_{j=1}^{n}\lambda_{oj}\alpha_{oj}$.

Combining the above five equations (26)-(30), we obtain the long-run availability

(31) $$p_A(\infty) \equiv \lim_{t\to\infty} p_A(t) = \frac{\lambda_1+\lambda_2}{\lambda_o\Big\{\lambda_1/(\lambda_o+\lambda_2) + \lambda_2/(\lambda_o+\lambda_1) + (\lambda_1+\lambda_2)\alpha_o\Big\} + \lambda_1\lambda_2\, C},$$

where

$$C = \frac{(\lambda_1^*+\lambda_2)\alpha_1+\bar{g}_1(\lambda_o+\lambda_2)}{(\lambda_1^*+\lambda_2)+(\lambda_o-\lambda_1^*)\bar{g}_1(\lambda_o+\lambda_2)} + \frac{(\lambda_1+\lambda_2^*)\alpha_2+\bar{g}_2(\lambda_o+\lambda_1)}{(\lambda_1+\lambda_2^*)+(\lambda_o-\lambda_2^*)\bar{g}_2(\lambda_o+\lambda_1)}.$$

3.4 Special cases

1) Consider the case in which the two units in U are identical. Then $\lambda_i=\lambda$, $\lambda_i^*=\lambda^*$, $g_i(s)=g(s)$, $\alpha_i=\alpha$ $(i=1,2)$, and (31) reduces to

(32) $$p_A(\infty) = \left[\frac{\lambda_o}{\lambda_o+\lambda} + \lambda_o\alpha_o + \frac{(\lambda+\lambda^*)\alpha+\bar{g}(\lambda_o+\lambda)}{(\lambda+\lambda^*)+(\lambda_o-\lambda^*)\bar{g}(\lambda_o+\lambda)}\right]^{-1}.$$

2) In the case when the subsystem $V = \{A_{oj};\ j=1,\dots,n\}$ is absent, setting $\lambda_o= 0$ in (31), we obtain

$$p_A(\infty) = \frac{\frac{1}{\lambda_1} + \frac{1}{\lambda_1}}{\frac{(\lambda_1^*+\lambda_2)\alpha_1+\bar{g}_1(\lambda_2)}{(\lambda_1^*+\lambda_2)-\lambda_1^*\bar{g}_1(\lambda_2)} + \frac{(\lambda_1+\lambda_2^*)\alpha_2+\bar{g}_2(\lambda_1)}{(\lambda_1+\lambda_2^*)-\lambda_2^*\bar{g}_2(\lambda_1)}} \tag{33}$$

This gives the long-run availability for a warm standby system having two dissimilar units.

3) Consider the particular case where $\mu_i(x)=\mu_i$ $(i=1,2)$. Substitution of $\alpha_i=\mu_i^{-1}$, $g_i(s)=\mu_i/(\mu_i+s)$ $(i=1,2)$ into (31) yields

$$p_A(\infty) = \left[1 + \lambda_o\alpha_o + \frac{\lambda_1}{\mu_1}\cdot\frac{\lambda_2}{\lambda_1+\lambda_2}\cdot\frac{\lambda_1^*+\lambda_2}{\lambda_1^*+\lambda_2+\mu_1} + \frac{\lambda_2}{\mu_2}\cdot\frac{\lambda_1}{\lambda_1+\lambda_2}\cdot\frac{\lambda_1+\lambda_2^*}{\lambda_1+\lambda_2^*+\mu_2}\right]^{-1} \tag{34}$$

It is noted that, in this case, the repair rate of each unit in V is not constant.

4. Time to system failure

In the preceding section, the Laplace transform of system availability and the long-run availability for our model have been derived. In this section, we shall deal with another important operational measure, the time to system failure, and then derive the MTSF. Let $\eta^{(i)}$ be the time to system failure, measured from an instant at which the process has entered the state $E_i (i=1,\ldots,4)$.

4.1 Differential equations and its solution

Define the following state probabilities:

$$p_j^{(i)}(t) = P\left\{E_j(t)\wedge[\eta^{(i)}> t] \mid E_i(0)\right\}, \quad (i,j=1,\ldots,4)$$

$$p_j^{(i)}(t,x)dx+o(dx) = P\left\{E_j(t)\wedge[x<\varepsilon(t)<x+dx]\wedge[\eta^{(i)}>t] \mid E_i(0)\right\},$$

$$(i=1,\ldots,4\ ;\ j=3,4)$$

$$p_f^{(i)}(t) = P\left\{\text{at time } t\text{, the system is in a failed state} \mid E_i(0)\right\}.$$

$$(i=1,\ldots,4)$$

The distribution of the time to system failure can be derived by arguments similar to those used in Section 3. To do this we consider the modified process which ceases as soon as the system failure occurs. The differential equation for this process are :

$$\left\{\frac{d}{dt} + (\lambda_o+\lambda_1+\lambda_2^*)\right\} p_1^{(i)}(t) = \int_o^t p_3^{(i)}(t,x)\mu_2(x)dx \ , \tag{35}$$

$$\left\{\frac{d}{dt} + (\lambda_o+\lambda_1^*+\lambda_2)\right\} p_2^{(i)}(t) = \int_o^t p_4^{(i)}(t,x)\mu_1(x)dx \ , \tag{36}$$

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + [\lambda_o+\lambda_1+\mu_2(x)]\right\} p_3^{(i)}(t,x) = 0 \ , \tag{37}$$

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + [\lambda_o+\lambda_2+\mu_1(x)]\right\} p_4^{(i)}(t,x) = 0 \ , \tag{38}$$

$$\left\{\frac{d}{dt} + \lambda_o\right\} p_f^{(i)}(t) = \lambda_o + \lambda_1 p_3^{(i)}(t) + \lambda_2 p_4^{(i)}(t) \ . \tag{39}$$

The boundary conditions are :

$$p_3^{(i)}(t,0) = \lambda_2^* p_1^{(i)}(t) + \lambda_2 p_2^{(i)}(t) \ , \tag{40}$$

$$p_4^{(i)}(t,0) = \lambda_1 p_1^{(i)}(t) + \lambda_1^* p_2^{(i)}(t) \ . \tag{41}$$

The initial conditions are :

$$p_1^{(i)}(0) = \delta_{i1} \ , \quad p_2^{(i)}(0) = \delta_{i2} \ , \quad p_3^{(i)}(0,x) = \delta_{i3}\delta(x) \ , \tag{42}$$
$$p_4^{(i)}(0,x) = \delta_{i4}\delta(x) \ ,$$

where $\delta(x)$ is the Dirac delta function and δ_{ik} is the Kronecker symbol. We denote by $\bar{p}_j^{(i)}(s)$ and $\bar{p}_f^{(i)}(s)$ the Laplace transform of $p_j^{(i)}(t)$ and $p_f^{(i)}(t)$, respectively. Then, by using arguments analogous to those in the previous section, we obtain

$$\begin{aligned}(s+\lambda_o)\bar{p}_f^{(i)}(s) &= \frac{\lambda_o}{s} + \lambda_1[\delta_{i3}+\lambda_2^*\bar{p}_1^{(i)}(s)+\lambda_2\bar{p}_2^{(i)}(s)]\ \frac{1-\bar{g}_2(s+\lambda_o+\lambda_1)}{s+\lambda_o+\lambda_2} \\ &+ \lambda_2[\delta_{i4}+\lambda_1\bar{p}_1^{(i)}(s)+\lambda_1^*\bar{p}_2^{(i)}(s)]\ \frac{1-\bar{g}_1(s+\lambda_o+\lambda_2)}{s+\lambda_o+\lambda_2} \ ,\end{aligned} \tag{43}$$

in which, $\bar{p}_1^{(i)}(s)$ and $\bar{p}_2^{(i)}(s)$ are given by

$$\bar{p}_1^{(i)}(s) = \Big\{[s+\lambda_o+\lambda_2+\overset{*}{\lambda}_1-\overset{*}{\lambda}_1\bar{g}_1(\cdot)][\delta_{i1}+\delta_{i3}\bar{g}_2(\cdot\cdot)] + \lambda_2\bar{g}_2(\cdot\cdot)[\delta_{i2}+\delta_{i4}\bar{g}_1(\cdot)]\Big\}/D(s), \tag{44}$$

$$\bar{p}_2^{(i)}(s) = \Big\{[s+\lambda_o+\lambda_1+\overset{*}{\lambda}_2-\overset{*}{\lambda}_2\bar{g}_2(\cdot\cdot)][\delta_{i2}+\delta_{i4}\bar{g}_1(\cdot)] + \lambda_1\bar{g}_1(\cdot)[\delta_{i1}+\delta_{i3}\bar{g}_2(\cdot\cdot)]\Big\}/D(s), \tag{45}$$

where $D(s) = \Big\{s+\lambda_o+\lambda_2+\overset{*}{\lambda}_1[1-\bar{g}_1(\cdot)]\Big\}\Big\{s+\lambda_o+\lambda_1+\overset{*}{\lambda}_2[1-\bar{g}_2(\cdot\cdot)]\Big\}-\lambda_1\lambda_2\bar{g}_1(\cdot)\bar{g}_2(\cdot\cdot)$

and suppressed arguments are $\cdot \equiv s+\lambda_o+\lambda_2$ and $\cdot\cdot \equiv s+\lambda_o+\lambda_1$.

The Laplace transform of the distribution of time to system failure

$$E\Big\{\exp[-s\eta^{(i)}]\Big\} = s\bar{p}_f^{(i)}(s)$$

can be derived from (43) with (44)-(45).

4.2 Mean time to system failure (MTSF)

Consider the system where the tandem group is absent in our model, i.e. , the system consists of only a two-unit warm standby group. Let $\xi^{(i)}$ be the time to system failure, analogous to $\eta^{(i)}$, in that system. Then it is easily shown that

$$E\Big\{\eta^{(i)}\Big\} = \frac{1}{\lambda_o}\Big[1-E\Big\{\exp[-\lambda_o\xi^{(i)}]\Big\}\Big] \quad . \tag{46}$$

On the other hand, putting $\lambda_o=0$ in (43) gives

$$\begin{aligned} E\Big\{\exp[-s\xi^{(i)}]\Big\} &= \lambda_1[\delta_{i3}+\overset{*}{\lambda}_1\bar{\bar{p}}_1^{(i)}(s)+\lambda_2\bar{\bar{p}}_2^{(i)}(s)]\,\frac{1-\bar{g}_2(s+\lambda_1)}{s+\lambda_1} \\ &+ \lambda_2[\delta_{i4}+\lambda_1\bar{\bar{p}}_1^{(i)}(s)+\overset{*}{\lambda}_2\bar{\bar{p}}_2^{(i)}(s)]\,\frac{1-\bar{g}_1(s+\lambda_2)}{s+\lambda_2} \end{aligned}, \tag{47}$$

where

$$\bar{\bar{p}}_1^{(i)}(s) = \Big\{[s+\lambda_2+\overset{*}{\lambda}_1-\overset{*}{\lambda}_1\bar{g}_1(s+\lambda_2)][\delta_{i1}+\delta_{i3}\bar{g}_2(s+\lambda_1)] + \lambda_2\bar{g}_2(s+\lambda_1)[\delta_{i2}+\delta_{i4}\bar{g}_1(s+\lambda_2)]\Big\}/\,\bar{D}(s) ,$$

$$\bar{\bar{p}}_2^{(i)}(s) = \Big\{ [s+\lambda_1+\lambda_2^*-\lambda_2^*\bar{g}_2(s+\lambda_1)][\delta_{i2}+\delta_{i4}\bar{g}_1(s+\lambda_2)]$$

$$+ \lambda_1\bar{g}_1(s+\lambda_2)[\delta_{i1}+\delta_{i3}\bar{g}_2(s+\lambda_1)]\Big\} / \bar{D}(s) ,$$

and

$$\bar{D}(s) = \Big\{s+\lambda_2+\lambda_1^*[1-\bar{g}_1(s+\lambda_2)]\Big\}\Big\{s+\lambda_1+\lambda_2^*[1-\bar{g}_2(s+\lambda_1)]\Big\}$$

$$- \lambda_1\lambda_2\bar{g}_1(s+\lambda_2)\bar{g}_2(s+\lambda_1) .$$

Substitution of (47) with $s=\lambda_o$ into (46) gives the MTSF $E\{\eta^{(i)}\}$ for our model. For instance, in case i=1, it holds that

$$E\{\eta^{(1)}\} = \frac{1}{\lambda_o}\Big\{1-\lambda_1[\lambda_2^*\bar{\bar{p}}_1^{(1)}(\lambda_o) + \lambda_2\bar{\bar{p}}_2^{(1)}(\lambda_o)]\frac{1-\bar{g}_2(\lambda_o+\lambda_1)}{\lambda_o+\lambda_1}$$

$$- \lambda_2[\lambda_1\bar{\bar{p}}_1^{(1)}(\lambda_o)+\lambda_1^*\bar{\bar{p}}_2^{(1)}(\lambda_o)]\frac{1-\bar{g}_1(\lambda_o+\lambda_2)}{\lambda_o+\lambda_2}\Big\}$$

(48)

$$= \Big\{(\lambda_o+\lambda_1+\lambda_2)[1+\lambda_1^*\cdot\frac{1-\bar{g}_1(\lambda_o+\lambda_2)}{\lambda_o+\lambda_2}] + [\lambda_2^*(\lambda_o+\lambda_1^*+\lambda_2)$$

$$+(\lambda_1\lambda_2-\lambda_1^*\lambda_2^*)\bar{g}_1(\lambda_o+\lambda_2)]\cdot\frac{1-\bar{g}_2(\lambda_o+\lambda_1)}{\lambda_o+\lambda_1}\Big\} / \bar{D}(\lambda_o) .$$

Similarly, it is shown that

(49)

$$E\{\eta^{(2)}\} = \Big\{(\lambda_o+\lambda_1+\lambda_2)[1+\lambda_2^*\cdot\frac{1-\bar{g}_2(\lambda_o+\lambda_1)}{\lambda_o+\lambda_1}] + [\lambda_1^*(\lambda_o+\lambda_1+\lambda_2^*)$$

$$+(\lambda_1\lambda_2-\lambda_1^*\lambda_2^*)\bar{g}_2(\lambda_o+\lambda_1)]\cdot\frac{1-\bar{g}_1(\lambda_o+\lambda_2)}{\lambda_o+\lambda_2}\Big\} / \bar{D}(\lambda_o) .$$

(50) $$E\{\eta^{(3)}\} = \frac{1-\bar{g}_2(\lambda_o+\lambda_1)}{\lambda_o+\lambda_1} + \bar{g}_2(\lambda_o+\lambda_1)E\{\eta^{(1)}\} .$$

(51) $$E\{\eta^{(4)}\} = \frac{1-\bar{g}_1(\lambda_o+\lambda_2)}{\lambda_o+\lambda_2} + \bar{g}_1(\lambda_o+\lambda_2)E\{\eta^{(2)}\} .$$

4.3 Special cases

1) Consider the case where U is a two-unit hot standby system. Then $\lambda_i^* = \lambda_i$ (i=1,2), and

$$E\{\eta^{(1)}\} = E\{\eta^{(2)}\}$$

(52) $$= \frac{1 + \frac{\lambda_1}{\lambda_o+\lambda_2}[1-\bar{g}_1(\lambda_o+\lambda_2)] + \frac{\lambda_2}{\lambda_o+\lambda_1}[1-\bar{g}_2(\lambda_o+\lambda_1)]}{\lambda_o+\lambda_1[1-\bar{g}_1(\lambda_o+\lambda_2)] + \lambda_2[1-\bar{g}_2(\lambda_o+\lambda_1)]}$$

Setting $\lambda_o=0$ in this, we obtain the formula (5) in Gaver [1] .

2) Consider the case where U is a two-unit cold standby system. Then, $\lambda_i^*=0$, and

(53) $$E\{\eta^{(1)}\} = \frac{(\lambda_o+\lambda_1+\lambda_2)+\frac{\lambda_1\lambda_2}{\lambda_o+\lambda_1}\bar{g}_1(\lambda_o+\lambda_2)[1-\bar{g}_2(\lambda_o+\lambda_1)]}{(\lambda_o+\lambda_1)(\lambda_o+\lambda_2)-\lambda_1\lambda_2\bar{g}_1(\lambda_o+\lambda_2)\bar{g}_2(\lambda_o+\lambda_1)},$$

(54) $$E\{\eta^{(2)}\} = \frac{(\lambda_o+\lambda_1+\lambda_2)+\frac{\lambda_1\lambda_2}{\lambda_o+\lambda_2}\bar{g}_2(\lambda_o+\lambda_1)[1-\bar{g}_1(\lambda_o+\lambda_2)]}{(\lambda_o+\lambda_1)(\lambda_o+\lambda_2)-\lambda_1\lambda_2\bar{g}_1(\lambda_o+\lambda_2)\bar{g}_2(\lambda_o+\lambda_1)}.$$

3) In the case when the subsystem V is absent, setting $\lambda_o=0$ in (48)-(51), we obtain

(55) $$E\{\eta^{(1)}\} = \Big\{(\lambda_1+\lambda_2)+(\lambda_1\lambda_2-\lambda_1^*\lambda_2^*)\bar{g}_1(\lambda_2)[1-\bar{g}_2(\lambda_1)]/\lambda_1 + \lambda_1^*(\lambda_1+\lambda_2)[1-\bar{g}_1(\lambda_2)]/\lambda_2+\lambda_2^*(\lambda_1^*+\lambda_2)[1-\bar{g}_2(\lambda_2)]/\lambda_1\Big\} / \bar{D}(0) ,$$

(56) $$E\{\eta^{(3)}\} = \frac{1-\bar{g}_2(\lambda_1)}{\lambda_1} + \bar{g}_2(\lambda_1)E\{\eta^{(1)}\} ,$$

where $\bar{D}(0)=\{\lambda_2+\lambda_1^*[1-\bar{g}_1(\lambda_2)]\}\{\lambda_1+\lambda_2^*[1-\bar{g}_2(\lambda_1)]\}-\lambda_1\lambda_2\bar{g}_1(\lambda_2)\bar{g}_2(\lambda_1)$. $E\{\eta^{(2)}\}$ and $E\{\eta^{(4)}\}$ are analogous to $E\{\eta^{(1)}\}$ and $E\{\eta^{(3)}\}$, respectively, and these yield the MTSF for a warm standby system having two dissimilar units. Moreover, if we put $\lambda_i=\lambda$, $\lambda_i^*=\lambda^*$ and $g_i(s)=g(s)$ in (55), then we find a well-known result

(57) $$E\{\eta^{(1)}\} = \frac{1}{\lambda} + \frac{1}{(\lambda+\lambda^*)[1-g(\lambda)]} .$$

(see e.g., Gnedenko[2], equation (6.2.4)). On the other hand, if we put $\lambda_i^*=0$ in (55), then

$$E\left\{n^{(1)}\right\} = \frac{1}{\lambda_1} + \frac{\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\bar{g}_1(\lambda_2)}{1-\bar{g}_1(\lambda_2)\bar{g}_2(\lambda_1)} \quad , \tag{58}$$

which coincides with the formula (19) in Osaki [3] .

5. Conclusion

As a natural generalization of our model, it may be thought of the case where subsystem U consists of three or more dissimilar units. However, as is clearly foreseen, the possible states of the system are too cumbersome to solve the corresponding differential equations and its complexity is caused by the condition "dissimilar" . If all units in subsystem U are identical and the other assumptions are similar to those stated in Section 2, then the problem can be solved and the results will be appeared in the near future.

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