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ON SOME THEOREMS OF KNAPSACK FUNCTION

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Abstract

The two properties of Knapsack Function—breakpoint property and periodicity property—are theoretically investigated. And an example is presented to each of them.

1. Introduction

Among the all integer linear programming (abreviated as ILP), integer linear programming with all the constants (including coefficients of constraint matrix) nonnegative, one constrained, is called Knapsack Problem [3]–[7], [15]. Knapsack Problem with all the variables restricted to 0–1 is sometimes called 0–1 Knapsack Problem [13], [14]. In [12], the author reported a solution procedure of an ILP having a similar structure with 0–1 Knapsack Problem. Here in section (2), the author will show, extending the solution procedure proposed by Dreyfus, S. E. and K. L. Prather [3], a solution procedure of an ILP with all the constants nonnegative which is viewed as an ILP of Knapsack type. Theorem 1 and Theorem 2 provide the validity of this procedure which was not given in [3] even for the Knapsack case. Although the periodicity of the Knapsack function was for the first time found by Gilmore, P.C. and R.E. Gomory [4], it is hard to follow up their proof (See [10]). In [11] Hu, T.C. stated this fact with a complete proof in the case $\rho_1>\rho_2$.[1] I also prove this fact in Theorem 4 section (3), in the case of $\rho_1=\cdots=\rho_k>\rho_{k+1}\geqq\cdots\geqq\rho_n$. This with the proof of Hu, T.C. offers an elementary, but complete proof for the periodicity property of any Knapsack Function. I also tried to find the small Knapsack length b from which Knapsack Function has periodicity property.

2. ILP with All the Constants Nonnegative

Notations. Let $N=\{1, 2, 3, \cdots\}$ be the set of natural numbers, $N_0=N\cup\{0\}$, $I=\{\cdots, -1, 0, 1, \cdots\}$=the set of integers, $N_0^n=N_0\times\cdots\times N_0$=$n$-fold direct product of N_0, I^m=m-fold direct product of I for m, $n\in N$. Notation "$x\geqq 0$ integer" means "each component of x is nonnegative integer." And "iff" means "if and only if."

We concentrate on the following ILP with b regarded as a parameter, $b\in I^m$. $F(b)$: max cx subject to $w_ix\leqq b_i(1\leqq i\leqq m)$, $x\geqq 0$ integer

where $c=(c_1, \cdots, c_n)$, $x=(x_1, \cdots, x_n)^T$, $w_i=(w_{i1}, \cdots, w_{in})$, $b=(b_1, \cdots, b_m)^T$ and T denotes transpose operation.

As an ILP with all the constants nonnegative, we assume

Assumption 1. c_j, $w_{ij}\in N_0$ $(1\leqq i\leqq m,\ 1\leqq j\leqq n)$ and all these constants including m, $n\in N$ are fixed.

Therefore optimal objective function value of $F(b)$ is a function of b. So define

Definition 1.

$$f(b)=\begin{cases}\max cx \text{ subject to } w_ix\leqq b_i(1\leqq i\leqq m),\ x\geqq 0 \text{ integer} & \\ & \text{for } b\in N_0^m\subseteq I^m\\ -\infty & \text{for } b\in I^m\backslash N_0^m\,.\end{cases}$$

[1] The meaning of ρ_j is given in Assumption 5.

Definition 2. $x \in N_0^n$ is called a feasible solution for $F(b)$ (or simply, feasible) iff $w_i x \leqq b_i (1 \leqq i \leqq m)$ and is called an optimal feasible solution for $F(b)$ (or simply, optimal) iff x is feasible and attains $f(b)$ (*i.e.* $cx = f(b)$).

If $w_{ij_0}=0(1 \leqq i \leqq m)$ and $c_{j_0}=0$ we can disregard x_{j_0} in the definition of $f(b)$. Moreover if $w_{ij_0}=0(1 \leqq i \leqq m)$ and $c_{j_0}>0$ we can make $f(b)$ as large as possible. Therefore, hereafter we can assume without loss of generality

Assumption 2. For any j $(1 \leqq j \leqq n)$ there exists i $(1 \leqq i \leqq m)$ such that $w_{ij}>0$.

And for any optimal x, if there exists j_0, $c_{j_0}=0$ then $(x_1, \cdots, x_{j_0-1}, 0, x_{j_0+1}, \cdots, x_n)^T$ is also an optimal. So we can assume without loss of generality

Assumption 3. $c_j>0$ for any j $(1 \leqq j \leqq n)$.

Next we define (See [3])

Definition 3. $[w_{ij}, \infty)=\{w: w_{ij} \leqq w$ and w is a real number$\}$, $w_{\cdot j}=(w_{1j}, \cdots, w_{mj})^T$, $\prod_{i=1}^{m}[w_{ij}, \infty)=m$-fold direct product of $[w_{ij}, \infty)$ $(1 \leqq i \leqq m)$. And we call $b \in I^m$ breakpoint and write b: b.p. iff $f(b)>\max_{1 \leqq i \leqq m} f((b_1, \cdots, b_{i-1}, b_i-1, b_{i+1}, \cdots, b_m)^T)$.

Definition 4. Write b: n.b.p. iff b is not a breakpoint.

From Assumption 1, 2, 3, $0 \leqq f(b)$ for any $b \in N_0^m$. Moreover

Lemma 1. (1) $f(b) \geqq \max_{1 \leqq i \leqq m} f((b_1, \cdots, b_{i-1}, b_i-1, b_{i+1}, \ldots, b_m)^T)$ for any $b \in N_0^m$.

(2) $0=(0, \cdots, 0)$: b.p.

(3) If $N_0^m \ni b$: b.p. then for any x: optimal, $w_i x = b_i (1 \leqq i \leqq m)$.

Lemma 2. For $b \in N_0^m$, $w_{\cdot j} \leqq b$ for some j iff $f(b)>0$.

Remark. $w_{\cdot j} \leqq b$ for some j is equivalent to $b \in \bigcup_{j=1}^{n} \prod_{i=1}^{m} [w_{ij}, \infty)$. So for $b \in N_0^m$, $b \in \bigcup_{j=1}^{n} \prod_{i=1}^{m} [w_{ij}, \infty)$ (respectively $b \notin \bigcup_{j=1}^{n} \prod_{i=1}^{m} [w_{ij}, \infty)$) iff $f(b)>0$ (respectively $f(b)=0$).

Lemma 3. For $b \in N_0^m$, if $f(b)>0$ then $f(b)=\max_{1\leqq j\leqq n}\{f(b-w_{\cdot j})+c_j\}$.

In order to obtain $f(b)$ for b in $\bigcup_{j=1}^{n}\prod_{i=1}^{m}[w_{ij}, \infty)$, we may proceed step by step the origin owing to Lemma 3. But the following Lemma 4 enables us to go back only through the breakpoints.

Lemma 4. For any b: b.p. & $b \in \bigcup_{j=1}^{n}\prod_{i=1}^{m}[w_{ij}, \infty)$, $f(b)=\max_{1\leqq j\leqq n, b-w_{\cdot j}: \text{ b.p.}}\{f(b-w_{\cdot j})+c_j\}$.

Proof. $f(b)=\max_{1\leqq j\leqq n}\{f(b-w_{\cdot j})+c_j\}$ by Lemma 2 and 3. Suppose that the maximum is attained at $b-w_{\cdot j}$: n.b.p. then there exists i such that (2) $b1=(b_1, \cdots, b_{i-1}, b_i-1, b_{i+1}, \cdots, b_m)^T$, $f(b)=f(b-w_{\cdot j})+c_j=f(b1-w_{\cdot j})+c_j$.

On the other hand b is b.p. so $f(b)>f(b1)>0$ (for, if $f(b1)=0$ then one of b_i-1-w_{ij}, b_k-w_{kj} $(k\neq i)$ becomes negative, so that $0<f(b)=-\infty+c_j<0$. Contradiction.). Applying Lemma 3 for $b1$ we obtain $f(b)>f(b1)=\max_{1\leqq l\leqq n}\{f(b1-w_{\cdot l})+c_l\}\geqq f(b1-w_{\cdot j})+c_j$ and by (2) $f(b1-w_{\cdot j})+c_j=f(b)$ contradiction. Q.E.D.

In order to obtain a solution procedure, we define and prove

Definition 5. For $b \in N_0^m$ let $R(b)=\{y: y \in N_0^m, y_i\leqq b_i(1\leqq i\leqq m)\}\backslash\{b\}$ $B(b)=\{y: y \in R(b)$ & y: b.p.$\}$ (Note that $b \notin R(b)$ and $b \notin B(b)$.)

Theorem 1. Assume that we have calculated $B(b_0)$ and $f(b)$ for all $b \in B(b_0)$.

(1) If b_0: b.p. then we can get the optimal value $f(b_0)$ and optimal solution which gives $f(b_0)$ by Lemma 4.

(2) If b_0: n.b.p. then finding $b_{\max}$ such that $f(b_{\max})=\max_{b\in B(b_0)}f(b)$, we see that $f(b_{\max})=f(b_0)$. So for the optimal solution which gives $f(b_0)$ we can take that of $b_{\max}$.

Proof. First part is Lemma 4 itself. If b_0: n.b.p. then after finding $b_{\max}$ as stated in Theorem 1, we see that there exist no b.p. in $\{y: y \in N_0^m, (b_{\max})_i\leqq y_i\leqq(b_0)_i(1\leqq i\leqq m)\}\backslash\{b_{\max}\}$. For, it there existed $\hat{b}$ then from the definition of b.p. $f(b_{\max})<f(\hat{b})$, $\hat{b} \in B(b_0)$ which contradicts the definition of $b_{\max}$. Moreover as b_0: n.b.p. $f(b_{\max})=f(b_0)$. Q.E.D.

Definition 6. Provided that $B(b_0)$ has been calculated and b_0: b.p., we call $PB(b_0)=\{b\colon b-w_{\cdot j}\in B(b_0)\cup\{b_0\}$ for some $j(1\leqq j\leqq n)\}\backslash(B(b_0)\cup\{b_0\})$ the set of potential breakpoints generated by b_0. And we call the element of $PB(b_0)$, potential breakpoint (p.b.p.) for b_0.

The meaning of p.b.p. will be clear. We can easily obtain

Lemma 5. There exist no b.p. in $(N_0^m\backslash\bigcup_{j=1}^{n}\prod_{i=1}^{m}[w_{ij},\infty))\backslash\{0\}$ (that is there exist no p.b.p. for the origin in this set.).

Theorem 2. Assume that $B(b)$ has been calculated for $b\in\bigcup_{j=1}^{n}\prod_{i=1}^{m}[w_{ij},\infty)$. b: b.p. iff $(f(b)=)\max_{1\leqq j\leqq n,\, b-w_{\cdot j}\in B(b)}\{f(b-w_{\cdot j})+c_j\}>\max_{y\in B(b)}f(y)$.

Remark. Therefore under the assumption of Theorem 2, we see that if $\max_{1\leqq j\leqq n,\, b-w_{\cdot j}\in B(b)}\{f(b-w_{\cdot j})+c_j\}\leqq\max_{y\in B(b)}f(y)$ then b: n.b.p..

Proof. *If* part is trivial. *Only if* part; For $y\in B(b)$, $y_i\leqq b_i(1\leqq i\leqq m)$ so that $f(y)\leqq f(b)$. As b: b.p. and $b\neq y$, $f(b)>\max_{y\in B(b)}f(y)$. Q.E.D.

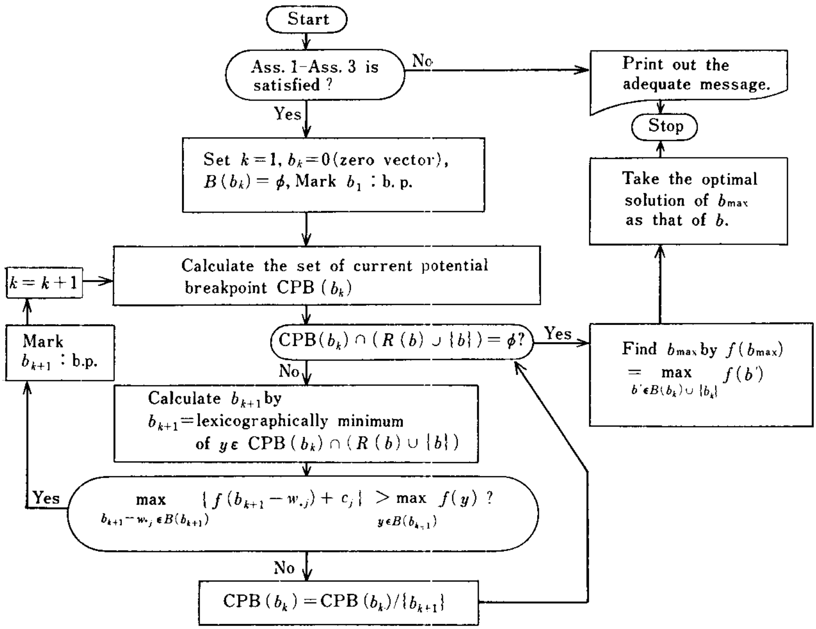

Fig. 1. Solution procedure.

Definition 7. As usual for $b, b' \in N_0^m$, we say that b is lexicographically equal or smaller than b' iff $b=b'$ or $b_1<b'_1$ or there exists $k(1 \leqq k \leqq m-1)$ such that $b_1=b'_1, \cdots, b_k=b'_k, b_{k+1}<b'_{k+1}$. And the set of current potential breakpoint $CPB(b_k)$ be such that $CPB(b_k)=PB(b_k) \setminus$ ({previously established b.p.} ∪ {previously established n.b.p.}).

Concluding all the preceeding results, we obtain a solution procedure for an ILP with all the constants nonnegative. (See Fig. 1).

It is easy to check that this procedure reduces to that of Dreyfus and Prather when $m=1$.

Example. $m=2$, $n=3$, $c=(6, 9, 7)$, $w_1 \doteqdot (4, 3, 3)$, $w_2=(1, 2, 3)$, $b=(8, 4)^T$. In the following table $M_1=\max\{f(b_{k+1}-w_{.j})+c_j\colon\ 1 \leqq j \leqq n,\ b_{k+1}-w_{.j}$

Table 1

k	Previously established breakpoint b_k	$B(b_k)$	p.b.p. generated at iteration k i.e. $b_k+w_{.j}$ $j=1$	$j=2$	$j=3$	b_{k+1}	M_1	M_2	$f(b_{k+1})$	Optimal feasible solution for b_{k+1} when b_{k+1}: b.p.
1	0 0	ϕ	4(3) 1	3(2) 2	3_2^{**} 3	3 2	9	0	9	0 1 0
2	3 2	0 0	7(4) 3	6(4) 4	6* 5	3 3	7	9	/	/
						4 1	6	0	6	1 0 0
3	4 1	0 0	8(6) 2	7(4) 3	7_5^{**} 4	6 4	18	9	18	0 2 0
4	6 4	0 3 4 0, 2, 1	10* 5	9* 6	9* 7	7 3	15	9	15	1 1 0
5	7 3	0 3 4 0, 2, 1	11* 4	10* 5	10* 6	7 4	13	18	/	/
						8 2	12	9	12	2 0 0

$\in B(b_{k+1})\}$, $M_2=\max\{f(y):\ y\in B(b_{k+1})\}$. It is evident to determine $CPB(b_k)$ from already generated potential breakpoints and the elimination rule Solution Procedure indicates. The signal in the right upper corner of p.b.p. means as follows.

S1 (α): eliminated from iteration $\alpha-1$ to α because this is established to be a b.p..

S2 $*$: eliminated because this is not in $R(b)\cup\{b\}$.

S3 $**_\alpha$: eliminated from iteration $\alpha-1$ to α because this is revealed to be a n.b.p..

At iteration 5 calculation is stopped because $CPB(b_5)\cap(R(b)\cup\{b\})=\phi$ with the optimal solution $x_1=0$, $x_2=2$, $x_3=0$ and the optimal value $f(b)=18$. (See Table 1).

3. Periodicity Property of Knapsack Function

As $m=1$ we write $w_{.1}\equiv w=(w_1,\ \cdots,\ w_n)$. Ass. 1–Ass. 3 reduce to

Assumption 4. $c_j,\ w_j\in N(1\leqq j\leqq n)$

and the problem

$$F(b):\ \max\{cx:\ wx\leqq b,\ x\geqq 0 \text{ integer}\},$$

where we set $f(b)=\max\{cx:\ wx\leqq b,\ x\geqq 0\text{ integer}\}$ which is called Knapsack function [4]. Lemma 3 reduces to

Lemma 6. For $b\geqq w_a\equiv \min_{1\leqq j\leqq n} w_j$, there exists optimal x with $x_{j_0}>0$ iff $f(b)=f(b-w_{j_0})+c_{j_0}$.

Following the line of Hu, T.C. [11], we can assume without loss of generarity

Assumption 5. $\rho_1\geqq\rho_2\geqq\cdots\geqq\rho_n$ where $\rho_j=c_j\div w_j(1\leqq j\leqq n)$

Remark. Under this assumption either $\rho_1>\rho_2$ or there exists $k(2\leqq k\leqq n)$ $\rho_1=\cdots=\rho_k>\rho_{k+1}\geqq\cdots\geqq\rho_n$.

Definition 8. For $s\in N_0$ let $f(b;\ LEs)=\max\{cx:\ wx\leqq b,\ x\geqq 0 \text{ integer},\ x_1\leqq s\}$.

With this Definition we can prove in a same way as Hu did when $s=0$ [11],

Lemma 7. In the case $\rho_1>\rho_2$, for $s\in N_0$, $b\in N$, if $b\geqq\rho_1 w_1\div(\rho_1-\rho_2)+w_1 s$

then $f(b;\ LEs)<f(b)$ (that is, for any optimal $\bar{x}$ for $F(b)$, $\bar{x}_1>s$).

Until we arrive at the end of this paper, let us assume and define

Assumption 6. $\rho_1=\cdots=\rho_k>\rho_{k+1}\geqq\cdots\geqq\rho_n(2\leqq k\leqq n,\ n\geqq 2)$.

Definition 9. Let a be the greatest common divisor of $w_1, \cdots, w_k$ and d_i be such that $w_i=ad_i$ $(1\leqq i\leqq k)$ (Note that $d_1, \cdots, d_k$ are mutually prime.). Then $f(b)=\max\{\rho_1 a(\sum_{j=1}^{k} d_j x_j)+\sum_{j>k}^{n} c_j x_j:\ a(\sum_{j=1}^{k} d_j x_j)+\sum_{j>k}^{n} w_j x_j\leqq b,\ x\geqq 0 \text{ integer}\}$. So define $f_a(b)=\max\{\rho_1 at+\sum_{j>k}^{n} c_j x_j:\ at+\sum_{j>k}^{n} w_j x_j\leqq b,\ (t, x_{k+1}, \cdots, x_n)\geqq 0 \text{ integer}\}$.

Applying Lemma 6 and Lemma 7 to $f_a(b)$

Lemma 8. (1) $f(b)\leqq f_a(b)$.

(2) If $b\geqq\rho_1 a\div(\rho_1-\rho_{k+1})$ then $f_a(b)=f_a(b-a)+\rho_1 a$.

(3) If $b\geqq\rho_1 a\div(\rho_1-\rho_{k+1})+as$ then for any optimal $\bar{x}$ for $f_a(b)$, $\bar{t}>s$.

To find small b from which $f(b)=f(b-a)+\rho_1 a$ we prepare

Lemma 9. For u, v such that $u<v$,

(1) if u is integer then there exists integer in the interval $[u, v]$,

(2) if u is not integer then there exists integer in $[u, v]$ iff $[u]+1\leqq v$ where $[u]$ is the greatest integer which is less than or equal to u.

Lemma 10. For mutually prime $e_1, e_2\in N$ with $e_1x_1^0+e_2x_2^0=1$, it holds for $t\in N$ that $(-tx_2^0\div e_1)\in I$, or $[-tx_2^0\div e_1]+1\leqq tx_1^0\div e_2$ iff there exist x_1, $x_2\in N_0$ such that $t=e_1x_1+e_2x_2$.

Proof. *Only if* part; As $(tx_1^0\div e_2)-(-tx_2^0\div e_1)=t(e_1x_1^0+e_2x_2^0)\div e_1e_2=t\div e_1e_2>0$, by Lemma 9 there exists $p\in I$, $(-tx_2^0\div e_1)\leqq p\leqq(tx_1^0\div e_2)$. Letting $x_1=tx_1^0-e_2p$ and $x_2=tx_2^0+e_1p$, $e_1x_1+e_2x_2=t$ $(x_1, x_2\in N_0)$. *If* part can be proved similary. Q.E.D.

Remark. Condition $-tx_2^0/e_1\in I$, or $[-tx_2^0/e_1]+1\leqq tx_1^0/e_2$ is satisfied if $t\geqq e_1e_2$.

Assuming $d_1\leqq\cdots\leqq d_k$ (if not, reindex the x_j so as to satisfy this condition), we have three cases. (a) $d_1=\cdots=d_k$, (b) $1=d_1<d_k$, (c) $1<d_1<d_k$. In case (a) $f(b)=f(b-a)+\rho_1 ad_1=f(b-a)+c_1$ for $b\geqq c_1/(\rho_1-\rho_{k+1})$ is easily derived. In case (b) $f_a(b)=f(b)$ for any $b\in N_0$ is also easily

derived. And as $f_a(b)=f_a(b-a)+\rho_1 a$ for $b\geqq\rho_1 a/(\rho_1-\rho_{k+1})$ we have $f(b)=f(b-a)+\rho_1 a$ for $b\geqq\rho_1 a/(\rho_1-\rho_{k+1})$. To attack case (c) we prepare Condition (A) $1<d_1\leqq\cdots\leqq d_k$ and $d_1<d_k$ and that there exist γ, δ such that d_γ, d_δ are mutually prime.

Remark. Condition (A) is always true when $k=2$.

Lemma 11. Under Condition (A), let $d_\gamma x_\gamma^0+d_\delta x_\delta^0=1$, $r=\min\{t\colon t\in N, -tx_\delta^0/d_\gamma\in I$ or $[-tx_\delta^0/d_\gamma]+1\leqq tx_\gamma^0/d_\delta\}$ then for $t\in N$, $t\geqq r$ there exist $x_1, \cdots, x_k\in N_0$ such that at$=w_1x_1+\cdots+w_kx_k$.

Proof. Corresponding d_γ, d_δ to e_1, e_2 in Lemma 10, there exist x_γ, x_δ such that $t=d_\gamma x_\gamma+d_\delta x_\delta$. Noting that $ad_\gamma=w_\gamma$, $ad_\delta=w_\delta$ and setting $x_i=0$ $(i\neq\gamma, \delta)$ we have at$=w_\gamma x_\gamma+w_\delta x_\delta=w_1x_1+\cdots+w_kx_k$. Q.E.D.

Remark. In fact, $r=d_\gamma d_\delta-(d_\gamma+d_\delta)+1$ [16].

Theorem 3. Under Condition (A) let $b_A=\langle\rho_1 a/(\rho_1-\rho_{k+1})+ar\rangle$. If $b\geqq b_A$ then $f(b)=f(b-a)+\rho_1 a$, where $\langle x\rangle$ is the least integer which is greater than or equal to x and r is defined in Lemma 11.

Proof. As $\langle x\rangle\geqq x$, $b\geqq\rho_1 a/(\rho_1-\rho_{k+1})+ar$. And by Lemma 7 for any optimal $\bar{x}$ for $f_a(b)$, $\bar{t}>r$ and by Lemma 11 there exist $x_1, \cdots, x_k\in N_0$ such that $at=w_1x_1+\cdots+w_kx_k=a(d_1x_1+\cdots+d_kx_k)$ which lead to $f_a(b)=f(b)$. Noting the fact $\rho_1 a/(\rho_1-\rho_{k+1})\leqq b$ and using Lemma 8 $f_a(b)=f_a(b-a)+\rho_1 a$. As $b-a\geqq\rho_1 a/(\rho_1-\rho_{k+1})+a(r-1)$ similar argument as above yields $f_a(b-a)=f(b-a)$. Q.E.D.

For $b\geqq b_A+a(d_j-1)=\tilde{b}_j$, $f(b)=f(b-a)+\rho_1 a=\cdots=f(b-w_j)+\rho_1 w_j=f(b-w_j)+c_j$ so that the optimal solution for $F(b)$ can be obtained by adding 1 to the jth component of the optimal solution for $F(b-w_j)$ $(1\leqq j\leqq k)$. This property is called "periodicity property." Finally, we can prove in an elementary way that any Knapsack function has this property. First,

Lemma 12. For any j $(2\leqq j\leqq k)$ there exists $u_j\in N$ such that for any $t\in N$, $t\geqq u_j$ there exist $x_1, \cdots, x_j\in N_0$ with $(d_1, \cdots, d_j)t=d_1x_1+\cdots+d_jx_j$, where $(d_1, \cdots, d_j)$ is the greatest common divisor of $d_1, \cdots, d_j$.

Proof. (By induction). When $j=2$, set $e_1=d_1/(d_1, d_2)$, $e_2=d_2/(d_1, d_2)$ and apply Lemma 10 with its Remark so that we can take $u_2=d_1d_2/$

$(d_1, d_2)^2$. This proves Lemma for $j=2$. Assuming Lemma is valid for j, let v_{j+1} be such that $((d_1, \cdots, d_{j+1})/(d_1, \cdots, d_j))v_{j+1} \geqq u_j$ and $((d_1, \cdots, d_{j+1})/(d_1, \cdots, d_j))v_{j+1}$: integer then for $t \geqq u_{j+1}$, $u_{j+1}=v_{j+1}+(d_1, \cdots, d_j)d_{j+1}/((d_1, \cdots, d_j), d_{j+1})^2$ (*i.e.* $t-v_{j+1} \geqq (d_1, \cdots, d_j)d_{j+1}/((d_1, \cdots, d_j), d_{j+1})^2$), applying this Lemma for $(d_1, \cdots, d_j)$ and d_{j+1} ($k=2$) there exist $x_{1\ldots j}$, $x_{j+1} \in N_0$ such that $(d_1, \cdots, d_{j+1})(t-v_{j+1})=(d_1, \cdots, d_j)x_{1\ldots j}+d_{j+1}x_{j+1}$. Noting $x_{1\ldots j}+((d_1, \cdots, d_{j+1})/(d_1, \cdots, d_j))v_{j+1} \geqq u_j$ and from the assumption of induction, there exist $x_1, \cdots, x_j \in N_0$ such that $(d_1, \cdots, d_j)(x_{1\ldots j}+((d_1, \cdots, d_{j+1})/(d_1, \cdots, d_j))v_{j+1})=d_1x_1+\cdots+d_jx_j$. So that $(d_1, \cdots, d_{j+1})t=d_1x_1+\cdots+d_{j+1}x_{j+1}$ for $t \geqq u_{j+1}$. Q.E.D.

By $(d_1, \cdots, d_k)=1$ and Lemma 12 we obtain

Theorem 4. If $b \geqq \langle \rho_1 a/(\rho_1-\rho_{k+1})+au_k \rangle$ then $f(b)=f(b-a)+\rho_1 a$, where u_k is given in Lemma 12.

Recalling the Remark beneath the Assumption 5 we have showed that the Knapsack function always has periodicity property.

Example. $n=4$, $c=(5, 7, 6, 5)$, $w=(5, 7, 13, 11)$. As $5/5=7/7>6/13>5/11$, $k=2$, $a=1$, $d_1=5$, $d_2=7$, Condition (A) is satisfied with $\gamma=1$, $\delta=2$. And we have $x_1^0=-4$, $x_2^0=3$, $r=29$, $b_A=\langle 13/7+29 \rangle=31$, $\tilde{b}_1=35$, $\tilde{b}_2=37$. According to the stopping criterion of references [8], [11], it is assured that from $k=31$, $b_k=42$ there is no use calculating further. So, in this example our stopping criterion is superior to that of [8], [11].

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