

## TRAFFIC LIGHT QUEUES WITH DEPARTURE HEADWAYS DEPENDING UPON POSITIONS

KATSUHISA OHNO\* and HISASHI MINE

*Kyoto University*

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### Abstract

This paper deals with fixed-cycle and semi-vehicle-actuated traffic light queues which have general arrivals of various types and departure headways depending upon positions. It should be noted that this condition on departure headways is substantiated empirically. Depending upon the properties of arrivals, a necessary and sufficient condition or sufficient conditions are derived under which stationary queue length distributions exist. A stationary queue length distribution of the fixed-cycle traffic light queue with constant departure headways depending upon positions and stationary and independent arrivals is obtained. Finally, a queue of vehicles in front of a pedestrian crossing controlled by a traffic light with push-buttons for pedestrians is discussed.

### 1. Introduction

Many authors have investigated queues of vehicles at intersections controlled by traffic lights, or traffic light queues. Traffic lights are classified according to their functions into fixed-cycle, vehicle-actuated and semi-vehicle-actuated traffic lights. Fixed-cycle traffic light queues

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with constant departure headways and binomial or Poisson arrivals were discussed by Beckmann, McGuire and Winsten [3], Webster [32], Haight [11], Newell [21], Buckley and Wheeler [5], Dunne [9] and others. Fixed-cycle traffic light queues with constant departure headways and stationary and independent arrivals were investigated by Miller [19], Newell [22], McNeil [17] and Siskind [28]. Darroch [6] also discussed the fixed-cycle traffic light queue with constant departure headways and stationary and independent arrivals and derived a necessary and sufficient condition for a stationary queue length distribution to exist. Mine and Ohno [20], [25] discussed a fixed-cycle traffic light queue with generally distributed starting delays, independent and identically distributed departure headways and independent or dependent arrivals, and derived a necessary and sufficient condition for a stationary distribution to exist. Vehicle-actuated traffic light queues were discussed by Garwood [10], Darroch, Newell and Morris [7], Newell [23] and Newell and Osuna [24]. Semi-vehicle-actuated traffic light queues were investigated by Haight [11], Little [14] and Mine and Ohno [20], [25].

All authors mentioned above suppose that departure headways of all vehicles except the first are constant or identically distributed. Ancker, Gafarian and Gray [1], however, analyzed data on departure headways of vehicles queueing up in a straight-through lane leading into a signalized intersection and showed that departure headways are mutually independent random variables with shifted Erlang density functions depending upon positions. In this paper, therefore, departure headways are assumed to be mutually independent random variables with general distribution functions depending upon positions (see Assumption 3 in Section 2). In Section 2, fixed-cycle traffic light queues are dealt with. For dependent arrivals, a sufficient condition for a stationary queue length distribution to exist is derived, and for independent arrivals, a necessary and sufficient condition is derived under which a stationary queue length distribution exists. In Section

3, a fixed-cycle traffic light queue with constant departure headways depending upon positions and with stationary and independent arrivals is dealt with and its stationary queue length distribution is obtained. In Section 4, a semi-vehicle-actuated traffic light queue is dealt with and a sufficient condition for a stationary queue length distribution to exist is derived. As an example, a queue of vehicles in front of a pedestrian crossing controlled by a traffic light with detectors or push-buttons for arriving pedestrians is discussed.

## 2. Fixed-Cycle Traffic Light Queue

Consider a queue of vehicles in a straight-through lane of an approach to an intersection controlled by a fixed-cycle traffic light. The signal sequence of traffic lights is amber, red, and green in Japan, U.S.A. and some countries; while in U.K. it is amber, red, red and amber shown together, and green. Since the signal sequence varies thus with countries, in order to keep up generality, the time interval from the end of a green period to the beginning of the next green period is called the *red period* throughout this paper. To begin with, suppose that the following two natural assumptions are satisfied:

1. One cycle of the traffic light controlling the queue is composed of one red period of fixed length  $r$  and one successive green period of fixed length  $g$ .

Hence the cycle length of the traffic light is  $T(=r+g)$ . It is to be noted that the green period is not the "effective green period" used popularly but the exact time interval during which the traffic light shows the signal "green".

2. Vehicles which arrive in the green period and find the queue empty pass through the intersection without delay.

This assumption means that once the queue becomes empty in the green period, it remains empty till the end of the green period. This fact is regarded as the most distinctive feature of the traffic light queue.

Let some vehicles wait in the queue at the end of the red period. As the traffic light turns green, the  $i$ th ( $i=1, 2, \dots$ ) vehicle in the queue crosses the stop line at time  $t_i$  ( $0 < t_i \leq g$ ) from the beginning of the green period and passes through the intersection. The time interval  $t_1$  is called the *departure headway of position 1* or that of the first vehicle and the time interval between  $t_{i-1}$  and  $t_i$  ( $i=2, 3, \dots$ ) is called the *departure headway of position  $i$*  or that of the  $i$ th vehicle. It should be noted that departure headways are defined only for vehicles that stop by the signal "red" or slow down owing to the existing queue. Ancker, Gafarian and Gray [1] collected data on departure headways of vehicles queuing up on a straight-through middle lane of a three-lane road into an intersection with no downstream bottleneck, and analyzed carefully statistical properties of departure headways. Their results are that departure headways are mutually independent random variables with shifted Erlang density functions depending upon positions and that departure headways of the positions from 2 on are almost surely (a.s.) greater than a positive number (see, Table 10 in [1]). Although many authors have investigated traffic light queues, almost all of them assume that departure headways of the positions from 2 on are constant or, more generally, mutually independent and identically distributed random variables (m.i.i.d.r.v.s). Moreover in order to simplify analysis, they adopt the convention that the introduction of the terms "effectively red", "effectively green", and "lost time" or "starting delay" makes departure headways of all positions constant or m.i.i.d.r.v.s. The above results based upon experiment, however, show that this conventional assumption is not satisfied in the actual situation. This compels that the following assumption should be made on departure headways:

3. Departure headways are mutually independent random variables with distribution functions (d.f.s.)  $F_i$  depending upon position  $i$  and ones from some position on are a.s. greater than a positive number. This assumption is more general than both the conventional assump-

tion and the above-mentioned results obtained empirically, and seems to be in the nature of departure headways. Since the terms “effectively red”, “effectively green” and “lost time” are not employed in this paper, as remarked in Assumption 1, it is essential to make some assumption on the effect of the amber period in the red period; note that “not-green” is called red in this paper. Suppose that:

4. If the queue is not empty at the end of the green period, the vehicle at the head of the queue passes through the intersection with probability  $p$  ( $0 \leq p \leq 1$ ) and stops with probability  $1-p$ .

This assumption may be a first approximation of what actually happens.

Let the  $n$ th cycle of the traffic light begin at time  $nT$  and finish at time  $(n+1)T$ , where  $n=0, 1, 2, \dots$ . Denote by  $s_{ni}$  the departure headway of position  $i$  ( $i=1, 2, \dots$ ) in the  $n$ th cycle. Assumption 3 implies that the random variables  $s_{ni}$  ( $n=0, 1, 2, \dots, i=1, 2, \dots$ ) are mutually independent and, for fixed  $i$ ,  $s_{ni}$  ( $n=0, 1, 2, \dots$ ) are identically distributed with d.f.  $F_i$ . Put  $s_{n0}=0$  a.s. for  $n=0, 1, 2, \dots$  and define the saturation flow  $f_n$  of the  $n$ th green period as

$$(1) \quad f_n = \sup \left\{ k; \sum_{i=0}^k s_{ni} \leq g \right\}.$$

Assumption 3 implies that the saturation flows  $f_n$  are m.i.i.d.r.v.s and for a sufficiently large finite number  $K$ ,

$$(2) \quad f_n < K \quad \text{a.s.}$$

Define the random time  $t_{nm}$  ( $m=0, 1, 2, \dots$ ) as

$$t_{nm} = \inf \left\{ r + \sum_{i=0}^m s_{ni}, T \right\}.$$

Obviously,  $t_{n0}=r$  and  $t_{nm}=T$  for  $m=f_n+1, f_n+2, \dots$ . For  $m=1, 2, \dots, f_n$ , the random variable  $t_{nm}$  represents the time at which the  $m$ th vehicle crosses the stop line during the  $n$ th green period, if this vehicle is in the queue. Assumption 3 implies that for fixed  $m$ ,  $t_{nm}$  ( $n=0, 1, 2, \dots$ ) are mutually independent and identically distributed (m.i.i.d.) and that for a nonnegative integer  $k$ , the  $k$ -dimensional

random vectors  $(t_{n1}, t_{n2}, \dots, t_{nk})$  conditioned by the events  $\{f_n=k\}$  are also m.i.i.d. Let  $x_n(t)$  denote the number of vehicles in the straight-through lane arriving at the intersection in the time interval of length  $t$  from the beginning of the  $n$ th cycle, *i.e.* the time interval  $(nT, nT+t]$ . Define  $y_{nm}$  ( $m=0, 1, 2, \dots$ ) as follows:

$$(3) \quad y_{n0} = x_n(t_{n0}) = x_n(r)$$

and for  $m=1, 2, \dots$ ,

$$(4) \quad y_{nm} = x_n(t_{nm}) - x_n(t_{n(m-1)}).$$

These random variables  $y_{n0}, y_{nm}$  ( $m=1, 2, \dots, f_n$ ) and  $y_{nf_{n+1}}$  represent the numbers of vehicles arriving at the intersection in the  $n$ th red period, in the departure headway of position  $m$  and in the remaining green period  $(t_{nf_n}, T]$ , respectively.

Denote by  $L_n$  ( $n=0, 1, 2, \dots$ ) the queue length at the beginning of the  $n$ th cycle and by  $N_{nm}$  ( $m=0, 1, 2, \dots$ ) the queue length immediately after the random time  $t_{nm}$ . Then from Assumptions 1 and 2 it follows that

$$(5) \quad N_{n0} = L_n + y_{n0}$$

and for  $m=1, 2, \dots, f_n$ ,

$$(6) \quad N_{nm} = \begin{cases} N_{n(m-1)} + y_{nm} - 1, & \text{if } N_{n(m-1)} > 0, \\ = 0, & \text{otherwise.} \end{cases}$$

Let  $\{e_n; n=0, 1, 2, \dots\}$  be a sequence of mutually independent random variables which take value 0 with probability  $(1-p)$  and value 1 with probability  $p$ . That is,  $\{e_n\}$  is the result of a sequence of Bernoulli trials in Assumption 4. Therefore by Assumptions 1 and 2 and (2),

$$(7) \quad L_{n+1} \equiv N_{nf_{n+1}} = \begin{cases} N_{nf_n} + y_{nf_{n+1}} - e_n, & \text{if } N_{nf_n} > 0, \\ = 0, & \text{otherwise.} \end{cases}$$

Combination of Equations (5) through (7) leads to the following expression:

$$L_{n+1} = L_n + \sum_{m=0}^{f_{n+1}} y_{nm} - e_n - f_n, \\ \text{if } \min \{N_{n0}, N_{n1}, \dots, N_{nf_{n+1}}\} > 0, \\ = 0, \text{ otherwise.}$$

Therefore on account of (3) and (4), the recurrence relation between  $L_n$  and  $L_{n+1}$  ( $n=0, 1, 2, \dots$ ) holds as follows:

$$(8) \quad L_{n+1} = L_n + x_n(T) - e_n - f_n, \\ \text{if } L_n + \min \left\{ \min_{0 \leq m \leq f_n} \{x_n(t_{nm}) - m\}, x_n(T) - e_n - f_n \right\} > 0, \\ = 0, \text{ otherwise.}$$

Now define random variables  $u_n$  and  $v_n$  ( $n=0, 1, 2, \dots$ ) as

$$(9) \quad u_n = x_n(T) - e_n - f_n$$

and

$$(10) \quad v_n = \min \left\{ \min_{0 \leq m \leq f_n} \{x_n(t_{nm}) - m\}, x_n(T) - e_n - f_n \right\}.$$

It is clear by these definitions and (2) that

$$(11) \quad u_n \geq v_n \text{ a.s.}$$

and

$$(12) \quad v_n \geq -K \text{ a.s.}$$

Hence Relation (8) reduces to the following general model: for  $n=0, 1, 2, \dots$ ,

$$(13) \quad L_{n+1} = L_n + u_n, \text{ if } L_n + v_n > 0, \\ = 0, \quad \text{otherwise,}$$

where  $u_n$  and  $v_n$  satisfy (11) and (12). This model in which  $u_n$  and  $v_n$  satisfy (11) is a generalization of the  $GI/G/1$  queueing process discussed by Lindley [13] and Loynes [16] and has been investigated in [20] and [25]. Let  $E\{\cdot\}$  denote the expectation of  $\cdot$ . Some of the results obtained in [20] and [25] can be summarized as:

**Lemma 1.** Suppose that  $\{(u_n, v_n); n=0, 1, 2, \dots\}$  is strictly stationary and  $\{u_n\}$  obeys the strong law of large numbers and that  $u_n$  and  $v_n$  satisfy (11). If  $E\{u_n\} < 0$ , then the d.f. of  $L_n$  defined iteratively by (13) converges to a unique honest d.f. independent of any initial distribution of  $L_0$ , as  $n$  tends to infinity. In particular, if random vectors  $(u_n, v_n)$  are m.i.i.d and  $u_n$  and  $v_n$  satisfy (12) and if

$$(14) \quad \Pr \{v_n > 0\} > 0,$$

then  $E\{u_n\} < 0$  is also a necessary condition under which the d.f. of  $L_n$  converges to a unique honest d.f. independent of any initial dis-

tribution of  $L_0$ , as  $n$  tends to infinity.

Stochastic properties of the arrival processes in the  $n$ th cycle,  $x_n(t)$  ( $0 < t \leq T$ ) are left unspecified as yet. Vehicles arriving at the intersection during the  $n$ th red period have to wait until the beginning of the  $n$ th green period at the shortest. Therefore it suffices for the following analysis of this section to specify stochastic properties of the arrival process in the  $n$ th green period  $x_n(t)$  ( $r \leq t \leq T$ ). In this paper, the sequence of arrival processes in green periods  $\{x_n(t)$  ( $r \leq t \leq T$ );  $n=0, 1, 2, \dots\}$  is called *strictly stationary*, if for arbitrary nonnegative integers  $k_n$  ( $n=0, 1, 2, \dots$ ) and arbitrarily fixed  $\tau_{ni}$  ( $n=0, 1, \dots, i=1, 2, \dots, k_n$ ) such that  $r \leq \tau_{n1} \leq \dots \leq \tau_{nk_n} \leq T$ , the sequence of random vectors  $\{(x_n(r), x_n(\tau_{n1}), \dots, x_n(\tau_{nk_n}), x_n(T))\}$  is strictly stationary. This means that for  $l < m < \dots < n$ , any nonnegative integer  $h$  and arbitrary sets  $A, B, C$ ,

$$\begin{aligned} & \Pr \{ (x_l(r), x_l(\tau_{l1}), \dots, x_l(\tau_{lk_l}), x_l(T)) \in A, (x_m(r), x_m(\tau_{m1}), \\ & \quad \dots, x_m(\tau_{mk_m}), x_m(T)) \in B, \dots, (x_n(r), x_n(\tau_{n1}), \dots, \\ & \quad x_n(\tau_{nk_n}), x_n(T)) \in C \} \\ & = \Pr \{ (x_{l+h}(r), x_{l+h}(\tau_{l1}), \dots, x_{l+h}(\tau_{lk_l}), x_{l+h}(T)) \in A, \\ & \quad (x_{m+h}(r), x_{m+h}(\tau_{m1}), \dots, x_{m+h}(\tau_{mk_m}), x_{m+h}(T)) \in B, \dots, \\ & \quad (x_{n+h}(r), x_{n+h}(\tau_{n1}), \dots, x_{n+h}(\tau_{nk_n}), x_{n+h}(T)) \in C \}. \end{aligned}$$

To begin with, suppose that the arrival processes in green periods satisfy the following assumption:

- A. The sequence of arrival processes in green periods  $\{x_n(t)$  ( $r \leq t \leq T$ );  $n=0, 1, 2, \dots\}$  is strictly stationary and independent of departure headways. In particular,  $\{x_n(T)\}$  is ergodic and metrically transitive (see Doob [8], page 457) and  $E\{x_n(T)\} = \lambda T < \infty$ .

This assumption appears to be quite reasonable and practical. Since  $x_n(t)$  ( $r \leq t \leq T$ ) are independent of random vectors  $(f_n, t_{n1}, \dots, t_{nf_n})$ , it follows from (10) that for any integer  $k$ ,

$$(15) \quad \Pr \{v_n = k\} = \sum_{i,j} \Pr \{e_n = i\} \int \Pr \{ \min \{x_n(r), x_n(\tau_{n1}) - 1, \dots, x_n(\tau_{nj}) - j, x_n(T) - i - j\} = k \} d_r P \{f_n = j, t_{n1} \leq \tau_{n1},$$

$$\dots, t_{nj} \leq \tau_{nj}\}.$$

Assumptions 3 and 4 imply that the random vectors  $(f_n, t_{n1}; \dots, t_{n\tau_n})$  are m.i.i.d. and so are random variables  $e_n$ . Consequently, on account of Assumption A, the sequence of random vectors  $\{(u_n, v_n)\}$  defined by (9) and (10) is strictly stationary. Moreover,  $\{u_n\}$  obeys the strong law of large numbers, because  $f_n$  and  $e_n$  are m.i.i.d. and  $\{x_n(T)\}$  is ergodic (Doob [8], page 465). Hence the queue length  $L_n$  at the beginning of the  $n$ th cycle satisfies Model (13) in which  $\{(u_n, v_n)\}$  is strictly stationary,  $\{u_n\}$  obeys the strong law of large numbers, and  $u_n$  and  $v_n$  satisfy (11).

Denote by  $A*B$  the convolution of a d.f.  $A$  with a d.f.  $B$ . Clearly, by Assumption 3,

$$(16) \quad H(g) \equiv E\{f_n\} = \sum_{i=1}^{\infty} F_1 * \dots * F_i(g),$$

where  $H(\cdot)$  is the renewal function of the renewal sequence  $\{s_{ni}; i=1, 2, \dots\}$  (see, Smith [30]). Therefore by Assumptions 4 and A,

$$E\{u_n\} = \lambda T - H(g) - p.$$

This equation and Lemma 1 lead immediately to a sufficient condition under which the distribution of the queue length  $L_n$  at the beginning of the  $n$ th cycle converges to a unique stationary distribution independent of any initial distribution of  $L_0$ , as  $n$  tends to infinity. It is clear that if the queue length at the beginning of each cycle has a stationary distribution, then the queue length at an arbitrarily fixed time from the beginning of each cycle has a stationary distribution, too. This consequence is said briefly that the traffic light queue has a stationary distribution.

**Theorem 1.** Suppose that Assumptions 1 through 4 and A are satisfied. The fixed-cycle traffic light queue has a unique stationary distribution independent of any initial distribution, if

$$(17) \quad \lambda T < H(g) + p,$$

where  $H(g)$  is given by (16).

Suppose temporarily that the *conventional assumption on departure*

headways is satisfied; that is, for d.f.  $L$  and mean  $l$  of starting delays,  $s_{n1}$  is distributed with d.f.  $L * F$  with mean  $(l+m)$  and for  $i=2, 3, \dots$ ,  $s_{ni}$  is distributed with d.f.  $F$  with mean  $m$ . Then, according to Smith [29],

$$(18) \quad H(g) = (g - l + E(\zeta)) / m - 1,$$

where  $\zeta$  is the residual useful life, that is, the remaining departure headway of the vehicle served at the end of the green period. Thus, (17) can be rewritten as

$$(19) \quad \lambda m T < g - l + \{E(\zeta) - (1-p)m\}.$$

This condition agrees in substance with the result obtained in [25], in which the total remaining departure headway of vehicles in the queue is dealt with and the "preemptive resume" service discipline is assumed instead of Assumption 4. Because if the vehicle at the head of the queue fails to stop for the amber signal and passes through the intersection, then  $\zeta$  is thought of as effectively green, and if it stops, then  $(s_{.i} - \zeta)$  is thought of as effectively red; consequently, the mean effective green time is

$$g + pE(\zeta) - \{l + (1-p)(m - E(\zeta))\}.$$

That is, the right member of (19) is equal to the mean effective green time.

In the above, the sufficient condition for the fixed-cycle traffic light queue to have a stationary distribution has been obtained under the assumption that the arrival processes in green periods are permitted to be dependent. In the sequel of this section, the arrival processes in green periods are restricted in order to derive a necessary and sufficient condition under which a stationary distribution exists. Arrival processes in green periods  $x_n(t)$  ( $r \leq t \leq T$ ) are called *mutually independent*, if for arbitrary integers  $k_n$  and sequences  $\{\tau_{ni}; i=1, 2, \dots, k_n\}$  such that  $r \leq \tau_{n1} \leq \dots \leq \tau_{nk_n} \leq T$ , random vectors  $(x_n(r), x_n(\tau_{n1}), \dots, x_n(\tau_{nk_n}), x_n(T))$  are mutually independent. Moreover, arrival processes in green periods are called *identically distributed*, if for any integer  $k$  and sequence  $\{\tau_i; i=1, 2, \dots, k\}$  such that  $r \leq \tau_1 \leq$

$\dots \leq \tau_k \leq T$ , random vectors  $(x_n(r), x_n(\tau_1), \dots, x_n(\tau_k), x_n(T))$  are identically distributed. Suppose that the arrival processes in green periods satisfy the following assumption instead of Assumption A:

**B.** The arrival processes in green periods  $x_n(t)$  ( $r \leq t \leq T$ ) are m.i.i.d. with  $E\{x_n(T)\} = \lambda T (< \infty)$  and independent of departure headways.

This assumption requires none of stationarity, absence of after-effects and orderliness of the stream of vehicles arriving at the intersection (see Khintchine [12]). Conversely, if the stream of arriving vehicles is a stationary one without after-effects and has finite intensity  $\lambda$ , the arrival processes in green periods satisfy Assumption B. Thus, Assumption B seems to be still reasonable. Since the random vectors  $(f_n, t_{n1}, \dots, t_{nf_n})$  are m.i.i.d. and so are random variables  $e_n$ , it follows from (15) and Assumption B that the random vectors  $(u_n, v_n)$  are m.i.i.d. Hence the queue length  $L_n$  satisfies Model (13) in which the random vectors  $(u_n, v_n)$  are m.i.i.d. and  $u_n$  and  $v_n$  satisfy (11) and (12). In a similar way to Theorem 1, Lemma 1 leads directly to a necessary and sufficient condition under which the fixed-cycle traffic light queue has a stationary distribution.

**Theorem 2.** Suppose that Assumptions 1 through 4, B and Condition (14) are satisfied. The fixed-cycle traffic light queue has a unique stationary distribution independent of any initial distribution, if and only if

$$(20) \quad \lambda T < H(g) + p,$$

where  $H(g)$  is given by (16).

Since by (10)

$$v_n = \min\{x_n(r), x_n(t_{n1}) - 1, \dots, x_n(t_{nf_n}) - f_n, x_n(T) - e_n - f_n\},$$

Condition (14) is not satisfied for a very light traffic such that  $x_n(T) \leq e_n + f_n$  a.s. For the stationary stream without after-effects of arriving vehicles, however, Condition (14) is satisfied unless intensity  $\lambda$  vanishes. Condition (14) is not satisfied also for an artificially controlled traffic which guarantees  $x_n(r) = 0$  a.s.; for instance, the ideal synchronization of sequential traffic lights may assure  $x_n(r) = 0$  a.s.

This, however, is hindered by the variation of vehicles' velocities and incoming streams from upstream minor road. Thus Condition (14) is satisfied for a usual intersection controlled by a traffic light. Suppose that the conventional assumption on departure headways is satisfied. Then, in much the same way as the remark following Theorem 1, Condition (20) reduces to the form of (18). Moreover, Condition (18) with  $p$  substituted by one includes Darroch's condition [6] as a simple case.

### 3. Stationary Queue Length Distribution

In this section, a fixed-cycle traffic light queue with regular departure headways depending upon positions is dealt with and its stationary queue length distribution is derived in a familiar way used by Bailey [2], Newell [21], Darroch [6] and others. Suppose that Assumptions 1, 2 and 4 are satisfied and that a departure headway of position  $i$  ( $i=1, 2, \dots$ ) is a constant positive number  $s_i$ . Clearly Assumption 3 is satisfied. It can be assumed without loss of generality that  $s_1 < g$ . By (1), the saturation flow  $f_n$  of the  $n$ th green period is a constant positive integer  $M$  such that  $\sum_{i=1}^M s_i \leq g < \sum_{i=1}^{M+1} s_i$ . For convenience of notations, put

$$(21) \quad s_{M+1} = g - \sum_{i=1}^M s_i .$$

Suppose that the stream of vehicles in the straight-through lane arriving at the intersection is a stationary one without after-effects and has a finite and positive intensity  $\lambda\alpha$ , that is, a compound Poisson process with the mean number  $\lambda$  of jumps in unit time and with the mean height  $\alpha$  of the individual jump. This assumption implies that Assumption B is satisfied and  $x_n(t)$  ( $r \leq t \leq T$ ) has a probability generating function (p.g.f.) in the following form (see Khintchine [12], page 34): for a complex variable  $z$ ,

$$(22) \quad \exp \{ \lambda t (\gamma(z) - 1) \} ,$$

where  $\gamma(z)$  is an arbitrary p.g.f. given by

$$(23) \quad \eta(z) = \sum_{k=1}^{\infty} p_k z^k$$

and

$$(24) \quad \alpha \equiv \eta'(1) < \infty.$$

It is assumed that

$$(25) \quad p_1 > 0$$

and that for a sufficiently small positive number  $\delta$ , the radius of convergence of  $\eta(z)$  is greater than  $(1+\delta)$ , that is,

$$(26) \quad \limsup_{k \rightarrow \infty} p_k^{1/k} < 1/(1+\delta).$$

For instance, the Borel-Tanner distribution  $p_k$  ( $k=1, 2, \dots$ ) (see Miller [18]) given by

$$p_k = k^{k-1} \gamma^{k-1} e^{-\gamma} / k!$$

satisfies (25) and (26) for positive parameter  $\gamma$  excluding the neighbourhood of 1, because the p.g.f. of this distribution has the radius of convergence  $e^{\gamma-1}/\gamma$ . Moreover suppose that for the positive probability  $p$  in Assumption 4,

$$(27) \quad \lambda \alpha T < M + p.$$

Then it follows from Theorem 2 that the distribution of the queue length  $L_n$  converges to a unique stationary distribution, as  $n$  tends to infinity.

Let  $\xi(z)$  be the p.g.f. of the random variable  $(1-e_n)$ , that is,

$$(28) \quad \xi(z) = p + qz,$$

where  $q=1-p$  and  $0 < p \leq 1$ . Denote by  $\tilde{L}_n(z)$ ,  $\tilde{N}_{nm}(z)$  and  $\tilde{y}_{nm}(z)$  the p.g.f.s. for  $L_n$ ,  $N_{nm}$  and  $y_{nm}$  ( $n=0, 1, 2, \dots, m=1, 2, \dots, M+1$ ), respectively. Then from (5) through (7) it follows that

$$\tilde{N}_{n0}(z) = \tilde{L}_n(z) \cdot \tilde{y}_{n0}(z),$$

for  $m=1, 2, \dots, M$ ,

$$\tilde{N}_{nm}(z) = \frac{\tilde{y}_{nm}(z)}{z} \tilde{N}_{nm-1}(z) + \left(1 - \frac{\tilde{y}_{nm}(z)}{z}\right) \tilde{N}_{nm-1}(0).$$

and

$$\tilde{L}_{n+1}(z) = \frac{\xi(z) \tilde{y}_{nM+1}(z)}{z} \tilde{N}_{nM}(z) + \left(1 - \frac{\xi(z) \tilde{y}_{nM+1}(z)}{z}\right) \tilde{N}_{nM}(0).$$

Therefore

$$\begin{aligned}
 (29) \quad \tilde{L}_{n+1}(z) &= \xi(z)\tilde{y}_{n0}(z)\left(\prod_{m=1}^{M+1}\frac{\tilde{y}_{nm}(z)}{z}\right)\tilde{L}_n(z) + \left(1 - \frac{\xi(z)\tilde{y}_{nM+1}(z)}{z}\right) \\
 &\quad \times \tilde{N}_{nM}(0) + \xi(z)\sum_{m=0}^{M-1}\left(1 - \frac{\tilde{y}_{nm+1}(z)}{z}\right)\left(\prod_{j=m+2}^{M+1}\frac{\tilde{y}_{nj}(z)}{z}\right) \\
 &\quad \times \tilde{N}_{nm}(0).
 \end{aligned}$$

Since from Theorem 2, as  $n$  tends to infinity,  $\tilde{L}_n(z)$  and  $\tilde{N}_{nm}(z)$  ( $m = 0, 1, \dots, M$ ) converge to the p.g.f.s. for the stationary distributions, say  $\tilde{L}(z)$  and  $\tilde{N}_m(z)$  ( $m = 0, 1, \dots, M$ ), letting  $n$  tend to infinity in (29) yields

$$\begin{aligned}
 (30) \quad (z^{M+1} - \xi(z)\prod_{m=0}^{M+1}\tilde{y}_m(z))\tilde{L}(z) &= (z^{M+1} - z^M\xi(z)\tilde{y}_{M+1}(z))\tilde{N}_M(0) \\
 &\quad + \xi(z)\sum_{m=0}^{M-1}(z - \tilde{y}_{m+1}(z))\left(\prod_{j=m+2}^{M+1}\tilde{y}_j(z)\right)z^m\tilde{N}_m(0),
 \end{aligned}$$

where by (3), (4), (21) and (22),

$$(31) \quad \tilde{y}_0(z) \equiv \tilde{y}_{n0}(z) = \exp\{\lambda r(\eta(z) - 1)\}$$

and for  $m = 1, 2, \dots, M + 1$ ,

$$(32) \quad \tilde{y}_m(z) \equiv \tilde{y}_{nm}(z) = \exp\{\lambda s_m(\eta(z) - 1)\}.$$

Put  $f(z)$  and  $g_m(z)$  ( $m = 0, 1, \dots, M$ ) as follows and rewrite them by use of (21), (28), (31) and (32):

$$(33) \quad f(z) \equiv z^{M+1} - \xi(z)\prod_{m=0}^{M+1}\tilde{y}_m(z) = z^{M+1} - (p + qz)\exp\{\lambda T(\eta(z) - 1)\},$$

for  $m = 0, 1, \dots, M - 1$ ,

$$\begin{aligned}
 (34) \quad g_m(z) &\equiv z^m\xi(z)(z - \tilde{y}_{m+1}(z))\left(\prod_{j=m+2}^{M+1}\tilde{y}_j(z)\right) \\
 &= z^m(p + qz)[z - \exp\{\lambda s_{m+1}(\eta(z) - 1)\}] \\
 &\quad \times \exp\left\{\lambda\sum_{j=m+2}^{M+1}s_j(\eta(z) - 1)\right\}
 \end{aligned}$$

and

$$\begin{aligned}
 (35) \quad g_M(z) &\equiv z^{M+1} - z^M\xi(z)\tilde{y}_{M+1}(z) = z^{M+1} - z^M(p + qz) \\
 &\quad \times \exp\{\lambda s_{M+1}(\eta(z) - 1)\}.
 \end{aligned}$$

Then Equation (30) reduces to

$$(36) \quad \tilde{L}(z) = \sum_{m=0}^M g_m(z) \tilde{N}_m(0) / f(z).$$

It should be noted that  $\tilde{L}(z)$  in (36) contains unknowns  $\tilde{N}_m(0)$ . These unknowns can be determined by a system of linear equations in the sequel.

Since  $\tilde{L}(z)$  is a p.g.f., it is analytic inside the unit circle and by Abel's theorem,  $\lim_{z \rightarrow 1-0} \tilde{L}(z) = 1$ . Therefore, by (24) and (33) through (36),

$$(37) \quad M + p - \lambda \alpha T = (p - \lambda \alpha s_{M+1}) \tilde{N}_M(0) + \sum_{m=0}^{M-1} (1 - \lambda \alpha s_{m+1}) \tilde{N}_m(0).$$

Moreover the numerator of (36) must have the same zeros inside the unit circle as the denominator  $f(z)$ . The following lemma follows directly from Rouché's theorem (see, for example, Titchmarsh [31], page 116):

**Lemma 2.** Suppose that Assumptions (25) through (27) are satisfied. Then the denominator  $f(z)$  has  $(M+1)$  distinct zeros, say,  $z_1, z_2, \dots, z_{M+1}$  on and inside the unit circle, where  $z_1 = 1$  and for  $k = 2, 3, \dots, M+1$ ,  $0 < |z_k| < 1$ .

From this lemma and (36) it follows that for  $k = 2, 3, \dots, M+1$ ,

$$(38) \quad \sum_{m=0}^M g_m(z_k) \tilde{N}_m(0) = 0.$$

Define an  $(M+1) \times (M+1)$  matrix  $A = (a_{kl})$  and  $(M+1)$  dimensional vectors  $\mathbf{x} = (x_l)$  and  $\mathbf{b} = (b_l)$  as follows:

$$\begin{aligned} a_{1l} &= 1 - \lambda \alpha s_l \text{ for } l = 1, 2, \dots, M, \quad a_{1M+1} = p - \lambda \alpha s_{M+1}, \\ a_{kl} &= g_{l-1}(z_k) \text{ for } l \neq 1, 2, \dots, M+1 \text{ and } k = 2, 3, \dots, M+1, \\ x_l &= \tilde{N}_{l-1}(0) \text{ for } l = 1, 2, \dots, M+1, \quad b_1 = M + p - \lambda \alpha T \text{ and} \\ b_l &= 0 \text{ for } l = 2, 3, \dots, M+1. \end{aligned}$$

Then the system of  $(M+1)$  linear equations (37) and (38) can be rewritten in the matrix form:

$$\mathbf{Ax} = \mathbf{b}.$$

This system of linear equations must have a unique solution and the matrix  $A$  is nonsingular, since from Theorem 2,  $\tilde{L}(z)$  is unique. There-

fore, for  $m=0, 1, \dots, M$ ,

$$(39) \quad \tilde{N}_m(0) = (M + p - \lambda\alpha T) \mathbf{A}_{1m+1} / |\mathbf{A}|,$$

where  $\mathbf{A}_{1m+1}$  is the cofactor of the element  $a_{1m+1}$ . Hence combination of Equations (36) and (39) leads to

$$(40) \quad \tilde{L}(z) = (M + p - \lambda\alpha T) \sum_{m=0}^M \mathbf{A}_{1m+1} g_m(z) / (|\mathbf{A}| f(z)).$$

If  $s_1 = s_2 = \dots = s_{M+1}$  and  $p=1$ , then this p.g.f. will agree with the Darroch's result [6], although he made less practical assumptions concerning on (22) and (23) than Assumptions (25) and (26).

Differentiating (40) yields

$$(41) \quad \tilde{L}'(1) = \left\{ (M + p - \lambda\alpha T) \sum_{m=0}^M \mathbf{A}_{1m+1} g_m''(1) - |\mathbf{A}| f''(1) \right\} / (2|\mathbf{A}| f'(1)).$$

While by (33) through (35),

$$(42) \quad f'(1) = M + p - \lambda\alpha T,$$

$$(43) \quad f''(1) = M(M+1) - \lambda T(2q\alpha + \lambda T\alpha^2 + \beta),$$

for  $m=0, 1, \dots, M-1$

$$(44) \quad g_m''(1) = 2(1 - \lambda\alpha s_{m+1}) \left( m + q + \lambda\alpha \sum_{j=m+2}^{M+1} s_j \right) - \lambda s_{m+1} (\lambda s_{m+1} \alpha^2 + \beta)$$

and

$$(45) \quad g_M''(1) = 2(1 - \lambda\alpha s_{M+1}) (M + q) - 2(M + 1)q - \lambda s_{M+1} (\lambda s_{M+1} \alpha^2 + \beta),$$

where  $\beta = \gamma''(1)$ . Therefore from (41) through (45) it follows that the expectation of the stationary queue length at the beginning of the cycle,  $\mathbf{E}\{L\}$  is given by:

$$\begin{aligned} \mathbf{E}\{L\} = & \left[ \sum_{m=1}^{M+1} \mathbf{A}_{1m} \{ (1 - \lambda\alpha s_m) (m - p) - \lambda s_m (\lambda s_m \alpha^2 + \beta) / 2 \} \right. \\ & + \sum_{m=1}^M \lambda\alpha (1 - \lambda\alpha s_m) \mathbf{A}_{1m} \sum_{j=m+1}^{M+1} s_j - q(M+1) \mathbf{A}_{1M+1} \left. \right] / |\mathbf{A}| \\ & - \{ M(M+1) + \lambda T(2q\alpha + \lambda T\alpha^2 + \beta) \} / 2(M + p - \lambda\alpha T). \end{aligned}$$

Let us be concerned with the expected delay per vehicle in a stationary state, say  $\mathbf{E}\{W\}$ . Denote by  $N(t)$  the queue length at time  $t$  from the beginning of the cycle in a stationary state. Then

$$(46) \quad \mathbf{E}\{W\} = \mathbf{E} \left\{ \int_0^T N(t) dt \right\} / (\lambda\alpha T).$$

While by Fubini theorem (see Loève [15], page 136),

$$(47) \quad \mathbf{E} \left\{ \int_0^T N(t) dt \right\} = \int_0^r \mathbf{E} N(t) dt + \sum_{m=1}^{M+1} \int_{t_{nm-1}}^{t_{nm}} \mathbf{E} N(t) dt$$

$$= r\mathbf{E}\{L\} + \lambda\alpha r^2/2 + \sum_{m=1}^{M+1} \{s_m \tilde{N}'_{m-1}(1) + \lambda\alpha s_m^2(1 - \tilde{N}_{m-1}(0))/2\}.$$

Moreover,

$$\tilde{N}'_0(1) = \mathbf{E}\{L\} + \lambda\alpha r$$

and for  $m=1, 2, \dots, M$ ,

$$\tilde{N}'_m(1) = \mathbf{E}\{L\} + \lambda\alpha r - \sum_{j=1}^M (1 - \tilde{N}_{j-1}(0))(1 - \lambda\alpha s_j).$$

Consequently from (39), (46) and (47),

$$\mathbf{E}\{W\} = \mathbf{E}\{L\}/(\lambda\alpha) + (M + p - \lambda\alpha T) \left[ \sum_{m=1}^M \mathbf{A}_{1m} \right.$$

$$\times \left\{ (1 - \lambda\alpha s_m) \sum_{j=m+1}^{M+1} s_j - \lambda\alpha s_m^2/2 \right\} - \lambda\alpha s_{M+1}^2 \mathbf{A}_{1M+1}/2 \left. \right] / \lambda\alpha T | \mathbf{A} |$$

$$+ \left\{ rg + \left( r^2 + \sum_{m=1}^{M+1} s_m^2 \right) / 2 \right\} / T - \sum_{m=1}^M (1 - \lambda\alpha s_m) \sum_{j=m+1}^{M+1} s_j / (\lambda\alpha T).$$

In the above, arrivals are assumed to be stationary. Now suppose that arrivals are nonstationary; that is, the p.g.f. of  $x_n(t)$  is assumed to be  $\exp \{ \lambda(t)(\gamma(z) - 1) \}$ , instead of (22). For these generalized arrivals,  $\tilde{L}(z)$ ,  $\mathbf{E}\{L\}$  and  $\mathbf{E}\{w\}$  can be obtained by much the same way as in the above.

#### 4. Semi-Vehicle-Actuated Traffic Light Queue

Consider a queue of vehicles in a straight-through lane of a major road leading into an intersection controlled by a semi-vehicle-actuated traffic light with detectors on a minor road. Denote by  $r_n$ ,  $g_n$  and  $T_n$  ( $n=0, 1, 2, \dots$ ) the lengths of the  $n$ th red period, the  $n$ th green period and the  $n$ th cycle, respectively. Stochastic properties of these random variables are determined by the control algorithm of the traffic light in front of the queue, stochastic properties of the stream of vehicles arriving at the intersection along the minor road, those of their departure headways, and so on. What controls the queue, however,

is not these many factors but random variables  $r_n$  and  $g_n$ . Therefore, in order to maintain generality of the following analysis, suppose that: 1'. The sequence of positive random vectors  $\{(r_n, g_n); n=0, 1, 2, \dots\}$  is strictly stationary and ergodic, and  $E(r_n)=r<\infty$  and  $E(g_n)=g<\infty$ . Consequently  $T_n$  have finite mean  $T(=r+g)$ . Moreover suppose that Assumptions 2 through 4 in Section 2 are satisfied and that departure headways are independent of  $r_n$  and  $g_n$ . Let the 0th cycle begin at time zero. Denote by  $x(t)$  ( $t>0$ ) the number of vehicles in the straight-through lane of the major road arriving at the intersection in time interval  $(0, t]$ . This process is called *the arrival process on the major road* in this section. Because the arrival processes in green periods used in the preceding sections are directly affected by random variables  $r_n$  and  $g_n$ . Put  $x_k=x(k+1)-x(k)$  for  $k=0, 1, 2, \dots$ . Suppose that

C. The arrival process on the major road  $x(t)$  ( $t>0$ ) is a stationary stream with finite intensity  $\lambda$  and independent of  $\{(r_n, g_n)\}$  and departure headways. In particular, the sequence  $\{x_k; k=0, 1, 2, \dots\}$  is ergodic.

Define the saturation flow  $f_n$  ( $n=0, 1, 2, \dots$ ) and the random time  $t_{nm}$  ( $m=0, 1, 2, \dots$ ) as

$$(48) \quad f_n = \sup \left\{ k : \sum_{i=0}^k S_{ni} \leq g_n \right\}$$

and

$$(49) \quad t_{nm} = \inf \left\{ r_n + \sum_{i=0}^m S_{ni}, T_n \right\}.$$

By Assumptions 1' and 3, the sequence of saturation flows  $f_n$  is strictly stationary. Besides, by (48), for arbitrary integers  $m_i, k_i$  ( $i=1, 2, \dots, i_0$ ), and  $n_j, l_j$  ( $j=1, 2, \dots, j_0$ ),

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{h=0}^{n-1} \Pr \{ f_{m_i} = k_i, f_{n_j+h} = l_j, i=1, \dots, i_0, j=1, \dots, j_0 \} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{h=0}^{n-1} \left[ \Pr \left\{ \sup \left\{ k; \sum_{r=0}^k S_{m_i r} \leq x_i \right\} = k_i, i=1, \dots, i_0 \right\} \right] \end{aligned}$$

$$\begin{aligned} &\times \Pr \left\{ \sup \left\{ k; \sum_{r=0}^k s_{n,r} \leq y_j \right\} = l_j, j=1, \dots, j_0 \right\} \\ &\times dP \{g_{m_i} \leq x_i, g_{n_j+h} \leq y_j, i=1, \dots, i_0, j=1, \dots, j_0\}. \end{aligned}$$

Since according to the necessary and sufficient condition for ergodicity (see Billingsley [4], page 17),

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{h=0}^{n-1} \Pr \{g_{m_i} \leq x_i, g_{n_j+h} \leq y_j, i=1, \dots, i_0, j=1, \dots, j_0\} \\ &= \Pr \{g_{m_i} \leq x_i, i=1, \dots, i_0\} \Pr \{g_{n_j} \leq y_j, j=1, \dots, j_0\}, \end{aligned}$$

it follows from Helly-Bray theorem (see Loève [15], page 182) that

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{h=0}^{n-1} \Pr \{f_{m_i} = k_i, f_{n_j+h} = l_j, i=1, \dots, i_0, j=1, \dots, j_0\} \\ &= \Pr \{f_{m_i} = k_i, i=1, \dots, i_0\} \Pr \{f_{n_j} = l_j, j=1, \dots, j_0\}. \end{aligned}$$

That is,  $\{f_n; n=0, 1, 2, \dots\}$  is ergodic, and obeys the strong law of large numbers (see, Doob [8], page 465). Note that  $f_n$  does not satisfy (2). Clearly, by (49),

$$(50) \quad x_n(t_{nm}) = x \left( \sum_{i=0}^{n-1} T_i + t_{nm} \right) - x \left( \sum_{i=0}^{n-1} T_i \right).$$

In the same way as in (8), it follows from 1', 2 and 4 that the following recurrence relation between  $L_n$  and  $L_{n+1}$  ( $n=0, 1, 2, \dots$ ) holds:

$$(51) \quad \begin{aligned} L_{n+1} &= L_n + x_n(T_n) - e_n - f_n, \\ &\text{if } L_n + \inf \left\{ \inf_{0 \leq m \leq f_n} \{x_n(t_{nm}) - m\}, x_n(T_n) - e_n - f_n \right\} > 0, \\ &= 0, \text{ otherwise.} \end{aligned}$$

Put for  $n=0, 1, 2, \dots$ ,

$$(52) \quad u_n = x_n(T_n) - e_n - f_n$$

and

$$(53) \quad v_n = \inf \left\{ \inf_{0 \leq m \leq f_n} \{x_n(t_{nm}) - m\}, x_n(T_n) - e_n - f_n \right\}.$$

Then,  $u_n$  and  $v_n$  satisfy (11), and not (12). For arbitrary integers  $n_j$  and  $k_j$  ( $j=1, \dots, j_0$ ), by (50) and (53),

$$\Pr \{v_{n_j} = k_j, j=1, \dots, j_0\} = \Pr \left\{ \inf \left\{ \inf_{0 \leq m \leq f_{n_j}} \{x_{n_j}(t_{n_j, m}) - m\}, \right. \right.$$

$$\begin{aligned}
 & \left. x_{n_j}(T_{n_j}) - e_{n_j} - f_{n_j} \right\} = k_j, \quad j=1, \dots, j_0 \Big\} \\
 & = \Pr \left\{ \inf_{0 \leq m \leq f_{n_j}} \left\{ x \left( \sum_{i=0}^{n_j-1} (\eta_i + \theta_i) + t_{n_j m} \right) - x \left( \sum_{i=0}^{n_j-1} \eta_i + \theta_i \right) - m \right\}, \right. \\
 & \quad \left. x \left( \sum_{i=0}^{n_j} \eta_i + \theta_i \right) - x \left( \sum_{i=0}^{n_j-1} \eta_i + \theta_i \right) - e_{n_j} - f_{n_j} \right\} = k_j, \\
 & \quad \left. j=1, \dots, j_0 \mid r_i \leq \eta_i, \quad g_i \leq \theta_i, \quad i=0, 1, \dots, n_{j_0} \right\} \\
 & \times dP \{ r_i \leq \eta_i, \quad g_i \leq \theta_i, \quad i=0, 1, \dots, n_{j_0} \}.
 \end{aligned}$$

Consequently from Assumptions 1', 3, 4 and C, for any positive integer  $h$ ,

$$\Pr \{ v_{n_j} = k_j, \quad j=1, \dots, j_0 \} = \Pr \{ v_{n_j+h} = k_j, \quad j=1, \dots, j_0 \}.$$

In a similar way it can be shown that  $\{(u_n, v_n)\}$  is strictly stationary. Since  $\{e_n\}$  and  $\{f_n\}$  obey the strong law of large numbers, it suffices to prove that  $\{x_n(T_n)\}$  obeys this law, in order to prove  $\{u_n\}$  obeys the law. Let  $y_n = \left[ \sum_{i=0}^{n-1} T_i \right]$ , where  $[\cdot]$  means the maximal integer not exceeding  $\cdot$ . By Assumption 1',  $\lim_{n \rightarrow \infty} y_n/n = \lim_{n \rightarrow \infty} \left\{ \sum_{i=0}^{n-1} T_i + \left( y_n - \sum_{i=0}^{n-1} T_i \right) \right\} / n = E(T_i) = T$  a.s. Therefore by Assumption C and (50),

$$\begin{aligned}
 (54) \quad & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} x_i(T_i) = \lim_{n \rightarrow \infty} \frac{1}{n} x \left( \sum_{i=0}^{n-1} T_i \right) \\
 & = \lim_{n \rightarrow \infty} \frac{y_n}{n} \cdot \frac{1}{y_n} \sum_{k=0}^{y_n-1} x_k + \frac{1}{n} \left( x \left( \sum_{i=0}^{n-1} T_i \right) - x(y_n) \right) \\
 & = \lambda T = E\{x_n(T_n)\} \text{ a.s.}
 \end{aligned}$$

Hence from (51) through (54) it follows that the queue length  $L_n$  at the beginning of the  $n$ th cycle satisfies Model (13) in which  $\{(u_n, v_n)\}$  is strictly stationary,  $\{u_n\}$  obeys the strong law of large numbers and  $u_n$  and  $v_n$  satisfy (11). The following theorem results immediately from Lemma 1:

**Theorem 3.** Suppose that Assumptions 1' through 4 and C are satisfied. The semi-vehicle-actuated traffic light queue has a unique limiting distribution independent of any initial distribution, if

$$(55) \quad \lambda T < \mathbf{E}\{H(g_n)\} + p,$$

where  $H(\cdot)$  is the renewal function of departure headways and is given by (16).

This theorem gives a sufficient condition under which the quite general semi-vehicle-actuated traffic light queue has a limiting distribution. Suppose temporarily that the conventional assumption on departure headways is satisfied. Let  $l_n$  and  $\zeta_n$  ( $n=0, 1, 2, \dots$ ) be the starting delay and the residual useful life in the  $n$ th cycle. Then in the same way as in (18),

$$\mathbf{E}\{H(g_n)\} = \mathbf{E}\{[g_n + \zeta_n - l_n]^+\} / m - 1,$$

where  $[\cdot]^+ = \max\{\cdot, 0\}$ . Therefore, (55) reduces to

$$\lambda m T < \mathbf{E}\{[g_n + \zeta_n - l_n]^+\} - (1-p)m.$$

To maintain generality of the control algorithm of the traffic light, the arrival process on the minor road and so on, only one Assumption 1' has been made for them. As an example which satisfies Assumption 1', consider a queue of vehicles in front of a pedestrian crossing controlled by a traffic light with detectors or push-buttons for arriving pedestrians. The control algorithm is:

5. The traffic light stays red for a fixed length  $r$  and turns green.

It turns red at fixed time  $a$  ( $\geq 0$ ) after any pedestrian arrives at the pedestrian crossing.

Moreover the following reasonable assumption is made:

6. Pedestrians arrive in groups at the pedestrian crossing. The interarrival times of successive groups are m.i.i.d.r.v.s. with d.f.  $G$  and positive mean  $\mu$ . The queue of pedestrians discharges a.s. during one red period.

Suppose that the first group of pedestrians arrives at time  $(-a)$ . Then the 0th cycle begins at time zero and each green period is composed of one residual useful life for interarrival times of pedestrians and one time interval of fixed length  $a$ . This implies that Assumption 1' is satisfied. Denote by  $R(\cdot)$  the renewal function of interarrival times: that is, for  $x \geq 0$ ,

$$(56) \quad R(x) = \sum_{k=1}^{\infty} G_k(x),$$

where  $G_k(x)$  is the  $k$ -fold convolution of  $G(x)$  with itself. Clearly,

$$(57) \quad T = \mu\{1 + R(a+r)\}.$$

Moreover, for  $x < a$ ,  $\Pr\{g_n \leq x\} = 0$  and for  $x \geq a$ ,

$$\begin{aligned} \Pr\{g_n \leq x\} &= G(r+x) - G(a+r) + \sum_{k=1}^{\infty} \int_0^{a+r} \{G(r+x-y) \\ &\quad - G(a+r-y)\} dG_k(y) \\ &= G(r+x) + \int_0^{a+r} G(r+x-y) dR(y) - R(a+r). \end{aligned}$$

Consequently

$$(58) \quad \begin{aligned} E\{H(g_n)\} &= \int_a^{\infty} H(x) dG(r+x) + \int_0^{a+r} \int_a^{\infty} H(x) \\ &\quad \times dG(r+x-y) dR(y). \end{aligned}$$

From Theorem 3, (57) and (58) it follows that if

$$\begin{aligned} \lambda\mu\{1 + R(a+r)\} &< p + \int_a^{\infty} H(x) dG(r+x) \\ &\quad + \int_0^{a+r} \int_a^{\infty} H(x) dG(r+x-y) dR(y), \end{aligned}$$

then the queue of vehicles in front of the pedestrian crossing has a unique limiting distribution, where  $H(x)$  and  $R(x)$  are given by (16) and (56), respectively.

## 5. Concluding Remarks

The queue of vehicles in the straight-through lane at the intersection controlled by a fixed-cycle or a semi-vehicle-actuated traffic light is discussed throughout in this paper. If  $g$  or  $g_n$  is interpreted as the length of the  $n$ th effective green period and if departure headways of vehicles turning right or left satisfy Assumption 3, then all theorems in this paper remain valid for the queue of vehicles turning right or left. Besides, with most vehicle-actuated traffic lights, maximum vehicle-extension periods are predetermined in order to prevent vehicles on intersecting roads from waiting indefinitely. Consequently,

vehicle-actuated traffic lights in effect become fixed-cycle ones, if the traffic is fairly heavy on all phases (see Webster and Cobbe [33]). Thus, Theorems 1 and 2 remain valid for the vehicle-actuated traffic light queue, if  $g$  and  $T$  are interpreted as lengths of the maximum green period and the maximum cycle. Those conditions shown in Theorems 1 through 3 may be essential for design of signalized intersections.

Consider an intersection with  $J$  approaches which is controlled by a fixed-cycle traffic light. Suppose that vehicles arriving on all approaches satisfy assumptions in Theorem 2. Then from Theorem 2, the queue of vehicles on the  $j$ th approach has a stationary distribution, if and only if

$$1 > \lambda_j T / \{H_j(g_j) + p_j\}$$

where suffix  $j$  refers to vehicles on the  $j$ th approach. Hence if and only if

$$\max_{j=1, \dots, J} [\lambda_j T / \{H_j(g_j) + p_j\}] < 1,$$

then all queues at the intersection have stationary distributions; that is, the intersection is undersaturated. Some relations between the above condition and optimal signal settings are shown in [26] and [27].

In Theorems 1 and 3, departure headways are assumed to be a.s. greater than a positive number. This assumption is not essential and can be dropped.

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## References

- [1] Ancker, C.J., A.V. Gafarian and R.K. Gray, "The Oversaturated Signalized Intersection—Some Statistics," *Transportation Sci.*, **2** (1968), 340-361.
- [2] Bailey, N.T.J., "On Queueing Processes with Bulk Service," *J. Roy. Statist. Soc. B.*, **16** (1954), 80-87.
- [3] Beckmann, M., C.B. McGuire and C.B. Winsten, *Studies in the Economics of Transportation*, Yale Univ. Press, New Haven, 1956.
- [4] Billingsley, P., *Ergodic Theory and Information*, John Wiley, New York, 1965.
- [5] Buckley, D.J. and R.C. Wheeler, "Some Results for Fixed-Time Traffic Signals," *J. Roy. Statist. Soc. B.*, **26** (1964), 133-140.
- [6] Darroch, J.N., "On the Traffic-Light Queue," *Ann. Math. Statist.*, **35** (1964), 380-388.
- [7] Darroch, J.N., G.F. Newell and R.W.J. Morris, "Queues for a Vehicle-Actuated Traffic Light," *Operations Res.*, **12** (1964), 882-895.
- [8] Doob, J.L., *Stochastic Processes*, John Wiley, New York, 1953.
- [9] Dunne, M.C., "Traffic Delay at a Signalized Intersection with Binomial Arrivals," *Transportation Sci.*, **1** (1967), 24-31.
- [10] Garwood, F., "An Application of the Theory of Probability to the Operation of Vehicular-Controlled Traffic Signals," *J. Math. Soc.*, **7** (1940), 65-77.
- [11] Haight, F.A., "Overflow at a Traffic Flow," *Biometrika*, **46** (1959), 420-424.
- [12] Khintchine, A.Y., *Mathematical Methods in the Theory of Queueing*, Charles Griffin, London, 1960.
- [13] Lindley, D.V., "The Theory of Queues with a Single Server," *Proc. Cambridge Phil. Soc.*, **48** (1952), 277-289.
- [14] Little, J.G., Jr., "Queueing of Side-Street Traffic at a Priority Type Vehicle-Actuated Signal," *Transportation Res.*, **5** (1971), 295-300.
- [15] Loève, M., *Probability Theory*, Van Nostrand, New York, 1963.
- [16] Loynes, R.M., "The Stability of a Queue with Non-Independent Inter-Arrival and Service Times," *Proc. Cambridge Phil. Soc.*, **58** (1962), 497-520.
- [17] McNeil, D.R., "A Solution to the Fixed-Cycle Traffic Light Problem for Compound Poisson Arrivals," *J. Appl. Prob.*, **5** (1968), 624-635.
- [18] Miller, A.J., "A Queueing Model for Road Traffic Flow," *J. Roy. Statist. Soc. B.*, **23** (1961), 64-75.
- [19] Miller, A.J., "Settings for Fixed-Cycle Traffic Signals," *Operations Res. Quart.*, **14** (1963), 373-386.
- [20] Mine, H. and K. Ohno, "Traffic Light Queues as a Generalization to Queueing Theory," *J. Appl. Prob.*, **8** (1971), 480-493.
- [21] Newell, G.F., "Queues for a Fixed-Cycle Traffic Light," *Ann. Math. Statist.*, **31** (1960), 589-597.
- [22] Newell, G.F., "Approximation Methods for Queues with Application to the Fixed-Cycle Traffic Light," *SIAM Rev.*, **7** (1965), 223-240.

- [23] Newell, G.F., "Properties of Vehicle-Actuated Signals: I. One-way streets," *Transportation Sci.*, **3** (1969), 30-52.
- [24] Newell, G.F. and E.E. Osuna, "Properties of Vehicle-Actuated Signals: II. Two-way streets," *Transportation Sci.*, **3** (1969), 99-125.
- [25] Ohno, K. and H. Mine, "Traffic Light Queues with Dependent Arrivals as a Generalization to Queueing Theory," *J. Appl. Prob.*, **9** (1972), 630-641.
- [26] Ohno, K. and H. Mine, "Optimal Traffic Signal Settings. I: Criterion for Undersaturation of a Signalized Intersection and Optimal Signal Settings," *Transportation Res.*, **7** (1973), 243-267.
- [27] Ohno, K. and H. Mine, "Optimal Traffic Signal Settings. II: A Refinement of Webster's Method," *Transportation Res.*, **7** (1973), 269-292.
- [28] Siskind, V., "The Fixed Cycle Traffic Light Problem: A Note on a Paper by McNeil," *J. Appl. Prob.*, **7** (1970), 245-248.
- [29] Smith, W.L., "Renewal Theory and Its Ramifications," *J. Roy. Statist. Soc. B.*, **20** (1958), 243-302.
- [30] Smith, W.L., "On Some General Renewal Theorem for Nonidentically Distributed Variables," *Proc. 4th Berkeley Symp. Math. Statist. Prob.*, **2** (1961), 467-514.
- [31] Titchmarsh, E.C., *The Theory of Functions*, Oxford Univ. Press, London, 1939.
- [32] Webster, F.V., "Traffic Signal Settings," *Road Research Technical Paper*, No. 39, Her Majesty's Stationery Office, London, 1958.
- [33] Webster, F.V. and B.M. Cobbe, "Traffic Signals," *Road Research Technical Paper*, No. 56, Her Majesty's Stationery Office, London, 1966.