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REDUCTION METHODS FOR TANDEM QUEUING SYSTEMS

TAKEJI SUZUKI and TAKESHI KAWASHIMA
The Defense Academy

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Abstract

A tandem queuing system has finite stages $A_1, A_2, \cdots, A_n$ in series, where stage A_i consists of c_i parallel channels each having the same constant service time t_i. Customers arrive at the first stage and are served through all the stages in order, on a basis of first-come, first-served. The queue in front of the first stage A_1 is unlimited, but the intermediate queues at the other stages $A_2, \cdots, A_n$ may be limited or unlimited. Then, it follows that for any arrival process (a) the time spent in the system by each customer is independent of the order of the stages and of the allowable sizes of the intermediate queues and (b) the problems of flow-time for the system can be reduced to corresponding problems for a system of fewer stages.

1. Introduction

The tandem queuing systems with arbitrary input and constant service time are treated by Avi-Itzhak [1] and Friedman [2]. The first deals with a system where every stage consists of the same number of servers and intermediate queues are finite. Involving with the general systems where the number of servers is not the same in all stages, he only considers a system of two stages as an example. The second concerns with two tandem queuing systems Model I and II, called by him. In these models all intermediate queues are unlimited. Model I is the case where each stage has arbitrary number of channels, and Model II is the same as Model I except that one of the stages is a single channel having variable service time with values larger than a definite constant. They showed that the time spent in the given system by each customer is independent of the order of the stages and also propose reduction methods by which the system can be reduced to one with fewer stages.

In this paper, we concern with a general system including Avi-Itzhak's and Friedman's Model I, and obtain two theorems covering their results.

Let us consider a tandem queuing system with n stages A_1, A_2, $\cdots$, A_n under the following assumptions: (a) customers arrive according to an arbitrary process of arrivals, (b) they are served on a basis of "first-come, first-served", (c) every one of them is processed through all the stages according to the order of the stages, and (d) stage A_i consists of c_i parallel channels each having the same constant service time $t_i>0$ $(i=1, 2, \cdots, n)$.

Now, we denote by q_i $(i=1, 2, \cdots, n)$ the queue-size allowable before the ith stage. The notation $[A_1A_2\cdots A_n]$ is used to represent the system with $q_1=\infty$ and $0\leqq q_i<\infty$ for $i\neq 1$, and $(A_1A_2\cdots A_n)$ for the system with $q_i=\infty$ for all i.

Introducing some intermediate stages with zero service times to the system $[A_1A_2\cdots A_n]$, as shown by [1], all intermediate queues can be equated to zero (that is, $q_i=0$, $i\neq 1$). We will start, however, our analysis without such the modification.

At the next section we prove that for any sequence of customer arrival times the time spent by a customer in the system $(A_1A_2\cdots A_n)$ is the same as he would have in the system $[A_1A_2\cdots A_n]$.

2. Relations between $(A_1A_2\cdots A_n)$ and $[A_1A_2\cdots A_n]$

We assume a sequence of customers $C(1)$, $C(2)$, $C(3)$, $\cdots$ indexed in order of arrival, such that $C(k)$ arrives at the first stage A_1 at time $a(k)$. In the system $(A_1A_2\cdots A_n)$ (or $[A_1A_2\cdots A_n]$), the customer $C(k)$ departs from each stage A_i at time $z_i(k)$ (or $z'_i(k)$). For simplicity, we put $z(k)=z_n(k)$ (or $z'(k)=z'_n(k)$). Further, channels of these systems are all free at the first arrival time and then customers are assigned on a cyclic policy to the channels at each stage. Here if we number the c channels from 1 to c, then the customer $C(k)$ is assigned to the mth channel provided that $k\equiv m$ (modulo c). Throughout the following statements, we put $z_i(k)=z'_i(k)=-\infty$ if $k\leqq 0$.

Lemma 2.1. For any given input sequence $\{a(k)\}$,

$$z_i(k)\leqq z'_i(k) \quad (i=1, 2, \cdots, n;\ k=1, 2, \cdots).$$

Proof. For the system $[A_1A_2\cdots A_n]$ we have

$$z'_1(k)\geqq \max\{z'_1(k-c_1)+t_1,\ a(k)+t_1\},$$
$$z'_i(k)\geqq \max\{z'_i(k-c_i)+t_i,\ z'_{i-1}(k)+t_i\} \quad (i=2, 3, \cdots, n).$$

For the system $(A_1A_2\cdots A_n)$ we have

$$z_1(k)= \max\{z_1(k-c_1)+t_1,\ a(k)+t_1\},$$
$$z_i(k)= \max\{z_i(k-c_i)+t_i,\ z_{i-1}(k)+t_i\} \quad (i=2, 3, \cdots, n).$$

Clearly, $z_i(1)=z'_i(1)$ $(i=1, 2, \cdots, n)$. We assume the induction hypothesis that

$$z_j(m)\leqq z'_j(m) \quad (m\leqq k-1,\ j=1, 2, \cdots, n \text{ and } m=k,\ j\leqq i-1).$$

Then, from the above relations we obtain

$$z_i(k)\leqq z'_i(k).$$

Lemma 2.2. For any given input sequence $\{a(k)\}$,

$$z'(k)\leqq \max\{z'(k-c_i)+t_i \quad (i=1, 2, \cdots, n),\ a(k)+\sum_{i=1}^{n} t_i\}. \tag{2.1}$$

Proof. For the system $[A_1A_2\cdots A_n]$, the waiting time and the blocking time of the customer $C(k)$ at the stage A_i are denoted by $w_i(k)$ and $b_{i+1}(k)$ respectively, where $b_1(k)=0$. And we put $B_i(k)=b_i(k)+w_i(k)$. If $B_i(k)=0$ for all i, then

$$z'(k)=a(k)+\sum_{i=1}^{n} t_i . \tag{2.2}$$

If there is some i such that $B_i(k)>0$ and $B_j(k)=0$ for all $j>i$, then we have $z'(k)=z_i'(k)+\sum_{j=i+1}^{n} t_j$ and $z_i'(k)=z_i'(k-c_i)+t_i$. Since, in general, $z'(k-c_i)\geqq z_i'(k-c_i)+\sum_{j=i+1}^{n} t_j$, we see that

$$z'(k)=z_i'(k-c_i)+\sum_{j=i+1}^{n} t_j+t_i\leqq z'(k-c_i)+t_i . \tag{2.3}$$

From (2.2) and (2.3), we obtain (2.1).

Theorem 2.1. For any given input sequence $\{a(k)\}$,

$$z(k)=\max\{z(k-c_i)+t_i \quad (i=1, 2, \cdots, n),\ a(k)+\sum_{i=1}^{n} t_i\}, \tag{2.4}$$

$$z'(k)=\max\{z'(k-c_i)+t_i \quad (i=1, 2, \cdots, n),\ a(k)+\sum_{i=1}^{n} t_i\}, \tag{2.5}$$

$$z(k)=z'(k) \quad (k=1, 2, 3, \cdots). \tag{2.6}$$

Proof. At first we will prove (2.4) by induction. Clearly,

$$z_1(k)=\max\{z_1(k-c_1)+t_1,\ a(k)+t_1\} \text{ for all } k.$$

We assume that

$$z_r(k)=\max\{z_r(k-c_i)+t_i \quad (i=1, 2, \cdots, r),\ a(k)+\sum_{i=1}^{r} t_i\} \tag{2.7}$$

for all k and $r\leqq j-1$,

and by considering the following two cases we will show that the relation for $r=j$ holds.

Case 1. When $k\leqq c_j$, we see that $z_j(k)=z_{j-1}(k)+t_j$. And $z_j(k)$ may be written by the following,

$$z_j(k)=\max\{z_{j-1}(k-c_i)+t_j+t_i \quad (i=1, 2, \cdots, j-1),\ a(k)+\sum_{i=1}^{j} t_i\}$$

using (2.7). Further, from the relations $z_j(k-c_i)=z_{j-1}(k-c_i)+t_j$ and $z_j(k)\geqq z_j(k-c_j)+t_j$, we have

$$\begin{aligned} z_j(k)&=\max\{z_j(k-c_i)+t_i \quad (i=1, 2, \cdots, j-1),\ a(k)+\sum_{i=1}^{j} t_i\} \\ &=\max\{z_j(k-c_i)+t_i \quad (i=1, 2, \cdots, j),\ a(k)+\sum_{i=1}^{j} t_i\}. \end{aligned}$$

Case 2. When $k>c_j$, to start with, we assume the induction hypothesis that

(2.8) $$z_j(h)=\max\{z_j(h-c_i)+t_i \quad (i=1, 2, \cdots, j),\ a(h)+\sum_{i=1}^{j} t_i\}$$
for all $h<k$.

Then, we have

(2.9) $$\begin{aligned} z_j(k) &= \max\{z_j(k-c_j)+t_j,\ z_{j-1}(k)+t_j\} \\ &= \max\{z_j(k-c_i-c_j)+t_j+t_i \quad (i=1, 2, \cdots, j-1), \\ &\quad z_j(k-2c_j)+2t_j,\ z_{j-1}(k)+t_j,\ a(k-c_j)+t_j+\sum_{i=1}^{j} t_i\} \\ &\quad \text{(from (2.8))} \\ &= \max\{z_j(k-c_i-c_j)+t_j+t_i,\ z_{j-1}(k-c_i)+t_j+t_i \quad (i=1, \\ &\quad 2, \cdots, j-1),\ z_j(k-2c_j)+2t_j,\ a(k-c_j)+t_j+\sum_{i=1}^{j} t_i, \\ &\quad z_j(k-c_j)+t_j,\ a(k)+\sum_{i=1}^{j} t_i\} \qquad \text{(from (2.7) and } z_j(k) \\ &\quad \geqq z_j(k-c_j)+t_j) \\ &= \max\{z_j(k-c_i)+t_i \quad (i=1, 2, \cdots, j),\ a(k)+\sum_{i=1}^{j} t_i\}. \end{aligned}$$

The last equation is derived from the first one of (2.9) and the inequality $z_j(k-c_j)\geqq \max\{a(k-c_j)+\sum_{i=1}^{j} t_i,\ z_j(k-2c_j)+t_j\}$. This completes the proof of (2.4).

Next, we will show (2.6). Clearly, $z(1)=z'(1)$. We assume $z(h)=z'(h)$ for all $h<k$. From (2.4) and (2.1), it follows that

$$\begin{aligned} z(k) &= \max\{z(k-c_i)+t_i \ (i=1, 2, \cdots, n),\ a(k)+\sum_{i=1}^{n} t_i\} \\ &= \max\{z'(k-c_i)+t_i \ (i=1, 2, \cdots, n),\ a(k)+\sum_{i=1}^{n} t_i\} \\ &\geqq z'(k). \end{aligned}$$

Using the lemma 2.1 we obtain (2.6). Finally, the relation (2.5) now follows from (2.4) and (2.6).

The relations (2.4) and (2.6) mean that for any given input sequence to the system $(A_1A_2\cdots A_n)$ and $[A_1A_2\cdots A_n]$ the output sequences are both independent of the order of the stages. Further, (2.6) means that these output sequences are identical. By use of these results we will able to controll the traffic flow in the system, that is, the flow time at the ith stage of a customer is varied by the permutation of the stages and the appropriate assignment of queue size at each stage without increase of the total flow time of the customer.

Also, only if we concern the time spent in the system by each customer the problem for the system $[A_1A_2\cdots A_n]$ is reduced to corresponding problem for the system $(A_1A_2\cdots A_n)$. The following section is devoted to research the system $(A_1A_2\cdots A_n)$.

3. Dominating Class of $(A_1A_2\cdots A_n)$

Let us consider any two stages A and B. If, in the system (AB), no customer ever waits at the stage B regardless of the input sequence to the stage A, then we say that the stage A dominates the stage B and write $A\gg B$. The statement $A\gg B$ is a property relative to stages A and B, and is independent of system context. For example, if $A\gg B$ then in the system $(\cdots B\cdots A\cdots)$ we may still say that $A\gg B$. We will use the notation (c, t) in order to denote the stage A, where c is the number of channels and t is the service time of each channel.

The following two lemmas were given by Friedman [2].

Lemma 3.1. $A_1=(c_1, t_1)$ dominates $A_2=(c_2, t_2)$ if and only if $[c_2/c_1]t_1\geqq t_2$, where [ ] denotes the greatest integer notation.

Lemma 3.2. (a) Dominance is transitive—that is, if $A_1\gg A_2$ and $A_2\gg A_3$, then $A_1\gg A_3$. (b) Dominance is persistent—that is, if A_i dominates A_j and they are nonadjacent stages in a system $(\cdots A_i\cdots A_j\cdots)$, then no waiting ever occurs at A_j, regardless of the input sequence to the system.

For a set of stages $A_1, A_2, \cdots, A_n$ we define the class of stages, say $D(A_1A_2\cdots A_n)$, such that

$$D(A_1A_2\cdots A_n)=\{B|B\gg A_i \ (i=1, 2, \cdots, n)\}.$$

Then, this class may be written as follows:

$$D(A_1A_2\cdots A_n)=D(A_1)\cap D(A_2)\cap\cdots\cap D(A_n). \tag{3.1}$$

Proposition 3.1. $A\gg B$ if and only if $D(A)\subset D(B)$.

Proof. At first, we assume that $A\gg B$. If $C\in D(A)$, then $C\gg A\gg B$.

according to lemma 3.2(a). That is, $C \in D(B)$. Conversely, if $D(A) \subset D(B)$, then $A \in D(B)$ referring to the fact $A \in D(A)$. That is, $A \gg B$.

If, in a system $(BA_1A_2\cdots A_n)$, no customer ever waits at any stage A_i regardless of the input sequence to the stage B, then we say that B dominates the system $(A_1A_2\cdots A_n)$ and write the fact as $B \gg (A_1A_2\cdots A_n)$. For a system $(A_1A_2\cdots A_n)$ we define the class of stages, say $D((A_1A_2\cdots A_n))$, such that

$$D((A_1A_2\cdots A_n))=\{B \mid B \gg (A_1A_2\cdots A_n)\}.$$

Further, we introduce the notation $f_k(A_1A_2\cdots A_n)$ to denote the flow-time $z(k)-a(k)$ of the customer $C(k)$ in the system $(A_1A_2\cdots A_n)$.

Theorem 3.1. For any given system $(A_1A_2\cdots A_n)$,

$$\begin{aligned}(3.2)\qquad & D((A_1A_2\cdots A_n))\\ &=D(A_1A_2\cdots A_n)\\ &=\{(c,\ t_c) \mid 1\leqq c\leqq \min(c_1,\ c_2,\ \cdots,\ c_n) \text{ and}\\ &\quad t_c\geqq \max([c_i/c]^{-1}t_i\ \ (i=1,\ 2,\ \cdots,\ n))\},\end{aligned}$$

where $A_i=(c_i,\ t_i)$ and $t_i>0$ for all i.

Proof. For any stage $B=(c,\ t) \in D(A_1A_2\cdots A_n)$, we have $B \gg A_i$ $(i=1,\ 2,\ \cdots,\ n)$ from (3.1). By lemma 3.1 we find that the stage B satisfies the conditions

$$c\leqq c_i \text{ and } t\geqq [c_i/c]^{-1}t_i \qquad \text{for all } i.$$

Next, we take any stage $B=(c,\ t)$ satisfying the above conditions and consider the system $(BA_1A_2\cdots A_n)$. By lemma 3.2 (b) we see that $B \gg (A_1A_2\cdots A_n)$, that is, $B \in D((A_1A_2\cdots A_n))$.

Now it remains to prove that $D((A_1A_2\cdots A_n)) \subset D(A_1A_2\cdots A_n)$. If $B \in D((A_1A_2\cdots A_n))$, then we have from theorem 2.1 that

$$\begin{aligned} & w_B(k)+\sum_{i=1}^{n} t_i+t_B\\ &=f_k(BA_1A_2\cdots A_n)\\ &=f_k(BA_{i1}A_{i2}\cdots A_{in}) \qquad (A_{i1},\ A_{i2},\ \cdots,\ A_{in} \text{ is any permutation of } A_1,\ A_2,\ \cdots,\ A_n)\\ &=w'_B(k)+w'_{i1}(k)+\cdots+w'_{in}(k)+\sum_{i=1}^{n} t_i+t_B,\end{aligned}$$

where $w_B(k)$ is the waiting time of the customer $C(k)$ at the stage B

in the system $(BA_1A_2\cdots A_n)$ and $w'_B(k)$, $w'_{ij}(k)$ are the waiting times at B and A_{ij} in the system $(BA_{i1}A_{i2}\cdots A_{in})$ respectively. t_B is the service time at the stage B. Since $w_B(k)=w'_B(k)$, we have $w'_{ij}(k)=0$ for all j. That is, $B\in D((A_{i1}A_{i2}\cdots A_{in}))$ for any permutation of A_1, A_2, $\cdots$, A_n. If, in particular, we consider the system $(BA_i\cdots)$, we see that no waiting ever occurs at A_i regardless of the input sequence to B. Therefore, $B\gg A_i$ for any i. This completes the proof.

For a set of stages $\{A_i=(c_i,\ t_i),\ i=1,\ 2,\ \cdots,\ n\}$ we define the minimal dominating class of stages, say $M(A_1A_2\cdots A_n)$, such that

$$M(A_1A_2\cdots A_n)=\{(c,\ t_c^*)\,|\,1\leqq c\leqq \min(c_1,\ c_2,\ \cdots,\ c_n) \ \text{ and }\ t_c^*=\max([c_i/c]^{-1}t_i \quad (i=1,\ 2,\ \cdots,\ n))\}.$$

Clearly, $M(A_1A_2\cdots A_n)\subset D(A_1A_2\cdots A_n)$.

Lemma 3.3. If $A_1=(c_1,\ t_1)\gg A_2=(c_2,\ t_2)$, then

$$f_k(A_1)-t_1\geqq f_k(A_2)-t_2.$$

Proof. For the system (A_1A_2) we have

$$f_k(A_1A_2)=f_k(A_1)+t_2,$$

while for the system (A_2A_1)

$$f_k(A_2A_1)=f_k(A_2)+w_1(k)+t_1$$
$$\geqq f_k(A_2)+t_1,$$

where $w_1(k)$ is the waiting time of $C(k)$ at the stage A_1. Since $f_k(A_1A_2)=f_k(A_2A_1)$ from theorem 2.1, which gives us the desired result.

Corollary of lemma 3.3. If $c_1=c_2$ in the lemma 3.3, then

$$f_k(A_1)\geqq f_k(A_2).$$

Proof. From lemma 3.1 we have $t_1\geqq t_2$. Using lemma 3.3 the corollary is followed.

Proposition 3.2. For fixed c,

$$t_c^*=\inf\{t\,|\,(c,\ t)\in D(A_1A_2\cdots A_n)\}.$$

Proof. We put $T_c=\inf\{t\,|\,(c,\ t)\in D(A_1A_2\cdots A_n)\}$. If $C=(c,\ t)\in D(A_1A_2\cdots A_n)$, then $C\gg A_i$ for all i. Using lemma 3.1, we have $T_c\geqq t_c^*$. Next, we take the stage $E=(c,\ t_c^*)$. Then, we have

$$[c_i/c]t_c^*=\max\{[c_1/c]^{-1}[c_i/c]t_1,\ \cdots,\ t_i,\ \cdots\}\geqq t_i \quad \text{for all } i.$$

That is, $E \in D(A_1A_2\cdots A_n)$. Therefore, $T_c \leqq t_c^*$.

Theorem 3.2. For a given system $(A_1A_2\cdots A_n)$ and any input sequence,

$$\max_{1\leqq i\leqq n}\{f_k(A_i)-t_i\}+\sum_{i=1}^{n} t_i \leqq f_k(A_1A_2\cdots A_n)$$

$$\leqq \min_{B\in M(A_1\cdots A_n)}\{f_k(B)-t_B\}+\sum_{i=1}^{n} t_i \qquad \text{for all } k.$$

Proof. In the system $(A_1A_2\cdots A_n)$ we have

$$\begin{aligned} f_k(A_1A_2\cdots A_n) &= f_k(A_iA_1\cdots A_{i-1}A_{i+1}\cdots A_n) \quad \text{(from theorem 2.1)}\\ &= f_k(A_iA_1\cdots A_{i-1}A_{i+1}\cdots A_{n-1})+w_n(k)+t_n\\ &\geqq f_k(A_iA_1\cdots A_{i-1}A_{i+1}\cdots A_{n-1})+t_n\\ &\geqq \cdots\cdots\cdots\cdots\\ &\geqq f_k(A_i)+\sum_{j\neq i} t_j\\ &= f_k(A_i)-t_i+\sum_{j=1}^{n} t_j \qquad \text{for any } i, \end{aligned}$$

where $w_n(k)$ is the waiting time of the customer $C(k)$ at the stage A_n. Thus, the inequality of the left-hand side in the theorem holds.

Next, to prove the inequality of the right-hand side we consider the system $(A_1A_2\cdots A_nB)$ for any $B\in D(A_1A_2\cdots A_n)$. Then, we have

$$f_k(A_1A_2\cdots A_nB)=f_k(A_1A_2\cdots A_n)+w_B(k)+t_B,$$

where $w_B(k)$ is the waiting time of the customer $C(k)$ at the stage B. While for the system $(BA_1A_2\cdots A_n)$ we have

$$f_k(BA_1A_2\cdots A_n)=f_k(B)+\sum_{i=1}^{n} t_i.$$

From the above two relations and theorem 2.1, it is derived that

$$\begin{aligned} f_k(A_1A_2\cdots A_n) &= f_k(B)-t_B+\sum_{i=1}^{n} t_i-w_B(k)\\ &\leqq \{f_k(B)-t_B\}+\sum_{i=1}^{n} t_i. \end{aligned}$$

Therefore, we obtain

$$f_k(A_1A_2\cdots A_n)\leqq \min_{B\in D(A_1\cdots A_n)}\{f_k(B)-t_B\}+\sum_{i=1}^{n} t_i.$$

Using lemma 3.3, the desirable inequality follows at once.

4. System Reduction

Let us consider any system

(4.1) $$a(k)\overset{\infty}{\to}A_1\overset{q_2}{\to}A_2\overset{q_3}{\to}\cdots\overset{q_n}{\to}A_n\overset{\infty}{\to}z(k),$$

where each allowable queue size q_i ($i=2, 3, \cdots, n$) may be finite or infinite. Then by theorem 2.1 we may consider this system as follows without changing the flow-time sequence $\{z(k)-a(k),\ k=1, 2, \cdots\}$,

(4.2) $$a(k)\overset{\infty}{\to}A_1\overset{\infty}{\to}A_2\overset{\infty}{\to}\cdots\overset{\infty}{\to}A_n\overset{\infty}{\to}z(k).$$

If one stage A_j dominates all other stages, then using again theorem 2.1 we may place A_j first in the system (4.2) without changing flow-time sequence too, and by lemma 3.2(b) all waiting will occur only at the new first stage. From the view of flow-time, system (4.2) behaves like, or reduces to, the essentially single stage form

(4.3) $$a(k)\overset{\infty}{\to}A_j\to(\infty, t)\overset{\infty}{\to}z(k),$$

where t is the sum of the service times of all dominated stages, and (∞, t) symbolizes the stage with the infinite channels and service time t. The reduction procedure on the system of the form (4.2) with a stage dominating all others was illustrated by Friedman [2] in detail. However, in general, it is not possible to reduce a given system (4.1) to the single-stage form of (4.3). In such a case, the reduction procedure based on theorem 3.2 is applied to the given system to form a single-stage system.

The steps of the reduction procedure are as follows:

Step 1. Order the stages by increasing channel number c_i and, when channel numbers are equal, by decreasing service time.

Step 2. Within each sequence of stages with equal channel number, the first stage dominates all other stages. Move the dominated stages to the rear of the system.

Step 3. Apply the full dominance test (lemma 3.1) to the remaining stages, and move all dominated stages to the rear.

Step 4. Replace the dominated stages (whose order is immaterial) by (∞, t_d), where t_d is the sum of their service times. The result gives us the reduced form

(4.4) $$a(k)\overset{\infty}{\to}A_{p1}\overset{\infty}{\to}A_{p2}\overset{\infty}{\to}\cdots\overset{\infty}{\to}A_{pr}\to(\infty, t_d)\overset{\infty}{\to}z(k)$$

of the original system (4.1).

Step 5. Take up a stage $B=(c_B, t_B) \in M(A_{p1}A_{p2}\cdots A_{pr})$ and construct the system

(4.5) $$a(k) \xrightarrow{\infty} B \rightarrow (\infty, t-t_B) \xrightarrow{\infty} \bar{z}(k),$$

where t is the sum of all service times in the original system and $\bar{z}(k)$ is the departure time of the customer $C(k)$ in the new system (4.5). Then $\bar{z}(k) \geqq z(k)$ for all k.

Step 6. Select a stage $A_{pi}=(c_{pi}, t_{pi})$ of $A_{p1}, A_{p2}, \cdots, A_{pr}$ and construct the system

(4.6) $$a(k) \xrightarrow{\infty} A_{pi} \rightarrow (\infty, t-t_{pi}) \xrightarrow{\infty} \underline{z}(k),$$

where $\underline{z}(k)$ is the departure time of $C(k)$ in the resulting system (4.6). Then $\underline{z}(k) \leqq z(k)$ for all k.

Example. An example is given to demonstrate the reduction procedure. Let us consider the following system as an original:

$$a(k) \xrightarrow{\infty} A_1(4, 13) \xrightarrow{3} A_2(1, 3) \xrightarrow{\infty} A_3(2, 7) \xrightarrow{2} A_4(1, 2) \xrightarrow{\infty} z(k).$$

In the system the seven-step procedure is illustrated as follows:

Step 0. $a(k) \xrightarrow{\infty} (4, 13) \xrightarrow{\infty} (1, 3) \xrightarrow{\infty} (2, 7) \xrightarrow{\infty} (1, 2) \xrightarrow{\infty} z(k),$

Step 1. $a(k) \xrightarrow{\infty} \underbrace{(1, 3) \xrightarrow{\infty} (1, 2)}_{c=1} \xrightarrow{\infty} \underbrace{(2, 7)}_{c=2} \xrightarrow{\infty} \underbrace{(4, 13)}_{c=4} \xrightarrow{\infty} z(k),$

Step 2. $a(k) \xrightarrow{\infty} (1, 3) \xrightarrow{\infty} (2, 7) \xrightarrow{\infty} (4, 13) \rightarrow [(1, 2)] \xrightarrow{\infty} z(k),$

Step 3. $a(k) \xrightarrow{\infty} (1, 3) \xrightarrow{\infty} (2, 7) \rightarrow [(4, 13), (1, 2)] \xrightarrow{\infty} z(k),$

Step 4. $a(k) \xrightarrow{\infty} (1, 3) \xrightarrow{\infty} (2, 7) \rightarrow (\infty, 13+2) \xrightarrow{\infty} z(k),$
which is of the form (4.4),

Step 5. $a(k) \xrightarrow{\infty} (1, 7/2) \rightarrow (\infty, 25-7/2) \xrightarrow{\infty} \bar{z}(k),$

Step 6. $a(k) \xrightarrow{\infty} (1, 3) \rightarrow (\infty, 25-3) \xrightarrow{\infty} \underline{z}(k),$
or $a(k) \xrightarrow{\infty} (2, 7) \rightarrow (\infty, 25-7) \xrightarrow{\infty} \underline{z}(k).$

From the final forms in the steps 5 and 6 the flow time $f_k(A_1A_2A_3A_4)$ of the customer $C(k)$ is bounded above and below as follows:

$$\max\{f_k(A_2)-3, f_k(A_3)-7\}+25 \leqq f_k(A_1A_2A_3A_4)$$
$$\leqq f_k((1, 7/2))-7/2+25 \qquad \text{for all } k.$$

When the input process is Poissonian with parameter λ, the expected flow time f of any customer in the steady state is estimated in the following, where $\lambda \leqq 2/7$,

$$\{\text{maximum value of expected waiting times in } M(\lambda)/D(1/3)/1 \text{ and } E_2(\lambda/2)/D(1/7)/1\}+25$$
$$\leqq f$$
$$\leqq \{\text{expected waiting time in } M(\lambda)/D(2/7)/1\}+25.$$

It is well known that the expected waiting time in the system $M(\lambda)/D(\mu)/1$ is $\lambda/\{2\mu(\mu-\lambda)\}$. Thus, the above inequality gives us how to approximate the expected flow time in the steady state of any customer in the original complex system. In the system where all allowable queues are infinite, the waiting time of any customer and the queue length at each stage are respectively estimated according to the procedure presented by Friedman [2], but we will not go in detail.

References

[1] Avi-Itzhak, B., "A Sequence of Service Stations with Arbitrary Input and Regular Service Times," *Management Sci.*, **11** (1965), 565-571.

[2] Friedman, H.D., "Reduction Methods for Tandem Queuing Systems," *Opns. Res.*, **13** (1965), 121-131.