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A POLAR MODEL FOR A COOPERATIVE-COMPETITIVE DECISION BEHAVIOR IN AN ORGANIZATION

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Summary

A cooperative behavior of two suborganizations having different preferences in the same organization is represented in terms of the polar sets. This representation provides an algorithm to compromize the two preferences for optimization and risk aversion.

1. Introduction

The polar set has been studied by a few mathematicians from the interest in the duality between convex polytopes [6]. It is introduced into OR field by Balas in context of generating cutting planes for integer and nonlinear programming algorithms [1] [2]. Unlike his and Burdet's interest in its geometric properties to provide strong cuts [1] [2] [4] [5], this paper discusses its managerial interpretation. It is interpreted as the set of prices to guarantee a certain level of the revenue in any possible situation. While the sales amount is a decision variable, it is often subject to an uncontrollable change. On the other hand the price remains perfectly decidable in most cases. Therefore the price must be decided not only on the planned sales amount but also on the pessimistic one. The polar sets some of which are newly introduced herein are shown to provide the decision procedure for the problem.

In this decade a multilevel structured organization has attracted attentions with reference to the decentralization [3]. Here in this paper, however, two suborganizations are in the same level in the sense that each has its own activity, and each of them takes the initiative over, but does not control, the other.

2. Definitions and Properties of Polars

In the sequel R^k denotes the k-dimensional real space, R^k_+ denotes the positive orthant of R^k excluding the origin, $x,\ p \in R^n$, $S,\ T \subset R^n$, $S,\ T \neq \phi$, $u,\ l \in R^1$ such that $l<u$.

Def. 1. The generalized polar [1]: $P(S,\ u)=\{p \mid p^T x \leqq u,\ \forall x \in S\}$

Def. 2. The inverse polar: $P_I(S,\ l)=\{p \mid p^T x \geqq l,\ \forall x \in S\}$

Def. 3. The bibounded polar: $P_B(S,\ u,\ l)=\{p \mid l \leqq p^T x \leqq u,\ \forall x \in S\}$

Def. 4. [6] A convex polytope S_1 is dual to another convex polytope S_2 if the vertices of S_1 $1-1$ correspond to the faces of S_2 and the faces of S_1 $1-1$ correspond to the vertices of S_2. (A convex polytope is a bounded convex polyhedron.)

Let $S=\{x \mid a_i^T x \leqq b_i,\ \forall i=1,\ \cdots,\ m\}$ and $T=\{p \mid c_i^T p \leqq d_i,\ \forall i=1,\ \cdots,\ m'\}$ where $a_i,\ c_i \in R^n$ and $b_i,\ d_i \in R^1_+$.

Lemma 1 [1]. If S is a convex polytope then $P(S,\ u)$ is a convex polyhedron (possibly unbounded).

Lemma 2 [1]. If S is a convex polytope whose interior contains the origin, then $P(S,\ u)$ is a convex polytope dual to S. And $P(S,\ u) =\{p \mid p^T y_i \leqq u$ for each vertex y_i of $S\}$.

Lemma 3. If S is a convex polytope then $P_I(S,\ l)$ is a convex polyhedron (possibly unbounded). (Clear from Lemma 1 and Def. 2)

Theorem 1. $P_B(S,\ u,\ l)=P(S,\ u) \cap P_I(S,\ l)$.

Proof. Clear from Def. 1, Def. 2 and Def. 3.

Theorem 2. $P_B(S,\ u,\ l)$ is a convex polytope.

Proof. Since $p^T x$ in Def. 3 is upper and lower bounded, no component of vector p is unbounded. Thus by Theorem 1 $P_B(S,\ u,\ l)$ is a bounded convex polyhedron.

Theorem 3. If, given S, $P(S, u)$ is a convex polytope dual to S, then $P(S, u)$ is constructed from S by the following procedure.

Step 1. For each i, obtain n independent vectors $\bar{x}_i^k \in R^n$ $(k=1, \cdots, n)$ such that $a_i^T \bar{x}_i^k = b_i$.

Step 2. For each i, obtain the solution $v_i=(v_{i1}, \cdots, v_{in})$ to the system of linear equations $v_i^T \bar{x}_i^k = u$, $k=1, \cdots, n$.

Step 3. $P(S, u)=\{p=(p_1, \cdots, p_n) | p_j - \sum_{i=1}^{m} \theta_i v_{ij}=0,\ j=1, \cdots, n,\ \sum_{i=1}^{m} \theta_i=1,\ \theta_i \geqq 0,\ \theta_i \in R^1,\ i=1, \cdots, m\}$.

Proof. From the duality of S and $P(S, u)$, $S=P(P(S, u), u')$ for appropriate $u' \in R^1$. So the vertices of $P(S, u)$ 1—1 corresponds to faces of S. Thus step 1 and 2 correspond to the latter part of Lemma 2, and form the vertices of $P(S, u)$. By Lemma 1 $P(S, u)$ can be constructed as a convex combination of the vertices. QED.

Remark to step 1. The constraining hyperplanes of S are explicitly given while the vertices of S are unavailable. Therefore constructing $P(S, u)$ from the constraints of S is preferable. The points randomly selected on a hyperplane are interdependent with probability 0. Furthermore the dependence is easily checked, and another set of points is easily obtained.

Theorem 4. If S is a simplex not containing the origin, then there exists l (or u) for given u (or l respectively) such that $P_B(S, u, l)$ is a simplex dual to S.

Proof. Since the assumption $b_i \in R_+^1$ for all i in defining S precludes the parallel among the $n+1$ constraining hyperplanes of $P(S, u)$, a simplex S' is formed by them among which at least n hyperplanes are actively binding $P(S, u)$. S' is dual to S. $0 \notin S$ implys $p \notin$ Interior $P(S, u)$ for all $p \in S'$. If exactly n hyperplanes are actively binding $P(S, u)$, then all the vertices of S' except one (say v_0) lies outside $P(S, u)$, $P(S, u)$ is a cone with v_0 as the apex, and $P_I(S, l)$ for $l=u$ is a S'-truncated cone symmetric to $P(S, u)$ with respect to v_0. In other words S' separates $P(S, u)$ and $P_I(S, l)$ for $l=u$. Decreasing l shifts $P_I(S, l)$ through S' towards $P(S, u)$ until v_0 comes in Interior

$P_I(S, l)$. In a certain range of l, $P(S, u) \cap P_I(S, l)$ form a simplex S''. S'' is similar to S' and is dual to S. If the hyperplanes are all actively binding $P(S, u)$, then exactly one vertex (say v_0') of S' lies outside $P(S, u)$, $P_I(S, l)$ for $l=u$ is a cone with v_0' as the apex, and $P(S, u)$ is a S'-truncated cone symmetric to $P_I(S, l)$ with respect to v_0'. The analogous arguments hold. Exchanging u and l completes the proof. QED.

Theorem 5. $P_B(\bigcap_k S_k, u, l) \supset \bigcup_k P_B(S_k, u, l)$.

Proof. $P(S_1 \cap S_2, u) \supset P(S_1, u) \cup P(S_2, u)$ [4]. Analogously $P_I(S_1 \cap S_2, l) \supset P_I(S_1, l) \cup P_I(S_2, l)$. By Theorem 1 and the property of sets, $P_B(S_1 \cap S_2, u, l) = P(S_1 \cap S_2, u) \cap P_I(S_1 \cap S_2, l) \supset \{P(S_1, u) \cup P(S_2, u)\} \cap \{P_I(S_1, l) \cup P_I(S_2, l)\} \supset \{P(S_1\ u) \cap P_I(S_1, l)\} \cup \{P(S_2, u) \cap P_I(S_2, l)\} = P_B(S_1, u, l) \cup P_B(S_2, u, l)$. Repeating the process completes the proof. QED.

When the condition of Lemma 2 does not hold, some faces of S may correspond to no vertex of $P(S, u)$. More specifically, v_i obtained in step 2 of Theorem 3 may not satisfy $v_i^T y_j \leqq u$ for some vertex y_j of S. In this case the constraining faces f_i^k are obtained in $1-1$ correspondence to the vertices y_k^i of face i $(k=1, \cdots, N \geqq n)$ where $f_i^k = \{p | p^T y_k^i = u\}$. Then a number of quasivertices q_i^{kr} are appropriately obtained on f_i^k such that $q_i^{krT} y_j \leqq u$. $P(S, u)$ may be formed (or approximated) in aid of f_i^k (or q_i^{kr} respectively).

$P_I(S, l)$ or its approximation are constructed in an analogous way. Thus $P_B(S, u, l)$ is constructable.

Another way to approximate $P_B(S, u, l)$ is to apply Theorem 4. Selecting $n+1$ constraints of S and letting the others temporalily inactive forms a simplex S_1 containing S. Exchanging the active and the inactive constraints generates a series of simplices $\{S_k\}$. To S_k corresponds $P_B(S_k, u, l)$. Note $\bigcap_k S_k = S$. The union of $P_B(S_k, u, l)$ over k approximates $P_B(S_k, u, l)$ from inside as Theorem 5 shows.

3. Interpretation of Polars

Let us discuss a problem to decide the sales amounts and the prices of a series of commodities. A series of commodities means the commodities which are interdependent or mutually conflicting in demand and supply. In a deterministic case of the demand a centralized organization may decide the sales amounts x and the prices p by solving a bilinear programming problem $\max_{p,x} p^T x | x \in S,\ p \in T$.

Suppose only the pessimistic and the optimistic (possibly plus the expected) values of the sales amounts of the commodities are available with the distributions unknown. In this case the risk aversion may be to decide the prices which assure a certain level l of the revenue from the commodities in the pessimistic case, *i.e.*, the prices vector p such that $p^T x \geqq l,\ \forall x \in S$ which forms $P_l(S, l)$. A ceiling u may be set on the revenue to avoid the excessive profits. The ceiling u may be just an arbitrary large value. Thus $P_B(S, u, l)$ is obtained. If the total costs should be under u, the costs vector p should be $p^T x \leqq u,\ \forall x \in S$ which forms $P(S, u)$. Setting a floor l forms $P_B(S, u, l)$.

4. A Cooperative-Competitive Decision Behavior

Consider two suborganizations (or two specialized decision makers) which contribute to the whole organization in different ways. They have different but not contradictory preferences. Therefore they are primarily cooperative and secondly competitive. For example, a suborganization responsible for production level considers the expected revenue more important than the risk aversion, and the other suborganization responsible for pricing the contrary. The following decision procedure may reflect a realistic behavior of the two.

Step 0. S and T denote the feasible regions of the sales amounts and the prices of a series of commodities respectively.

Step 1. The set of risk averting prices is defined by $P_B(S, u, l) \cap T$. If it is empty, reconsider S, T, u, and l.

Step 2. Let $\bar{x}$ be the n-vector of the middle values of the optimistic and pessimistic values of the sales amounts of the commodities. (If the expected values are available, they are preferable.) Naturally $\bar{x} \in S$.

Step 3. Decide the "optimal" prices vector $\bar{p}$ for the commodities relative to the middle values $\bar{x}$ such that $\bar{p}\bar{x} = \max_p p\bar{x} | p \in P_B(S, u, l) \cap T$, given $\bar{x}$.

Step 4. Decide the "optimal" sales amount (or production level) vector $\bar{\bar{x}}$ of the commodities relative to the decided prices $\bar{p}$ such that $\bar{p}x = \max_x \bar{p}x | x \in S$, given $\bar{p}$.

The forecast demands $\bar{x}$ found the prices $\bar{p}$ which furthermore found the production levels $\bar{\bar{x}}$. Though the two suborganizations are equal in level, pricing belongs to a higher decision level where the risk aversion is also considered. This decision process may reflect more realistic behavior of organizations than the bilinear model $\max_{p,x} p^T x | x \in S,\ p \in T$ where an extreme centralization is assumed and the risk aversion is neglected. The prices and the sales amounts (production levels) decided in the proposed way are expected to be acceptable to both suborganizations with little dissatisfaction.

5. Extension

The revenue function may be $p^T Q x$ where Q is a $n \times n$ non-singular matrix (not necessarily definite). The set $\{p | p^T Q x \leqq u,\ \forall x \in S\}$ is termed a simple polaroid and is shown to be convex [4]. Hence the "optimal" prices are obtainable on a simple polaroid by an appropriate algorithm.

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